# Comments on "Noise-induced transitions in a simplified model of the thermohaline circulation"

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Starting from the classical Stommel (1961) two-box model for the North Atlantic meridional overturning circulation, Timmermann and Lohmann (2000, hereafter TL) consider the following pair of equations:

$$\frac{d}{dt}y = -|1-y|y + \mu_0 + y\epsilon \tag{1}$$

$$\frac{d}{dt}\epsilon = -\frac{\epsilon}{\tau} + \frac{\sigma}{\tau}\xi_t \tag{2}$$

In the above, $y$ represents the scaled salinity difference between the two boxes, $\mu_0$ is the scaled salinity forcing, and $\epsilon$ is an Ornstein-Uhlenbeck process with variance $\sigma^2/2\tau$ and autocorrelation e-folding time $\tau$. TL associate the process $\epsilon$ with the temperature difference between the two-boxes. The representation of the temperature gradient by an Ornstein-Uhlenbeck process is motivated by the findings of Lohmann and Schneider (1998) that in the Stommel model temperature differences vary on a much shorter timescale than salinity differences. As $\tau \to 0$, the process $\epsilon$ becomes Gaussian white noise. In this limit, equation (1) becomes a (Stratonovich) stochastic differential equation with multiplicative noise, that is, with a diffusion dependent on the state of the system (Gardiner, 1997). The probability density function (pdf) of $y$ then satisfies a one-dimensional Fokker-Planck equation (FPE), the stationary version of which can be integrated exactly. However, for $\tau \neq 0$, the process $y$ is no longer Markov and does not have an associated FPE. Instead, it is the two-dimensional process $(y, \epsilon)$ that is Markov with an associated FPE. Unfortunately, the stationary version of this FPE is in two variables and cannot be integrated analytically.

To circumvent this problem, TL employ the Unified Coloured Noise Approximation (UCNA, Jung and Hänggi, 1987) to reduce the 2-dimensional system to an approximate 1-dimensional system, whose associated Fokker-Planck equation is straightforwardly integrated. TL calculate the following UCNA expression for the stationary probability density function of $y$:

$$
\begin{aligned}
p_s(y,\tau) &= N_1 y^{-1}\left|1+\tau\left(\frac{\mu_0}{y}-y\right)\right| e^{\frac{2}{\Sigma^2}\left[-\frac{1}{2}\tau\left(-1+y+\frac{\mu_0}{y}\right)^2-\ln y+y-\frac{\mu_0}{y}\right]} \quad ,0<y\leq 1\\
p_s(y,\tau) &= N_2 y^{-1}\left|1+\tau\left(\frac{\mu_0}{y}+y\right)\right| e^{\frac{2}{\Sigma^2}\left[-\frac{1}{2}\tau\left(1-y+\frac{\mu_0}{y}\right)^2+\ln y-y-\frac{\mu_0}{y}\right]} \quad ,y>1
\end{aligned}
\tag{3}
$$

where $N_1, N_2$ are appropriate normalisations. Figures 6 and 7 of TL plot $p_s$ as a function of $y$ for different values of $\tau$ and $\sigma$. For small values of $\tau$, the results resemble those of the white noise limit. For increasing $\tau$, $p_s$ given by (3) displays qualitatively different behaviour. In particular, nodes and new extrema appear in the pdf. These results are interpreted as "noise-induced transitions", and the analogy of quantum-mechanical tunneling is used to describe the passage of the system across the node of the pdf. These features, however, are artificial and reflect the breakdown of the validity of the UCNA for $\tau$ of $O(1)$.

By construction, the UCNA assumes a timescale separation between the processes $y$ and $\epsilon$; it is only valid when $\epsilon$ varies much more rapidly than $y$. This implies that the UCNA is a small $\tau$ approximation, but how small is "small"? TL note that the domain of validity of the UCNA is given by the following pair of inequalities:

$$\kappa(y,\tau)=\frac{1}{\sqrt{\tau}}-\sqrt{\tau}\left(y-\frac{\mu_0}{y}\right)>>\tilde{t}^{-1} \quad ,0<y\leq 1 \tag{4}$$

$$\kappa(y,\tau)=\frac{1}{\sqrt{\tau}}-\sqrt{\tau}\left(-y-\frac{\mu_0}{y}\right)>>\tilde{t}^{-1} \quad ,y>1, \tag{5}$$

where $\tilde{t}$ is a "characteristic timescale" of variability of $y$. Defining a "typical" value of $y$, TL argue for the global (in $y$) satisfaction of these conditions. In fact, their validity must be considered locally in $y$. In particular, the UCNA certainly fails wherever $\kappa$ vanishes. For

$0 < y \leq 1$, this occurs for

$$y = \frac{1}{2\tau} + \sqrt{\left(\frac{1}{2\tau}\right)^2 + \mu_0} \tag{6}$$

(a second root is discarded because it occurs for $y < 0$, outside the domain of consideration). For $\tau < \frac{1}{1-\mu_0}$, this root also falls outside of $[0, 1]$. However, for $\tau \geq \frac{1}{1-\mu_0}$, $\kappa$ vanishes for $y \in [0, 1]$, and the UCNA fails within the domain of interest. In particular, it is clear from (3) that $\kappa$ vanishes at precisely the values $y$ where $p_s$ has a node. Thus, the emergence of zeros in the pdf of $y$ is an artifact of the breakdown of the UCNA. Presumably, the UCNA fails not just at the points at the zeros of $\kappa$, but in a surrounding neighbourhood, the size of which is determined by the timescale $\tilde{t}$. For $y > 1$, $\kappa$ never vanishes, and the UCNA should thus remain appropriate throughout this region. However, because the overall amplitude of the pdf is a function of its global structure, only the shape of the pdf for $y > 1$ should agree with the plots displayed in TL. Thus, for $\tau$ of $O(1)$, the UCNA can be shown to break down locally in $y$, with global consequences for the structure of the pdf.

Another approximate approach to determine the stationary distribution associated with (1),(2) is to integrate the equations numerically. Numerical integration of stochastic differential equations is discussed in Kloeden and Platen (1992). The simplest algorithm is a forward Euler discretisation. Denoting the discrete time step by $\delta$ so that

$$t_k = k\delta \tag{7}$$

the forward Euler discretisation of (1),(2) is

$$y_{t_k} = y_{t_{k-1}} + \delta(-|1 - y_{t_{k-1}}|y_{t_{k-1}} + \mu_0 + y_{t_{k-1}}\epsilon_{t_{k-1}}) \tag{8}$$

$$\epsilon_{t_k} = \epsilon_{t_{k-1}} - \delta\frac{\epsilon_{t_{k-1}}}{\tau} + \sqrt{\delta}\frac{\sigma}{\tau}W_{t_{k-1}} \tag{9}$$

where $W_{t_k}$ is a sequence of zero-mean, unit variance Gaussian random variables. Note that equations (1),(2) are an *additive* vector Langevin equation, not multiplicative as in the $\tau \to 0$ limit, because at no point in these equations is the white noise multiplied by a function of a state variable. Consequently, we do not need to distinguish between Ito and Stratonovich calculi in the discretisation of these equations.

Equations (8) and (9) were integrated for 15000 time units with a timestep $\delta = 0.05$ for parameter values $\tau$, $\sigma$ used in Figures 6 and 7 of TL. Figures 1 and 2 display Gaussian kernel density estimates of the stationary distribution functions. These results are robust to reduction of the stepsize $\delta$. Furthermore, inspection of the time series indicates that the record is long enough for estimation of the pdf from the time series to be appropriate.

Comparing Figures 1 and 2 with Figures 6 and 7 of TL, it is clear that for small values of $\tau$, the stationary pdfs produced by numerical integration and by the UCNA are in close agreement. However, for $\tau \sim O(1)$, there are marked differences. In particular, the pdfs produced by numerical integration do not display any nodes or new extrema. Instead, the result of raising $\tau$ for fixed $\sigma$ is seen to be a shift of the pdf toward the right-hand peak.

Calculating the stationary pdf of a system in a one-dimensional potential subject to coloured noise remains an unsolved problem in physics. A number of different approximations have been proposed, but they are valid only in the limit of small or of very large $\tau$ (Horsthemke and Lefever, 1984; Hänggi and Jung, 1995). To obtain the stationary pdfs of $y$ in the case where its timescale is of the same order of magnitude as $\epsilon$, at present we must take recourse to numerical methods. We note that an essential conclusion of TL is unchanged, namely that increasing $\sigma$ populates the left-hand peak of the stationary pdf of $y$ at the expense of the right-hand peak, while increasing $\tau$ has the opposite effect.

We also remark on the derivation and interpretation of equations (1),(2). The original pair of equations (5) and (7) of TL for the salinity and temperature differences read:

$$\frac{d}{dt}\Delta S = -\frac{c}{V}|\alpha\Delta T_o + \alpha\Delta T' - \beta\Delta S|\Delta S + \frac{S_0}{h}(P - E) \tag{10}$$

$$\frac{d}{dt}\Delta T' = -\gamma\Delta T' + \Sigma\xi_t, \tag{11}$$

where $V$ is the box volume, $c$ is the proportionality constant between the density gradient and the meridional flux, $\alpha$ and $\beta$ are respectively the thermal and haline expansion coefficients, $S_0$ is a reference salinity, $h$ is the box depth, $P - E$ is the salinity forcing, and $\Delta T_o$ is the time-mean meridional temperature difference. In equation (10), the process $\Delta T'$ appears only inside the absolute value sign. In equation (1), however, the process appears *outside* the absolute value sign. In fact, equation (1) does not follow from (10). Furthermore, it is not at all clear how (10) behaves in the limit considered by TL that $\Delta T'$ becomes a white noise process (Horsthemke and Lefever, 1984).

An alternate interpretation of (1) is as follows. We consider the Stommel (1961) model:

$$\frac{d}{dt}\Delta S = (-|q| + \epsilon)\Delta S + \frac{S_0}{h}(P - E) \tag{12}$$

$$\frac{d}{dt}\Delta T = (-|q| + \epsilon)\Delta T + \frac{F_{oa}}{C_p} \tag{13}$$

where the flux $q$ between the boxes is determined by the Stommel Ansatz:

$$q = \frac{c}{V}(\alpha\Delta T - \beta\Delta S), \tag{14}$$

and $C_p$ is the oceanic heat capacity. The freshwater forcing $P-E$ is an unspecified function of $\Delta T$, and the atmosphere-ocean heat flux $F_{oa}$ has the net effect of relaxing $\Delta T$ to some value $\Delta T_o$. The quantity $\epsilon$ in equations (12) and (13) is interpreted as a parameterisation of the transport of temperature and salinity between the boxes by eddies, which could constitute a non-trivial term in the heat and salinity budgets of each box. The process $\epsilon$ is *not* constrained to be positive, so eddy transport between the boxes may be upgradient (see e.g. Nakamura and Chao, 2000).

We model $\epsilon$ as an Ornstein-Uhlenbeck process and consider the limit that the relaxation of $\Delta T$ to $\Delta T_o$ is very fast relative to the timescale of the salinity dynamics. In this limit, we can make the approximation that $\Delta T$ is fixed at $\Delta T_o$ and $P - E$ is a constant. Then, scaling $\Delta S$ and $t$ as in TL, we obtain equation (1).

Recently, the stability of the climate system and the role of the thermohaline circulation in climate change has been investigated using models of different level of complexity (e.g., Manabe and Stouffer, 1994, 1995; Stocker and Schmittner, 1997). There is, however, some controversy about the reliability of model simulations (Rahmstorf, 1999; Rahmstorf and Ganopolski, 1999; Latif et al., 2000). One argument in favour of models of reduced complexity (e.g. Stocker and Schmittner, 1997) is that sensitivity studies can be performed in order to understand the basic underlying mechanisms. Furthermore, these models have been validated for the paleoclimate where the climate state is different from today's (Ganopolski et al., 1998). We note, however, that the noise level may enter into the estimation of climate sensitivity. Our investigations with a more conceptual model of the ocean circulation show that climate variability and the question of basic states are not separated from each other. The noise level as an inherent feature of each model should be considered as an important parameter affecting multiple equilibria (TL) and climate oscillations (Nicolis, 1982; Benzi et al., 1982).

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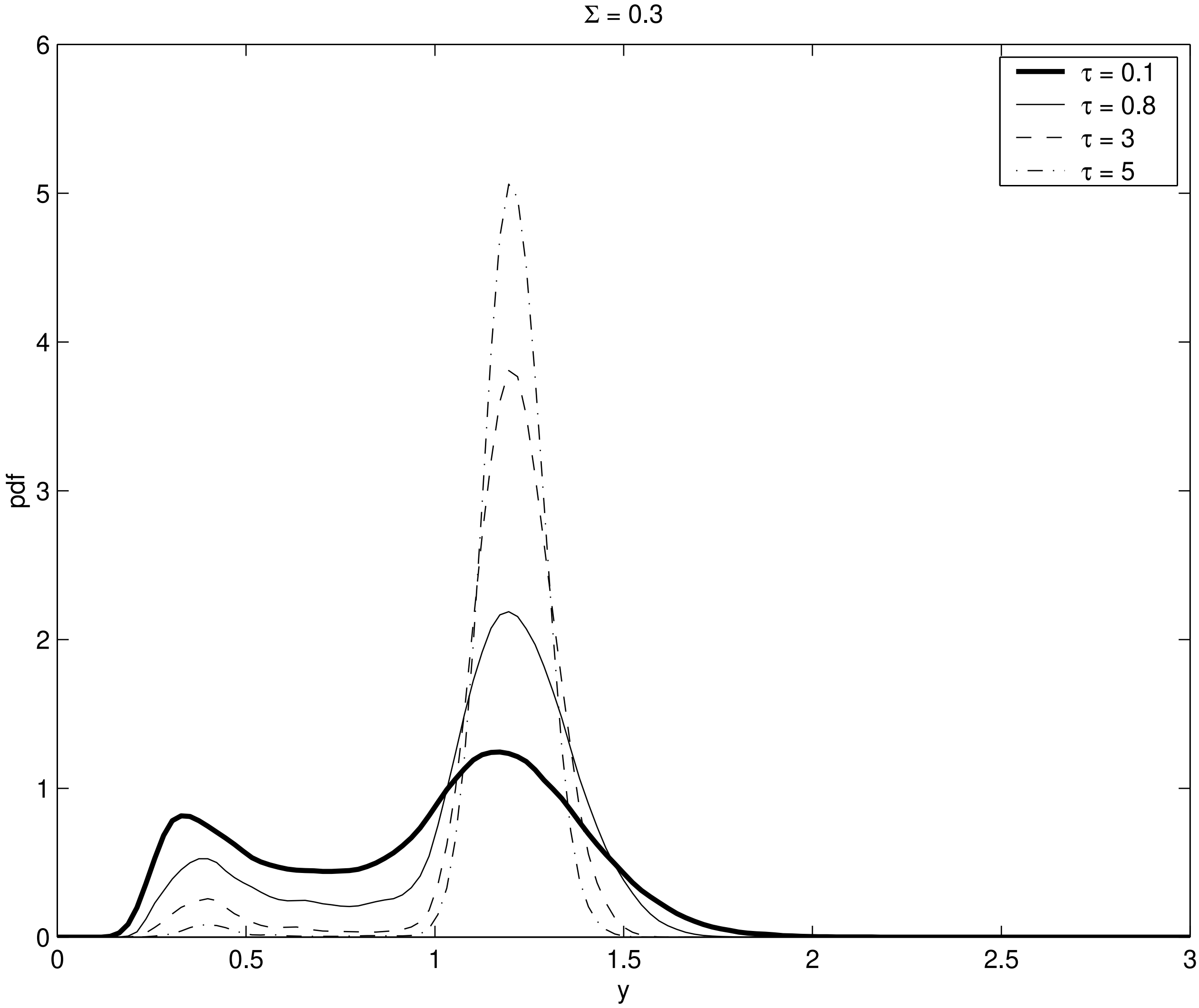


Figure 1: Gaussian kernel estimate of stationary pdf of (1),(2), produced by numerical integration. $\Sigma = 0.3$.

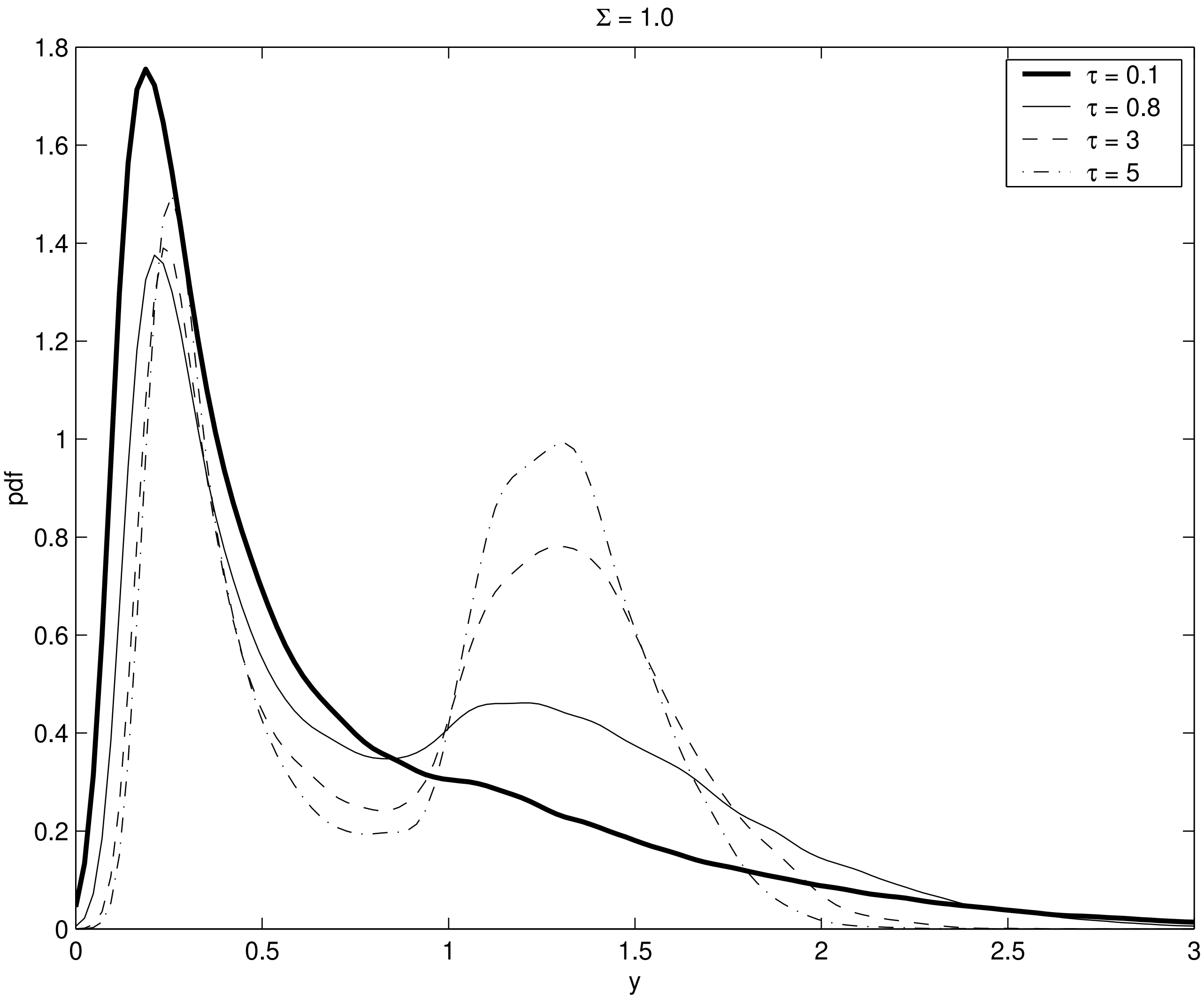


Figure 2: Gaussian kernel estimate of stationary pdf of (1),(2), produced by numerical integration. $\Sigma = 1$.