Climate Dynamics:

Concepts, Scaling and Multiple Equilibria

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Chapter 1

Introduction and Preparation

General framework: Climate dynamics from a fluid dynamics and complex systems approach

Paleoclimate reconstructions, in particular from ice cores have also shown that climate can change over relatively short periods such as a few years to decades (Fig. 1.1). Over the last century, humans have altered the composition of the Earth's atmosphere and surface to the extent that these factors measurably affect current climate conditions. The objective of the book is to examine fundamental concepts used to understand climate dynamics. Here, we will approach climate dynamics from a fluid dynamics and complex systems point of view.

In Chapter 2.3 I provide a framework to analyze the stability of dynamical systems. These systems provide the prototype of nonlinear dynamics, bifurcations, multiple equilibria. A bifurcation occurs when a parameter change causes the stability of an equilibrium. In his classic studies of chaotic systems, Lorenz has proposed a deterministic theory of climate change with his concept of the 'almost-intransitivity' of the highly non-linear climate systems. In the Lorenz equations exist the possibility of multiple stable solutions and internal variability, even in the absence of any variations in external forcing [Lorenz, 1976]. More complex models, e.g. Bryan [1986]; Dijkstra et al. [2004] also demonstrated this possibility. Chapter 4 deals with the general structure of fluid
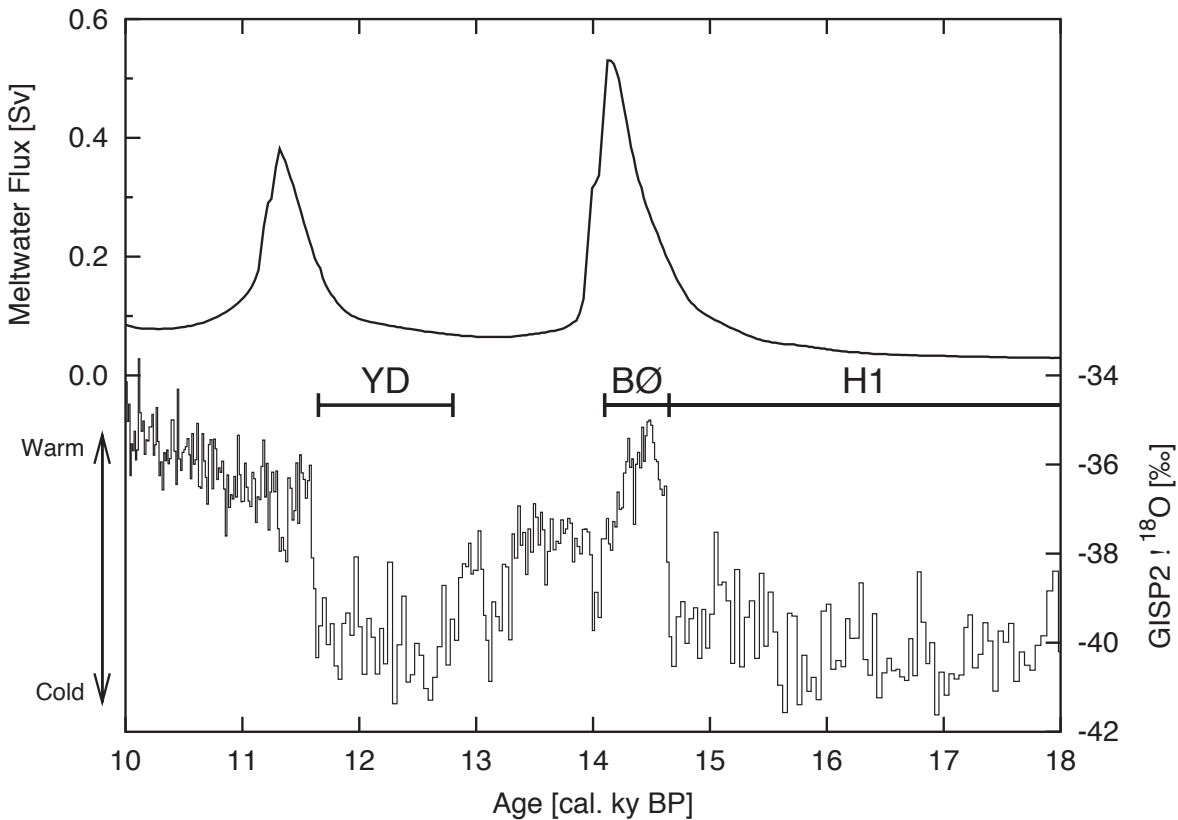
Figure 1.1: Top: Sea-level derived rate of deglacial meltwater discharge [Fairbanks, 1989]. Bottom: Oxygen isotope record from Greenland GISP2 ice core [Grootes and Stuiver, 1997], reflecting air temperature. Meltwater influx is maximum during the Bølling warm period (BØ) and minimum during Heinrich event 1 (H1) and Younger Dryas (YD). The time axis goes from right (18.000 years before present) to left (10.000 years before present). Source: Lohmann and Schulz [2000].

dynamical models. Like the ocean, the atmosphere is considered as a Newtonian Fluid. The concepts of scaling and vorticity are introduced. Chapter 5.3 deals with fluid dynamical applications, e.g. the Rayleigh-Bénard convection. A numerical solution of the Rayleigh-Bénard convection is provided in Chapter 3.4.

Chapter 6 repeats part of the main dynamics of large-scale oceanography. The Coriolis effect is one of the dominating forces for the large-scale dynamics of the oceans and the atmosphere. In meteorology and ocean science, it is convenient to use a rotating frame of reference where the Earth is stationary. The resulting flow can be derived from scaling arguments in a similar way as in text books of oceanography [Gill, 1982; Dijkstra, 2000; Olbers et al., 2012]. One fundamental

aspect of ocean dynamics are waves. A short theory is given and numerical examples are pro-vided. Furthermore, the deep ocean circulation is studied in a conceptual box model. Here, we introduce an interhemispheric box model of the deep ocean circulation to study the feedbacks in the climate system. The box model consists of four oceanic and three atmospheric boxes. The Atlantic deep ocean circulation is simulated by a simple model of meridional overturning. In the model of [Rooth, 1982] the Atlantic Ocean is described over both hemispheres and consider the North Atlantic and South Atlantic Ocean, respectively. This model is implemented in Chapter 7.2 and several applications can be performed such as the effect of meltwater on climate (Fig. 1.1). Such simple systems provide a general idea of the dominant processes in a complex system.

The structure of fluid dynamical models is valid for systems with many degrees of freedom, many collisions, and for substances which can be described as a continuum. The transition from the highly complex dynamical equations to a reduced system is an important step since it gives more credibility to the approach and its results. The transition is also necessary since the active entangled processes are running on spatial scales from millimetres to thousands of kilometres, and temporal scales from seconds to millennia. Therefore, the unresolve processes on subgrid scales have to be described. This is the typical problem in statistical physics: How can we obtain the macroscopic dynamics from the underlying (and often known microdynamical) theory? Two dif-ferent solutions are known, one is the so-called Mori-Zwanzig approach [Mori, 1965; Zwanzig, 1960, 1980] which relates the evolution of macroscopic variables to microscopic dynamics. The basic idea is the evolution of a system through a projection on a subset (macroscopic relevant part), where a randomness reflects the effects of the unresolved degrees of freedom. A particular exam-ple is the Brownian motion [Einstein, 1905; Langevin, 1908]. The other solution for the transition form the micro to macro-scales goes back to Boltzmann [1896]. The Boltzmann equation, also often known as the Boltzmann transport equation [Boltzmann, 1896; Bhatnagar et al., 1954; Cer-cignani, 1990] describes the statistical distribution of one particle in a fluid. It is one of the most important equations of non-equilibrium statistical mechanics, the area of statistical mechanics that deals with systems far from thermodynamic equilibrium. It is applied, for instance, when there

is an applied temperature gradient or electric field. Both, the Mori-Zwanzig and Boltzmann approaches play also a fundamental role in physics. The microscopic equations show no preferred time direction, whereas the macroscopic phenomena in the thermodynamics have a time direction through the enthropy. The underlying procedure is that part of the microscopic information is lost through coarse graining in space and time. Chapter 3 describes the approach from statistical mechanics towards the macroscopic theory. The Boltzmann equation and the Brownian motion are **the** approaches to understand the transition from micro to macro scales. For climate, this transition between the climate and weather scales has been formulated [Hasselmann, 1976; Leith, 1975], and later re-formulated in a mathematical context [Arnold, 2001; Chorin et al., 1999; Gottwald, 2010]. The effect of the weather on climate is seen by red-noise spectra in the climate system, showing one of the most fundamental aspects of climate, and serving also as a null hypothesis for climate variability studies. Chapter 3.4 deals with a fluid dynamical application, a 2D implementation of the Lattice Boltzmann Method (LBM) with the Bhatnagar-Gross-Krook (BGK) collision operator. The main structural parts of the program and several hints for the potential users are provided. While we do include a brief outline of the theory of LBM, detailed explanations are out of the scope of this book. Fore more details, please consult the references herein. The present code is intended to serve mainly as a showcase/practical introduction to Lattice Boltzmann Methods, hence advanced features and state-of-the-art algorithm improvements have been intentionally ommitted in favor of simplicity. One practical example, the Rayleigh-Benard convection [Rayleigh, 1916], is presented.

The content (first part) is designed for 12 lessons for a master course. The numerical examples may be helpful for the students who are already familiar with programming (they can improve the code and follow the main ideas of the code etc.), for those who are not familiar they can use it more as a black box and as a starting point for more research. Several task do not require that the complete code is understood, but one can change initial conditions or parameters in the problems. In the following, I list some exercises and introductionary material which can be used in the preparation of this course.

Exercise 1   –   Earth's curvature

1. The highest building on the campus of the University of Bremen is the so-called drop tower with a hight of h=110 metres (Fig. 13.2 upper panel). How far one can look onto the horizon under good weather conditions?

   Hint: Denote this distance by d. Remember the Earth's radius $a = 6378km$ and apply Pythogoras!

2. Why is the rule-of-thumb

$$d = \sqrt{2ha}$$

   a good approximation? (For h=10m this means d=11 km.) When h is in m, d in km, the formula can be written as

$$d = 3.5\sqrt{\frac{h}{m}} \text{ km.}$$

3. The town Bremerhaven where the Alfred Wegener Institute is located lies about 60 km north of Bremen. How big must a tower in Bremen be in order to see the coast in Bremerhaven? (Fig. 1.2 lower panel).

Exercise 2   –   Nabla

Calculate the following operations for the function

$$f(x, y, z) = x^3 + 3x - 4xz + z^4 \quad : \tag{1.1}$$

a)∇f,

b) calculate the divergence of the result!

c) Calculate the rotation of ∇f!

Figure 1.2: Upper panel: Drop tower in Bremen. Lower panel: Harbor in Bremerhaven, ca. 60 km north of Bremen.

1.1   Pendulum

The simple pendulum is another mechanical system that exhibits periodic motion. It consists of a particle-like bob of mass m suspended by a light string of length L that is fixed at the upper end, as shown in Fig. 1.3. The motion occurs in the vertical plane and is driven by the force of gravity. We shall show that, provided the angle Θ is small (less than about $10°$), the motion is that of a simple harmonic oscillator. The forces acting on the bob are the force T exerted by the string and the
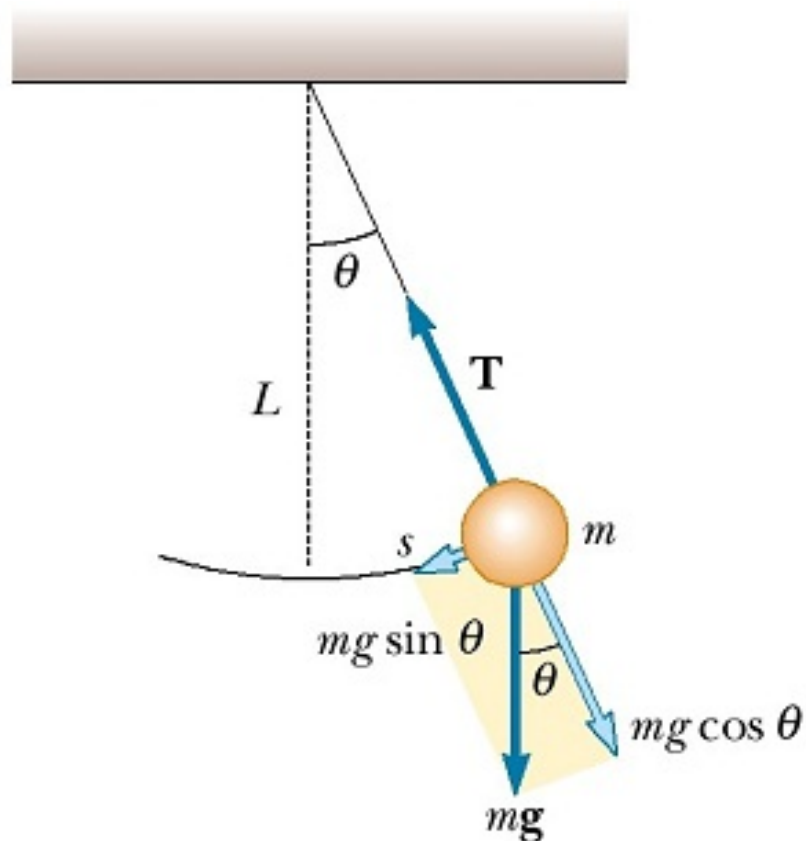
Figure 1.3: When Θ is small, a simple pendulum oscillates in simple harmonic motion about the equilibrium position $\Theta = 0$. The restoring force is $mg \sin \Theta$, the component of the gravitational force tangent to the arc.

gravitational force mg. The tangential component of the gravitational force, $mg \sin \Theta$, always acts toward $\Theta = 0$, opposite the displacement. Therefore, the tangential force is a restoring force,

and we can apply Newton's second law for motion in the tangential direction:

$$F = -mg \sin \Theta = m\frac{d^2s}{dt^2} \tag{1.2}$$

where s is the bob's displacement measured along the arc and the minus sign indicates that the tangential force acts toward the equilibrium (vertical) position. Because $s = L\Theta$ and L is constant, this equation reduces to the equation of motion for the simple pendulum.

$$\frac{d^2\Theta}{dt^2} = -\frac{g}{L}\sin \Theta \tag{1.3}$$

If we assume that Θ is small, we can use the approximation $\sin \Theta = \Theta$, thus the equation of motion for the simple pendulum becomes equation of motion for the simple pendulum

$$\frac{d^2\Theta}{dt^2} = -\frac{g}{L}\Theta \tag{1.4}$$

with solution

$$\Theta = \Theta_0 \cos(\omega t) \tag{1.5}$$

where $\omega = \sqrt{\frac{g}{L}}$ is the angular frequency.

The period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity. Because the period is independent of the mass, we conclude that all simple pendulums that are of equal length and are at the same location (so that g is constant) oscillate with the same period. The simple pendulum can be used as a timekeeper because its period depends only on its length and the local value of g. It is also a convenient device for making precise measurements of the free-fall acceleration. Such measurements are important because variations in local values of g can provide information on the location of oil and of other valuable underground resources.

Rule of thumb for pendulum length

It is useful to have a Rule of thumb for the period of the motion, the time for a complete oscillation (outward and return) is

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \text{can be expressed as} \quad L = \frac{g}{\pi^2}\frac{T^2}{4}. \tag{1.6}$$

If SI units are used (i.e. measure in metres and seconds), and assuming the measurement is taking place on the Earth's surface, then $g \approx 9.81 m/s^2$, and $g/\pi^2 \approx 1$ (0.994 is the approximation to 3 decimal places). Therefore, a relatively reasonable approximation for the length and period are,

$$L \approx \frac{T^2}{4},$$
$$T \approx 2\sqrt{L} \tag{1.7}$$

where T is the number of seconds between two beats (one beat for each side of the swing), and L is measured in metres.

Full problem without the approximation

If we consider the full problem without the approximation, the period is modified according to

$$T = 4\sqrt{\frac{L}{g}}K(k), \quad k = \sin\frac{\theta_0}{2} \tag{1.8}$$

where K is the complete elliptic integral of the first kind defined by

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1 - k^2\sin^2 u}}\,du\,. \tag{1.9}$$

For comparison of the approximation to the full solution, consider the period of a pendulum of length 1 m on Earth at initial angle 10 degrees is

$$4\sqrt{\frac{1 \text{ m}}{g}} \; K\left(\sin\frac{10°}{2}\right) \approx 2.0102 \text{ s.} \tag{1.10}$$

The linear approximation gives

$$2\pi\sqrt{\frac{1 \text{ m}}{g}} \approx 2.0064 \text{ s.} \tag{1.11}$$

The difference between the two values, less than **0.2%,** is much less than that caused by the variation of g with geographical location.

Foucault pendulum

Exercise 3   –   Foucault pendulum

The Foucault pendulum was the brainchild of the French physicist Leon Foucault. It was intended to prove that Earth rotates around its axis. Let us denote x, y the pendulum bob coordinates as seen by an observer on Earth. L is the length of the pendulum string and Θ is the pendulum angle. The pendulum moves, according to the restoring force from gravity. The string tension components can be expressed using small angle approximations, which also considerably simplify the problem, making it two-dimensional. The string tension due to the gravity force is

$$F_g = mg \begin{pmatrix} \sin\Theta \\ \sin\Theta \\ \cos\Theta \end{pmatrix} \approx mg \begin{pmatrix} x/L \\ y/L \\ 1 - z/L \end{pmatrix} \quad .$$

Figure 1.4: Foucault's famous pendulum in the Pantheon, Paris. What keeps it moving? Air resistance would normally stop the pendulum after a few hours – so an iron collar is installed on the wire surrounded by an electromagnet that attracts the collar as the bob swings out, then shuts off automatically as it swings back, thus, keeping pendulum going. The magnet is turned on and off by a switch which is activated when the support wire interrupts a beam of light shining across its path. Similar idea is followed by the Bremen Foucault's pendulum in our department.

Then, the horizontal dynamics can be described as

$$\ddot{x} = f\dot{y} - \frac{g}{L}x \tag{1.12}$$

$$\ddot{y} = -f\dot{x} - \frac{g}{L}y \tag{1.13}$$

where $f = 2\Omega \sin \varphi$.

1. Show the analytic solution to the Foucault pendulum problem introducing the complex number $\xi = x + i \cdot y$. Furthermore, call $\omega = \sqrt{\frac{g}{L}}$ is the angular frequency. Then,

$$\ddot{\xi} + if\dot{\xi} + \omega^2\xi = 0 \tag{1.14}$$

With the ansatz

$$\xi = H(t) \cdot \exp\left(-\frac{if}{2}t\right) \tag{1.15}$$

we obtain an equation for H

$$\ddot{H} + \left(\omega^2 + \frac{f^2}{4}\right)H = 0 \tag{1.16}$$

$$H(t) = C \exp\left[\pm it\sqrt{\omega^2 + \frac{f^2}{4}}\right] \tag{1.17}$$

and therefore

$$\xi = C \exp\left[it\left(-\frac{f}{2} \pm \sqrt{\omega^2 + \frac{f^2}{4}}\right)\right] \approx C \exp\left[it\left(-\frac{f}{2} \pm \omega\right)\right] \tag{1.18}$$

where C is a complex integration constant. The pendulum swing has a natural frequency

(also called pulsation) $\omega = \sqrt{g/L}$, which depends on the length of the pendulum string.[1] Looking at the last term in (1.23): At either the North Pole or South Pole, the plane of oscillation of a pendulum remains fixed relative to the distant masses of the universe while Earth rotates underneath it, taking one day to complete a rotation (frequency $\Omega = 2\pi/24h$). So, relative to Earth, the plane of oscillation of a pendulum at the North Pole undergoes a full clockwise rotation during one day, a pendulum at the South Pole rotates counterclockwise.[2]

When a Foucault pendulum is suspended at the equator, the plane of oscillation remains fixed relative to Earth. At other latitudes, the plane of oscillation precesses relative to Earth with a frequency $f/2 = \Omega \sin \varphi$ proportional to the sine of the latitude, where latitudes north and south of the equator are defined as positive and negative, respectively. For example, a Foucault pendulum at $30°$ S, viewed from above by an earthbound observer, rotates counterclockwise $360°$ in two days.

2. For Foucault's famous pendulum in Paris: The plane of the pendulum's swing rotated clockwise $11°$ per hour, making a full circle in 32.7 hours. What is the time period in Bremen, Germany?

3. Display the solution and compare it with the numerical solution with the following initial condition:

```
g = 9.81              # acceleration of gravity (m/s^2)
L = 67                # pendulum length (m) for the experiment in Paris
initial_x = L/100 # initial x coordinate (m)
initial_y = 0         # initial y coordinate (m)
initial_u = 0         # initial x velocity (m/s)
initial_v = 0         # initial y velocity (m/s)
Omega=2*pi/86400      # Earth's angular velocity of rotation (rad/s)
phi=49/180*pi         # 49 deg latitude in (rad)   for Paris 1851
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[1]For Foucault's famous pendulum: he suspended a 28 kg brass-coated lead bob with a 67 meter long wire from the dome of the Pantheon in Paris (about $49°$N). The natural frequency is $\sqrt{g/L} = 0.381/s$ related to a time period of 16 s.

[2]for the South Pole, there was indeed an experiment [Baker and Blackburn, 2005].
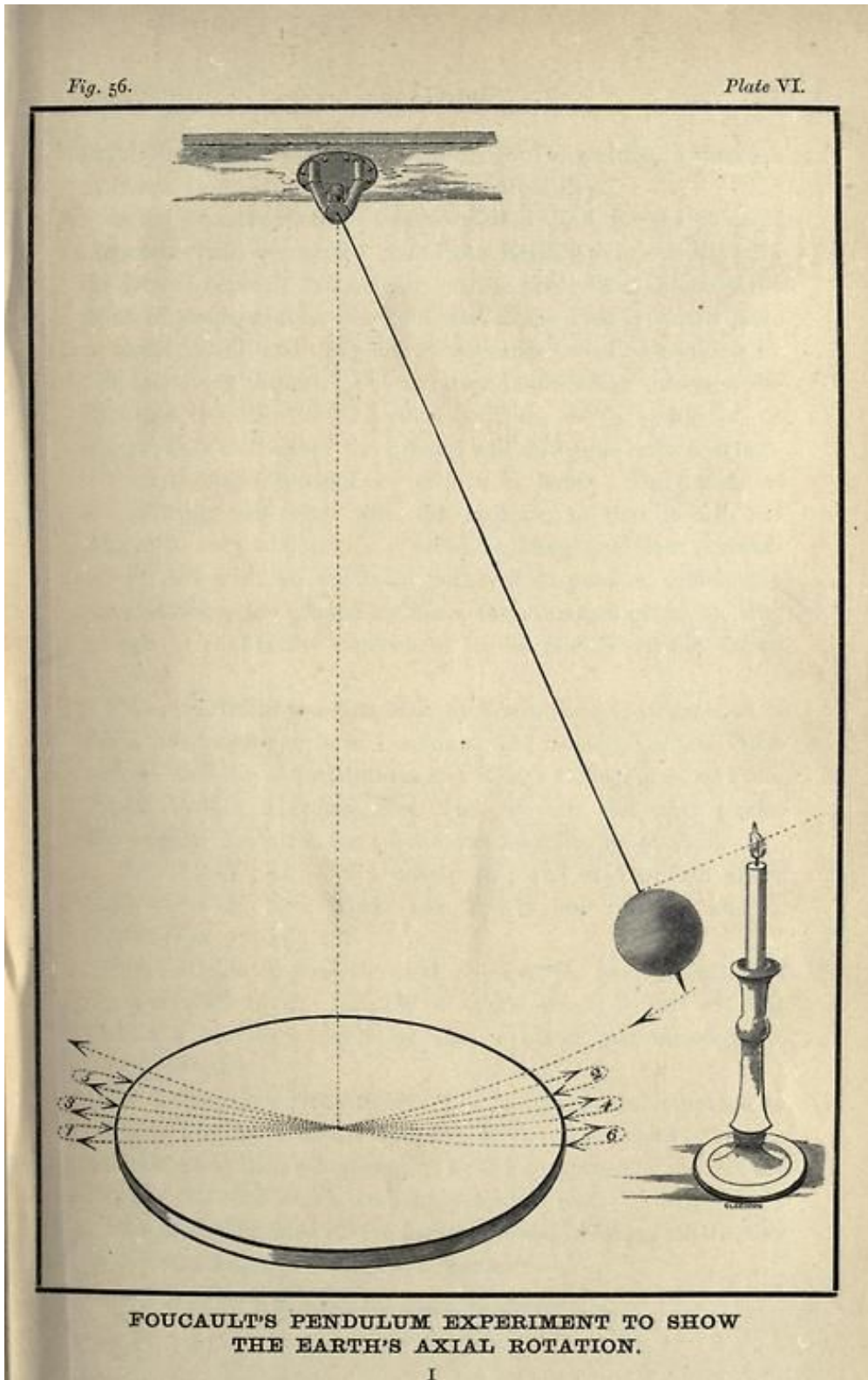
Figure 1.5: Foucault's pendulum experiment.

Exercise 4   –   Foucault pendulum 2

The horizontal dynamics of the Foucault pendulum with length L is given by

$$\ddot{x} \;=\; f\dot{y} - \frac{g}{L}x \tag{1.19}$$

$$\ddot{y} \;=\; -f\dot{x} - \frac{g}{L}y \tag{1.20}$$

with $f = 2\Omega \sin \varphi$. The length is typically on the order of 1-10 m.

a) Show that the solution is given by

$$x \;=\; x_0 \cos \omega^* t \tag{1.21}$$

$$y \;=\; x_0 \sin \omega^* t \tag{1.22}$$

$$\text{with} \quad \omega^* = \left(-\frac{f}{2} + \sqrt{\omega^2 + \frac{f^2}{4}} \right) \tag{1.23}$$

where x_0 is the initial condition, and $\omega = \sqrt{g/L}$.

b) Show that $\omega^2 \gg \frac{f^2}{4}$ and that

$$\omega^* \approx -\frac{f}{2} + \omega \quad . \tag{1.24}$$

c) Explain that the natural frequency (also called pulsation) ω can be used to measure gravity.

d) Show that the precession cycle can be used to determine the latitude! Discuss the special cases equator and South Pole!

1.2 Fourier transform

The Fourier transform decomposes a function of time (e.g., a signal) into the frequencies that make it up, similarly to how a musical chord can be expressed as the amplitude (or loudness) of its constituent notes. The Fourier transform of a function of time itself is a complex-valued function of frequency, whose absolute value represents the amount of that frequency present in the original function, and whose complex argument is the phase offset of the basic sinusoid in that frequency. The Fourier transform is called the frequency domain representation of the original signal. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time (see also https://en.wikipedia.org/?title=Fourier_transform).

The Fourier transformation of x is defined as

$$\hat{x}(\omega) = \int_{\mathsf{R}} x(t)e^{i\omega t}\, dt \tag{1.25}$$

and is denoted as a hat in the following.[3] And the inverse Fourier transformation of x is defined as

$$x(t) = \frac{1}{2\pi} \int_{\mathsf{R}} \hat{x}(\omega)e^{-i\omega t}\, d\omega \tag{1.26}$$

or with $\omega = 2\pi\nu$:

$$x(t) = \int_{\mathsf{R}} \hat{x}e^{-i2\pi\nu t}\, d\nu \quad . \tag{1.27}$$

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[3]Other common notations for the Fourier transform $\hat{x}(\omega)$: $\tilde{x}(\omega)$, $\check{x}(\omega)$, $F(\xi)$, $\mathcal{F}(x)(\omega)$, $(\mathcal{F}x)(\omega)$, $\mathcal{F}(x)$, $\mathcal{F}(\omega)$, $F(\omega)$. The sign of the exponential in the Fourier transform is something that we are concerned with for many years. Of course, there are two conventions that have been used with almost equal frequency, but I try to stick to one of them to avoid confusion. Here, we have used the convention of the positive sign in the exponential for the forward transform which represents the Fraunhofer diffraction pattern for a real-space object. This is consistent with assuming that a plane wave, going in positive direction in real space is written $\exp[i(\omega t - kx)]$ rather than a minus sign before the i, so that the phase advances with time.

Exercise 5   –   Fourier transformation

Tasks: Calculate the Fourier transformation of

1. $x(t + a)$ (time shift).

2. $x(t * a)$ (Scaling in the time domain).

3. $\frac{d}{dt}x(t)$ (time derivative).

4. $x(t) = \exp(-at^2)$ (Gaussian).

5. $x(t) = \delta(t)$ where the δ distribution is definedthrough the operator on any function y:
   $$y(t_0) = \int_{\mathsf{R}} y(t)\delta(t - t_0) \; dt$$

6. Show that for $x(t) = \exp(-iat)$, the Fourier transformation $\hat{x}(\omega) = 2\pi\delta(\omega - a)$. Hint: use the Fourier back transformation (1.25).

7. Calculate the Fourier transformation of a the periodic function $x(t) = \sin(\omega_0 t)$. Remember that $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$.

8. Prove the Uncertainty principle: the more concentrated $x(t)$ is, the more spread out its Fourier transform $\hat{x}(\omega)$ must be. In particular, the scaling property of the Fourier transform may be seen as saying: if we "squeeze" a function in t, its Fourier transform "stretches out" in ω. It is not possible to arbitrarily concentrate both a function and its Fourier transform.

9. Consider the sine and cosine transforms and show the following. Fourier's original formulation of the transform did not use complex numbers, but rather sines and cosines. Statisticians and others still use this form. An absolutely integrable function f for which Fourier inversion holds good can be expanded in terms of genuine frequencies (avoiding negative frequencies, which are sometimes considered hard to interpret physically) λ by

$$f(t) = \int_0^\infty [a(\lambda)\cos 2\pi\lambda t + b(\lambda)\sin 2\pi\lambda t] \, d\lambda. \qquad (1.28)$$

This is called an expansion as a trigonometric integral, or a Fourier integral expansion. The coefficient functions a and b can be found by using variants of the Fourier cosine transform

and the Fourier sine transform (the normalisations are, again, not standardised):

$$a(\lambda) \;=\; 2\int_{-\infty}^{\infty} f(t)\cos(2\pi\lambda t)dt \tag{1.29}$$

$$b(\lambda) \;=\; 2\int_{-\infty}^{\infty} f(t)\sin(2\pi\lambda t)dt. \tag{1.30}$$

Laplace transform

The Fourier transform is intimately related with the Laplace transform $F(s)$, which is also used for the solution of differential equations and the analysis of filters (https://en.wikipedia.org/wiki/Laplace_transform). We introduce the complex variable $s = -i\omega$.

$$\mathcal{L}\left\{x(t)\right\} = F(s) = \int_{0}^{\infty} e^{-st}x(t)dt \tag{1.31}$$

It follows (integration by parts for [1.32](#))

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} \;=\; sF(s) - x(0) \tag{1.32}$$

$$\mathcal{L}\left\{\exp(-at)\right\} \;=\; \frac{1}{s+a} \tag{1.33}$$

$$\mathcal{L}\left\{-\exp(-at) + \exp(-bt)\right\} \;=\; \frac{-1}{s+a} + \frac{1}{s+b} = \frac{a-b}{(s+a)(s+b)} \tag{1.34}$$

The Laplace transform of a sum is the sum of Laplace transforms of each term.

$$\mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\} \tag{1.35}$$

The Laplace transform of a multiple of a function is that multiple times the Laplace transformation

Function	Time domain $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace s-domain $F(s) = \mathcal{L}\{f(t)\}$
unit impulse	$\delta(t)$	1
delayed impulse	$\delta(t - \tau)$	$e^{-\tau s}$
unit step	$u(t)$	$\frac{1}{s}$
delayed unit step	$u(t - \tau)$	$\frac{1}{s}e^{-\tau s}$
exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$
sine	$\sin(\omega t) \cdot u(t)$	$\frac{\omega}{s^2+\omega^2}$
cosine	$\cos(\omega t) \cdot u(t)$	$\frac{s}{s^2+\omega^2}$
decaying sine wave	$e^{-\alpha t} \sin(\omega t) \cdot u(t)$	$\frac{\omega}{(s+\alpha)^2+\omega^2}$
decaying cosine wave	$e^{-\alpha t} \cos(\omega t) \cdot u(t)$	$\frac{s+\alpha}{(s+\alpha)^2+\omega^2}$
natural logarithm	$\ln(t) \cdot u(t)$	$-\frac{1}{s}\left[\ln(s) + \gamma\right]$
Convolution	$(f * g)(t) = \int_0^t f(\tau)g(t - \tau)\,d\tau$	$F(s) \cdot G(s)$

Table 1.1:   Laplace transformation (https://en.wikipedia.org/wiki/Laplace_transform.

of that function.

$$\mathcal{L}\{af(t)\} = a\mathcal{L}\{f(t)\} \tag{1.36}$$

Using this linearity, and various trigonometric, hyperbolic, and complex number (etc.) properties and/or identities, some Laplace transforms can be obtained from others quicker than by using the definition directly.

Exercise 6   –   Laplace transformation of mixed layer model

Solve the Imagine that the temperature of the ocean mixed layer is governed by

$$\frac{dT}{dt} = -\lambda T + Q(t)\,, \tag{1.37}$$

where λ is the typical damping rate of a temperature anomaly and $Q(t)$ a forcing.

1. Use the Laplace transformation to show

$$F(s) = \frac{Q(s) + T(0)}{s + \lambda} \quad . \tag{1.38}$$

where $Q(s) = \mathcal{L}\{Q(t)\}$

2. Consider the special case $Q(t) = \exp(i\omega_0 t)$, then $Q(s) = \frac{1}{s - i\omega_0}$. The forcing and the temperature is of course a real number, by representing is as a complex number we can simultaneously keep track of both phase components. Show

$$F(s) = \frac{T(0) + Q(s)}{s + \lambda} = \frac{T(0)}{s + \lambda} + \frac{1}{(s + \lambda)} \frac{1}{(s - i\omega_0)} \tag{1.39}$$

and via the Laplace back-transformation and (13.42, 13.47) that

$$T(t) = \exp(-\lambda t)T(0) + \frac{[\exp(i\omega_0 t) - \exp(-\lambda t)]}{\lambda + i\omega_0} \quad . \tag{1.40}$$

3. Calculate the real and complex part of (13.46).

4. Show: At low frequencies, the output is equal to the input. At high frequencies it rolls off as $1/\omega$ (it is a low-pass filter) and is out of phase by $90°$.

Let $x(t)$ be the input to a general linear time-invariant system, and $y(t)$ be the output, and the Laplace transform of $x(t)$ and $y(t)$ be $X(s)$ and $Y(s)$. Then, the output is related to the input by convolution with respect to the impulse response $h(t)$ by

$$y(t) = \int_0^\infty h(t')x(t - t')dt \tag{1.41}$$

Because of the convolution, the transfer function $H(s)$ is equal to the to the ratio of the Laplace

transforms of the input and output

$$H(s) = \frac{Y(s)}{X(s)}. \tag{1.42}$$

The impulse response of a linear transformation is the image of Dirac's delta function under the transformation, analogous to the fundamental solution of a partial differential operator. The general feature of the transfer function is that is the ratio of two polynomials. Since the polynomials can be constructed from knowledge of the roots, the location of the poles and zeros completetly characterizes the response of the system. The system is globally stable if all poles lie in the left half-plane with Re(poles) < 0. For example $\mathcal{L}\{\exp(-at)\} = \frac{1}{s+a}$, i.e. the system is stable if $Re(a) < 0$. Poles off the real axes are associated with oscillations. Summarizing, the convolution that gives the output of the system can be transformed to a multiplication in the transform domain, given signals for which the transforms exist

$$y(t) = (h * x)(t) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} h(t - \tau)x(\tau)\,\mathrm{d}\tau \stackrel{\text{def}}{=} \mathcal{L}^{-1}\{H(s)X(s)\}. \tag{1.43}$$

Transfer functions are commonly used in the analysis of systems such as single-input single-output filters, typically within the fields of signal processing, communication theory, and control theory. The term is often used exclusively to refer to linear, time-invariant systems. The descriptions below are given in terms of a complex variable, $s = \sigma - i\omega$, which bears a brief explanation. In many applications, it is sufficient to define $\sigma = 0$, which reduces the Laplace transforms with complex arguments to Fourier transforms with real argument ω. The applications where this is common are ones where there is interest only in the steady-state response.[4] The stability of linear systems will be discussed further in section .

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[4]In discrete-time systems, the relation between an input signal x(t) and output y(t) is dealt with using the z-transform, and then the transfer function is similarly written as $H(z) = \frac{Y(z)}{X(z)}$ and this is often referred to as the pulse-transfer function.

Exercise 7 – **Method of partial fraction expansion**

Consider a linear time-invariant system with transfer function

$$H(s) = \frac{1}{(s+\alpha)(s+\beta)}. \tag{1.44}$$

The impulse response is simply the inverse Laplace transform of this transfer function:

$$h(t) = \mathcal{L}^{-1}\{H(s)\}. \tag{1.45}$$

To evaluate this inverse transform, we begin by expanding $H(s)$ using the method of partial fraction expansion:

$$\frac{1}{(s+\alpha)(s+\beta)} = \frac{P}{s+\alpha} + \frac{R}{s+\beta}. \tag{1.46}$$

The unknown constants P and R are the residues located at the corresponding poles of the transfer function. Each residue represents the relative contribution of that singularity to the transfer function's overall shape. By the residue theorem, the inverse Laplace transform depends only upon the poles and their residues. To find the residue P, we multiply both sides of the equation by $s + \alpha$ to get

$$\frac{1}{s+\beta} = P + \frac{R(s+\alpha)}{s+\beta}. \tag{1.47}$$

Then by letting $s = -\alpha$, the contribution from R vanishes and all that is left is

$$P = \left.\frac{1}{s+\beta}\right|_{s=-\alpha} = \frac{1}{\beta - \alpha}. \tag{1.48}$$

Similarly, the residue R is given by

$$R = \left.\frac{1}{s+\alpha}\right|_{s=-\beta} = \frac{1}{\alpha - \beta}. \tag{1.49}$$

Note that

$$R = \frac{-1}{\beta - \alpha} = -P \tag{1.50}$$

and so the substitution of R and P into the expanded expression for H(s) gives

$$H(s) = \left(\frac{1}{\beta - \alpha}\right) \cdot \left(\frac{1}{s + \alpha} - \frac{1}{s + \beta}\right). \tag{1.51}$$

Finally, using the linearity property and the known transform for exponential decay (see in the Table 1.1 of Laplace transforms, above), we can take the inverse Laplace transform of H(s) to obtain:

$$h(t) = \mathcal{L}^{-1}\{H(s)\} = \frac{1}{\beta - \alpha}\left(e^{-\alpha t} - e^{-\beta t}\right), \tag{1.52}$$

which is the impulse response of the system. (This example will be used in section 10.2.3 with more details of the Laplace transformation.)

Exercise 8   –   Convolution

The same result can be achieved using the convolution property as if the system is a series of filters with transfer functions of $1/(s + a)$ and $1/(s + b)$. That is, the inverse of

$$H(s) = \frac{1}{(s + a)(s + b)} = \frac{1}{s + a} \cdot \frac{1}{s + b} \tag{1.53}$$

is

$$\mathcal{L}^{-1}\left\{\frac{1}{s + a}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s + b}\right\} \tag{1.54}$$

$$= e^{-at} * e^{-bt} = \int_0^t e^{-ax}e^{-b(t-x)} \, dx \tag{1.55}$$

$$= \frac{e^{-at} - e^{-bt}}{b - a}. \tag{1.56}$$

An integral formula for the inverse Laplace transform, is given by the line integral:

$$x(t) = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s) \, ds, \tag{1.57}$$

where the integration is done along the vertical line $Re(s) = \gamma$ in the complex plane such that γ is greater than the real part of all singularities of F(s). This ensures that the contour path is in the region of convergence. If all singularities are in the left half-plane, or F(s) is a smooth function on $-\infty < Re(s) < \infty$ (i.e., no singularities), then γ can be set to zero and the above inverse integral formula above becomes identical to the inverse Fourier transform.(https://en.wikipedia.org/wiki/Residue_theorem). The function f(t)=INVLAP(F(s)) offers a simple, effective and reasonably accurate way to achieve the result.[5] The transform F(s) may be any reasonable function of complex variable s^{α}, where α is an integer or non-integer real exponent. Thus, the function INVLAP can solve even fractional problems and invert functions F(s) containing rational, irrational or transcendental expressions. The function does not require to compute poles nor zeroes of F(s). It is based on values of F(s) for selected complex values of the independent variable s. The resultant computational error can be held arbitrarily low at the cost of CPU time (see Examples).

---

[5]It is based on the paper: J. Valsa and L. Brancik: Approximate Formulae for Numerical Inversion of Laplace Transforms, Int. Journal of Numerical Modelling: Electronic Networks, Devices and Fields, Vol. 11, (1998), pp. 153-166.

```
%%%%%%%%%%%%%%%%%%%%
% INVLAP  Numerical Inversion of Laplace Transforms
function [radt,ft]=INVLAP(Fs,tini,tend,nnt,a,ns,nd);
% Fs is formula for F(s) as a string
% tini, tend are limits of the solution interval
% nnt is total number of time instants
% a, ns, nd are parameters of the method
% if not given, the method uses implicit values a=6, ns=20, nd=19
% it is recommended to preserve a=6
% increasing ns and nd leads to lower error
% an example of function calling
% [t,ft]=INVLAP('s/(s^2+4*pi^2)',0,10,1001);
% to plot the graph of results write plot(t,ft), grid on, zoom on
FF=strrep(strrep(strrep(Fs,'*','.*'),'/','./'),'^','.^');
if nargin==4
  a=6; ns=20; nd=19;  end;    % implicit parameters
radt=linspace(tini,tend,nnt); % time vector
if tini==0  radt=radt(2:1:nnt);  end;  % t=0 is not allowed
tic        % measure the CPU time
for n=1:ns+1+nd                % prepare necessary coefficients
   alfa(n)=a+(n-1)*pi*j;
   beta(n)=-exp(a)*(-1)^n;
end;
n=1:nd;
bdif=fliplr(cumsum(gamma(nd+1)./gamma(nd+2-n)./gamma(n)))./2^nd;
beta(ns+2:ns+1+nd)=beta(ns+2:ns+1+nd).*bdif;
beta(1)=beta(1)/2;
for kt=1:nnt                   % cycle for time t
   tt=radt(kt);
   s=alfa/tt;                  % complex frequency s
   bt=beta/tt;
   btF=bt.*eval(FF);           % functional value F(s)
   ft(kt)=sum(real(btF));      % original f(tt)
end;
toc
%%%%%%%%%%%%%%
```

```
%%%%%%%%%%%%%%
% example 0:

[t1,ft1]=INVLAP('1/(s+4)',0.001,3,100);

plot(t1,ft1), grid on
xlabel('time '), ylabel('f(t)= exp(-\omega t)')
title('Laplace back from F(s)=1/(s+\omega)  with  \omega = 4')

[t1,ft1]=INVLAP('1/(s^2 +2* s + 5)',0.001,4,100);
plot(t1,ft1), grid on
xlabel('time '), ylabel('f(t) = 0.5   exp (-t)  sin (2 t)')
title('Laplace backtransformation from    F(s)= 1/(s^2 +2* s + 5) ')
```

```
[t1,ft1]=INVLAP('(3 * s^2 +7 *s +10)/(4*s + s *(s + 1)^2)',0.001,4,100);
plot(t1,ft1), grid on
xlabel('time '), ylabel('f(t) = 2+exp(-t) *cos(2t)+exp(-t)+sin(2t)')
title('Laplace back from  F(s)= (3 * s^2 +7 *s +10)/(4*s + s *(s + 1)^2) ')

%%%%%%%%%%%%%%
% example_1 - inversion of a irrational fraction F(s)
clear, close all
[t1,ft1]=INVLAP('tanh(s)/s',0.01,20,1000);
[t2,ft2]=INVLAP('tanh(s)/s',0.01,20,2000,6,280,59);
figure(3)
set(3,'color','white')
subplot(2,1,1)
plot(t1,ft1), grid on, zoom on
xlabel('t [s]'), ylabel('f(t)')
title('rectangular periodic wave')
subplot(2,1,2)
plot(t2,ft2), grid on, zoom on
xlabel('t [s]'), ylabel('f(t)')
title('improved accuracy')

%%%%%%%%%%%%%%
% example_2 - inversion of a fractional F(s)
clear, close all
[t1,ft1]=INVLAP('1/(sqrt(s)*s)',0.01,5,200,6,40,20);
[t2,ft2]=INVLAP('(20.5+3.7343*s^1.15)/(21.5+3.7343*s^1.15+0.8*s^2.2+0.5*s^0.9)/s',0
figure(4)
set(4,'color','white')
subplot(2,1,1)
plot(t1,ft1), grid on, zoom on
xlabel('t [s]'), ylabel('f(t)')
%title('
subplot(2,1,2)
plot(t2,ft2), grid on, zoom
xlabel('t [s]'), ylabel('f(t)')
title('step response of a fractional control system')
%%%%%%%%%%%%%%
% example_3 - inversion of a fractional F(s) in symbolic form
clear, close all
syms D1 alfa1 R1 s
% parameters of the network with constant phase element
I=0.25;  Rs=0.1;  R1=100;  D1=1;  alfa1=-0.7;
% F(s) in symbolic form
F1=I*(Rs+(R1*D1*s^alfa1/(R1+D1*s^alfa1)))*(1-exp(-4000*s))/s;
% F(s) as a string
F1=char(F1);
% parameters of the fractional control system
k=20.5;  a1=3.7343;  alfa1=1.15;  a2=0.8;  alfa2=2.2;  a3=0.5;  alfa3=0.9;
F2=(k+a1*s^alfa1)/(k+1+a1*s^alfa1+a2*s^alfa2+a3*s^alfa3)/s;
F2=char(F2);        % F(s) as a string
[t1,ft1]=INVLAP(F1,0.01,1e4,1000,6,39,89);
[t2,ft2]=INVLAP(F2,0.01,5,200);
```

```
figure(4)
set(4,'color','white')
subplot(2,1,1)
plot(t1,ft1), grid on, zoom on
xlabel('t [s]'), ylabel('f(t)')
title('response to the input current impulse')
subplot(2,1,2)
plot(t2,ft2), grid on, zoom
xlabel('t [s]'), ylabel('f(t)')
title('step response of a fractional control system')
```

1.3   Covariance and spectrum

A stationary process exhibits an autocovariance function of the form

$$Cov(\tau) = \langle (x(t+\tau) - \langle x \rangle)(x(t) - \langle x \rangle) \rangle \qquad (1.58)$$

where $\langle \dots \rangle$ denotes the statistical ensemble mean.[6] Normalized to the variance (i.e. the autoco-variance function at $\tau = 0$) one gets the autocorrelation function $C(\tau)$:

$$C(\tau) = Cov(\tau)/Cov(0) \quad . \qquad (1.59)$$

Many stochastic processes in nature exhibit short-range correlations, which decay exponentially:

$$C(\tau) \sim \exp(-\tau/\tau_0), \text{ for } \tau \to \infty \qquad (1.60)$$

These processes exhibit a typical time scale τ_0. For a white noise process ξ (as defined in 7.55), the autocorrelation function $C(\tau)$ is given by

$$C(\tau) = \delta(\tau) \quad . \qquad (1.61)$$

---

[6]For the covariance, one can have two processes $Cov(\tau) = \langle (x(t+\tau) - \langle x \rangle)(y(t) - \langle y \rangle) \rangle$.

Spectrum of the stochastic process

The Fourier transformation of the random variable x is

$$\hat{x}(\omega) = \int_{\mathbb{R}} x(t) e^{i\omega t} \, dt \quad = \lim_{T \to \infty} \int_{-T/2}^{T/2} x(t) e^{i\omega t} \, dt \tag{1.62}$$

and is also a ramdom variable, but its power spectral density $S(\omega)$ is not:

$$S(\omega) \quad := \quad \langle \hat{x}\hat{x}^+ \rangle = \langle |\hat{x}(\omega)|^2 \rangle \quad . \tag{1.63}$$

Using the **ergodic hypothesis**, the ensemble average $S(\omega) = \langle \hat{x}\hat{x}^+ \rangle$ can be expressed as the time average

$$\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \quad \hat{x}\hat{x}^+ \tag{1.64}$$

and therefore the spectrum can be expressed as

$$S(\omega) \quad = \quad \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{i\omega t} x(t) dt \quad \int_{-T/2}^{T/2} e^{-i\omega t'} x(t') dt' \quad . \tag{1.65}$$

The "total" integrated spectral density equals the variance of the series. Thus the spectral density within a particular interval of frequencies can be viewed as the amount of the variance explained by those frequencies. Mathematically, the spectral density is defined for both negative and positive frequencies. However, due to symmetry of the function $S(\omega)$ is quite often displayed for positive values only.

Let us calculate the inverse Fourier transformation of $S(\omega)$ and calculate the relation to the autocovariance function $Cov(\tau)$ of the stationary process $x(t)$:

$$\frac{1}{2\pi} \int_R S(\omega) \ e^{-i\omega\tau} d\omega$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_R d\omega \frac{e^{-i\omega\tau}}{2\pi} \int_{-T/2}^{T/2} e^{i\omega t} x(t) dt \int_{-T/2}^{T/2} e^{-i\omega t'} x(t') dt'$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \left(\frac{1}{2\pi} \int_R e^{i\omega(t-t'-\tau)} d\omega \right) x(t) x(t') dt dt'$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \delta(t - t' - \tau) x(t) x(t') dt dt' \tag{1.66}$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) x(t - \tau) dt \tag{1.67}$$

$$= \langle x(t) x(t - \tau) \rangle \quad = \quad Cov(\tau) \tag{1.68}$$

The transformation (1.66) comes from the Fourier transform of the $\delta-$function:

$$\int_R e^{-i\omega t} \delta(t) dt = 1 \quad \longrightarrow \quad \delta(t) = \frac{1}{2\pi} \int_R e^{i\omega t} d\omega \tag{1.69}$$

As the frequency domain counterpart of the autocovariance function of a stationary process, one can calculate the spectrum as

$$S(\omega) = \widehat{Cov(\tau)} \quad , \tag{1.70}$$

where the hat denotes again the Fourier transformation. This is the Wiener-Chinchin theorem, relating the sprectrum of a random process to its autocorrelation function (Fig. 10.1).

The white noise process

The white noise process is therefore a function with constant $S(\omega)$, since the autocovariance is a delta dunction (1.61). The color of a noise signal (a signal produced by a stochastic process) is generally understood to be some broad characteristic of its power spectrum. This sense of 'color' for noise signals is similar to the concept of timbre in music (which is also called "tone color"); however the latter is almost always used for sound, and may consider very detailed features of the spectrum. The practice of naming kinds of noise after colors started with white noise, a signal whose spectrum has equal power within any equal interval of frequencies. That name was given by analogy with white light, which was (incorrectly) assumed to have such a flat power spectrum over the visible range. Other color names, like pink, red, and blue were then given to noise with other spectral profiles, often (but not always) in reference to the color of light with similar spectra. Some of those names have standard definitions in certain disciplines, while others are very informal and poorly defined. Noise is somehow opposite to music where we hear distinct frequencies (see for the frequencies of music: https://en.wikipedia.org/wiki/Piano_key_frequencies).

In equal temperament, one starts from a reference such as the note A, which is usually taken to have frequency 440 Hz. All other notes have frequencies of the form 440 Hz $*a^n$ where n is the number of semitones between the note in question and the reference note A. The ratio of an equal-tempered semitone is $a = \sqrt[12]{2} = 1.05946 \, (a^{12} = 2)$. In equal temperament, enharmonic notes such as $C^{\#}$ and D^b are acoustically identical, they share the same frequency. Equal temperament was well-suited for the kind of music that was written from the eighteenth century onward, with its much greater range of modulations and chromatic harmonic vocabulary.

In Pythagorean tuning, intervals are derived by successions of perfect fifths, so the corresponding frequency ratios are powers of $3/2$. In conventional Western music, twelve perfect fifths in succession,

$$C - G - D - A - E - B - F^{\#} - C^{\#} - G^{\#} - D^{\#} - A^{\#} - E^{\#} - B^{\#},$$

are supposed to equal seven octaves (C = B#). However, since $(3/2)^{12}$ does not equal 2^7, twelve Pythagorean perfect fifths give an interval slightly larger than seven octaves. The difference is a small interval known as the Pythagorean comma, which corresponds to a ratio of $(3/2)^{12}$ to $2^7 \approx$ $1.013643.$ The system of equal temperament gradually became adopted because it removed the limitations on keys for modulation. The discrepancies between just and equaltempered intervals are small and easily accepted by most listeners.
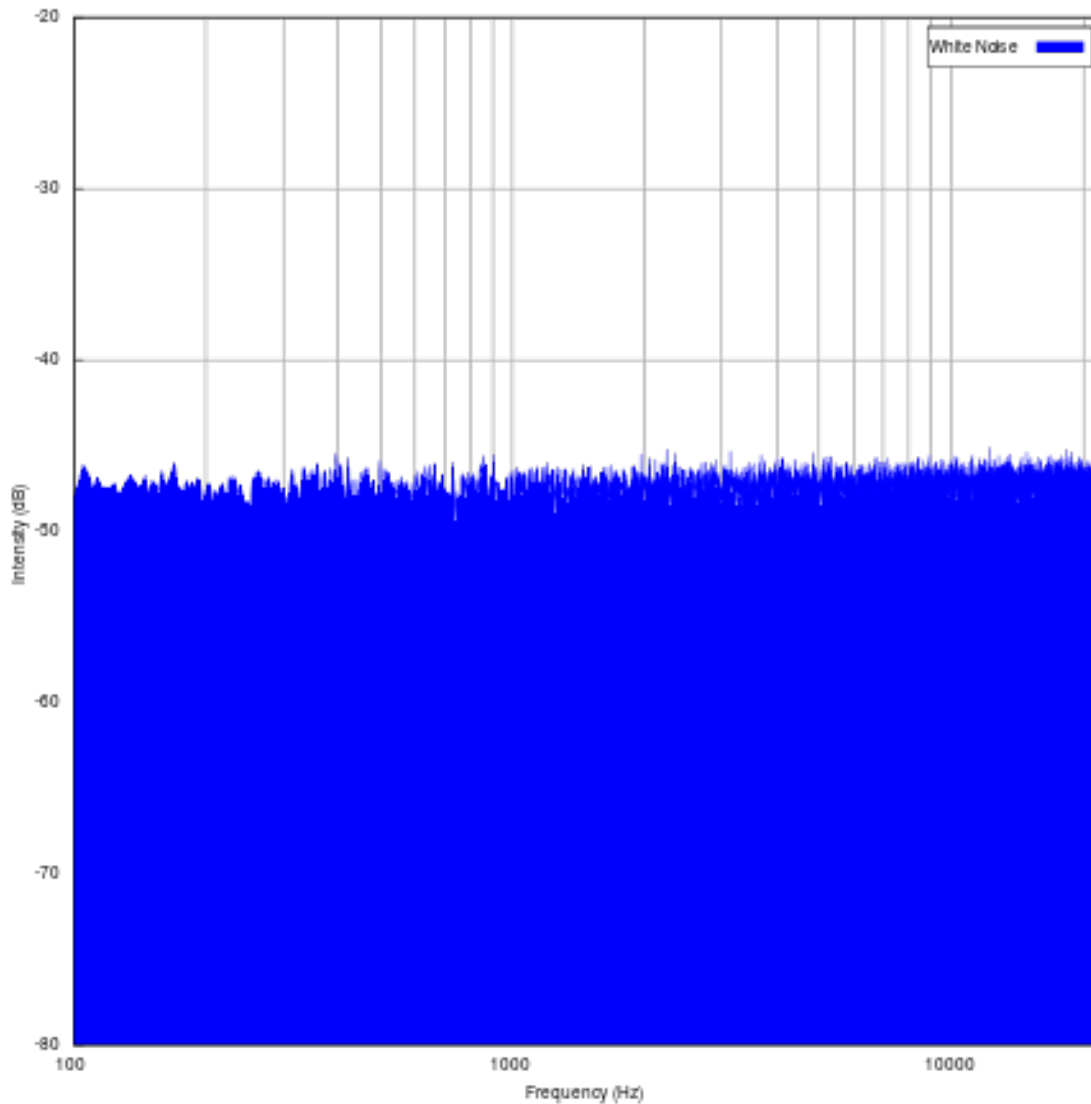
Figure 1.6: White noise spectrum. Flat power spectrum. (logarithmic frequency axis). For example, with a white noise audio signal, the range of frequencies between 40 Hz and 60 Hz contains the same amount of sound power as the range between 400 Hz and 420 Hz, since both intervals are 20 Hz wide. Note that spectra are often plotted with a logarithmic frequency axis rather than a linear one, in which case equal physical widths on the printed or displayed plot do not all have the same bandwidth, with the same physical width covering more Hz at higher frequencies than at lower frequencies. In this case a white noise spectrum that is equally sampled in the logarithm of frequency (i.e., equally sampled on the X axis) will slope upwards at higher frequencies rather than being flat.

1.4 Transport phenomena

As preparation of the course, you may repeat several mathematical formulations. It is important to notice that the fluid dynamical equations are generally formulated as a transport phenomenon. An important relation is: if X is a quantity of a volume element which travels from position $\vec{r}$ to $\vec{r} + d\vec{r}$ in a time dt, the total differential dX is then given by:

$$
\begin{aligned}
dX &= \frac{\partial X}{\partial x}dx + \frac{\partial X}{\partial y}dy + \frac{\partial X}{\partial z}dz + \frac{\partial X}{\partial t}dt \\
&\Rightarrow \\
\frac{dX}{dt} &= \frac{\partial X}{\partial x}v_x + \frac{\partial X}{\partial y}v_y + \frac{\partial X}{\partial z}v_z + \frac{\partial X}{\partial t}
\end{aligned}
\tag{1.71}
$$

This results in general to: $\dfrac{dX}{dt} = \dfrac{\partial X}{\partial t} + (\vec{v} \cdot \nabla)X$.

From this follows that also holds:

$$
\frac{d}{dt} \iiint X d^3 V = \frac{\partial}{\partial t} \iiint X d^3 V + \oiint X(\vec{v} \cdot \vec{n})d^2 A
\tag{1.72}
$$

where the volume V is surrounded by surface A. Some properties of the ∇ operator are:

$$
\begin{aligned}
&\mathbf{div}(\phi\vec{v}) = \phi\mathbf{div}\vec{v} + \nabla\phi \cdot \vec{v} && \mathbf{rot}(\phi\vec{v}) = \phi\mathbf{rot}\vec{v} + (\nabla\phi) \times \vec{v} && \mathbf{rot}\nabla\phi = \vec{0} \\
&\mathbf{div}(\vec{u} \times \vec{v}) = \vec{v} \cdot (\mathbf{rot}\vec{u}) - \vec{u} \cdot (\mathbf{rot}\vec{v}) && \mathbf{rot}\,\mathbf{rot}\vec{v} = \nabla\,\mathbf{div}\vec{v} - \nabla^2\vec{v} && \mathbf{div}\,\mathbf{rot}\tilde{v} = 0 \\
&\mathbf{div}\,\nabla\phi = \nabla^2\phi && \nabla^2\vec{v} \equiv (\nabla^2 v_1, \nabla^2 v_2, \nabla^2 v_3)
\end{aligned}
$$

Here, $\vec{v}$ is an arbitrary vector field and ϕ an arbitrary scalar field. Some important integral theorems are:

Gauss:
$$\oiint (\vec{v} \cdot \vec{n})d^2 A = \iiint (\mathrm{div}\vec{v})d^3 V$$

Stokes for a scalar field:
$$\oint (\phi \cdot \vec{e}_t)ds = \iint (\vec{n} \times \nabla\phi)d^2 A$$

Stokes for a vector field:
$$\oint (\vec{v} \cdot \vec{e}_t)ds = \iint (\mathrm{rot}\vec{v} \cdot \vec{n})d^2 A$$

This results in:
$$\oiint (\mathrm{rot}\vec{v} \cdot \vec{n})d^2 A = 0$$

Ostrogradsky:
$$\oiint (\vec{n} \times \vec{v})d^2 A = \iiint (\mathrm{rot}\vec{v})d^3 A$$

$$\oiint (\phi\vec{n})d^2 A = \iiint (\nabla\phi)d^3 V$$

Here, the orientable surface $\iint d^2 A$ is limited by the Jordan curve $\oint ds$.

Exercise 9 – Self test

1. Given
$$f(x, y, z, t) = x^2 + y^2 + z^2 \sin(\omega t).$$

   What are the partial derivatives with respect to the variables x and t?

2. What is the definition of ∇, Laplace, divergence, total (substantial) derivative, total differential for a function $f(x, y, z, t)$?

3. Calculate the rotation of ∇f.

4. Given the function $g(x) = ax^2 - 3x^4 + 2x \sin(\alpha x)$, please provide the Taylor expansion of g around $x = 0$ up to the 3rd order in x !

5. In the atmosphere, ocean, ice system, we are dealing with forces. Please list some relevant real and apparent forces.

6. What is the differential equation describing radioactive decay? Please provide also the solution with initial condition $x(t = 0) = x_0$. How is the half-life time defined?

7. The potential temperature of a parcel of fluid at pressure p is the temperature that the parcel would acquire if adiabatically brought to a standard reference pressure p_0, usually 100 kPa. The potential temperature of air is often given by

$$\Theta = T \left(p_0/p\right)^{R/c_p}$$

where T is the current absolute temperature of the parcel, R is the gas constant of air, and c_p is the specific heat capacity at a constant pressure. $\kappa = R/c_p = 2/7$ for an ideal diatomic gas. For a constant lapse rate $\frac{dT}{dz} = \gamma = const.$, why does the potential temperature Θ increase with height? Hint: Atmospheric pressure decreases with height.

1.5   General form of wave equations

The general form of the wave equation is:

$$\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2} \tag{1.73}$$

where q is the disturbance and c the *propagation velocity*. In general holds: $c = \nu\lambda$. By definition holds: $k\lambda = 2\pi$ and $\omega = 2\pi\nu$. Therefore,

$$c = \nu\lambda = 2\pi\nu/k = \omega/k \quad . \tag{1.74}$$

 In principle, there are two types of waves:

1. Longitudinal waves: for these holds $\vec{k} \parallel \vec{c} \parallel \vec{q}$. In a longitudinal wave the particle displacement is parallel to the direction of wave propagation. The animation (http://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html) shows a one-dimensional longitudinal plane wave propagating down a tube. The particles do not move down the tube with the wave; they simply oscillate back and forth about their individual equilibrium po-

sitions. Pick a single particle and watch its motion. The wave is seen as the motion of the compressed region (ie, it is a pressure wave), which moves from left to right. The second animation shows the difference between the oscillatory motion of individual particles and the propagation of the wave through the medium. The animation also identifies the regions of compression and rarefaction.

2. Transversal waves: for these holds $\vec{k} \parallel \vec{c} \perp \vec{q}$. In a transverse wave the particle displacement is perpendicular to the direction of wave propagation. The animation (http://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html) below shows a one-dimensional transverse plane wave propagating from left to right. The particles do not move along with the wave; they simply oscillate up and down about their individual equilibrium positions as the wave passes by. Pick a single particle and watch its motion. The S waves (Secondary waves) in an earthquake are examples of Transverse waves. S waves propagate with a velocity slower than P waves, arriving several seconds later.

3. Water waves: Water waves are an example of waves that involve a combination of both longitudinal and transverse motions. As a wave travels through the waver, the particles travel in clockwise circles. The radius of the circles decreases as the depth into the water increases. The animation (http://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html) below shows a water wave travelling from left to right in a region where the depth of the water is greater than the wavelength of the waves. I have identified two particles in yellow to show that each particle indeed travels in a clockwise circle as the wave passes.

The *phase velocity* is given by

$$c_{\mathrm{ph}} = \omega/k \quad . \tag{1.75}$$

The *group velocity* is given by:

$$c_{\mathrm{g}} = \frac{d\omega}{dk} = c_{\mathrm{ph}} + k\frac{dc_{\mathrm{ph}}}{dk} \tag{1.76}$$

If c_{ph} does not depend on ω holds: $c_{\mathrm{ph}} = c_{\mathrm{g}}$. In a dispersive medium it is possible that $c_{\mathrm{g}} > c_{\mathrm{ph}}$ or $c_{\mathrm{g}} < c_{\mathrm{ph}}$. If one wants to transfer information with a wave, e.g. by modulation of an electro-magnetic wave, the information travels with the velocity at with a change in the electromagnetic field propagates. This velocity is often almost equal to the group velocity.

For some media, the propagation velocity follows from:

- Pressure waves in a liquid or gas: $c = \sqrt{\kappa/\varrho}$, where κ is the modulus of compression.

- For pressure waves in a gas also holds: $c = \sqrt{\gamma p/\varrho} = \sqrt{\gamma RT/M}$.

Plane waves

The equation for a harmonic traveling plane wave is

$$q(\vec{x}, t) = \hat{q} \cos\left(\vec{k} \cdot \vec{x} \pm \omega t + \varphi\right) \quad .$$

When the situation is spherical or cylindrical symmetric, the the homogeneous wave equation can be solved. When the situation is spherical symmetric, the homogeneous wave equation is given by:

$$\frac{1}{c^2}\frac{\partial^2(rq)}{\partial t^2} - \frac{\partial^2(rq)}{\partial r^2} = 0$$

with general solution:

$$q(r, t) = C_1\frac{f(r - ct)}{r} + C_2\frac{g(r + ct)}{r}$$

When the situation has a cylindrical symmetry, the homogeneous wave equation becomes:

$$\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q}{\partial r}\right) = 0$$

This is a Bessel equation, with solutions which can be written as Hankel functions. For sufficient large values of r these are approximated by:

$$q(r, t) = \frac{\hat{q}}{\sqrt{r}} \cos(k(r \pm vt))$$

If an observer is moving w.r.t. the wave with a velocity c_{obs}, she/he will observe a change in frequency: the *Doppler effect*. This is given by: $\dfrac{\nu}{\nu_0} = \dfrac{c_{\text{f}} - c_{\text{obs}}}{c_{\text{f}}}$.

The general solution in one dimension

Starting point is the equation:

$$\frac{\partial^2 q(x, t)}{\partial t^2} = \sum_{m=0}^{N} \left(b_m \frac{\partial^m}{\partial x^m} \right) q(x, t)$$

where $b_m \in \mathbb{R}$. Substituting $q(x, t) = A e^{i(kx - \omega t)}$ gives two solutions $\omega_j = \omega_j(k)$ as dispersion relations. The general solution is given by:

$$q(x, t) = \int_{-\infty}^{\infty} \left(a(k) e^{i(kx - \omega_1(k)t)} + b(k) e^{i(kx - \omega_2(k)t)} \right) dk$$

Because in general the frequencies ω_j are non-linear in k there is dispersion and the solution cannot be written any more as a sum of functions depending only on $x \pm ct$: the wave front transforms.

Chapter 2

General concepts

2.1 Programming with R

Install R

The latest version of R for Linux, OS X and Windows is freely available on the CRAN webpage: http://cran.r-project.org (Fig. 2.1). Download and install the R version for your operating system (for many linux distributions R is also available in the package management system).Furthermore, look at the web page for R studio http://www.rstudio.com/, R studio is a free and open source user interface for R. One particular package is Shiny. This makes it super simple for R users like you to turn analyses into interactive web applications that anyone can use.

Examples to start

```
Please see the web page for  some information how to get R running:
http://www.r-project.org/
http://www.awi.de/en/go/paleo/methods/r/
```
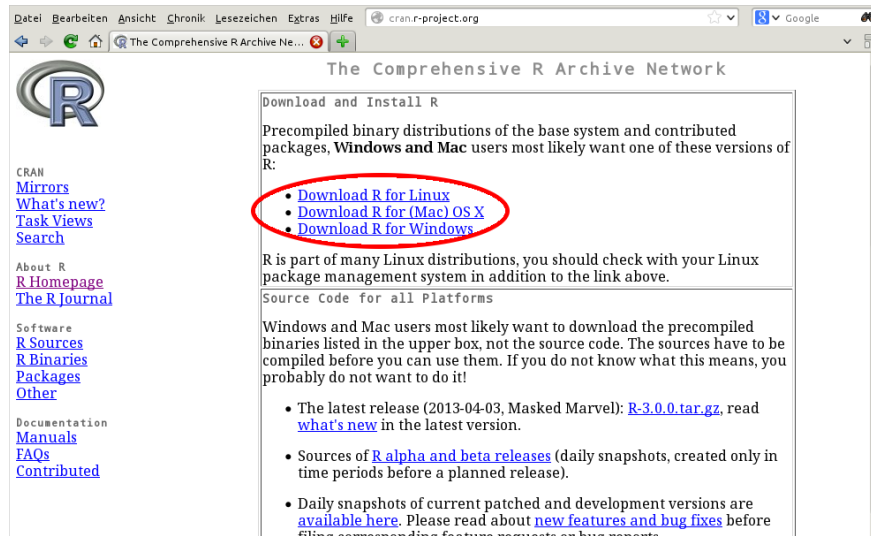
Figure 2.1: R is available for download from the CRAN webpage: http://cran.r-project.org.

```
Using R for Introductory Statistics:

http://cran.r-project.org/doc/contrib/Verzani-SimpleR.pdf

############ this letter is used for comments

#?function   shows the help for a function
?sin

#There is no definition needed for simple (scalar) variables
#but instead of =, <- is used
#just assign name<-value
a<-1

#Print the number on the screen:
a #prints only on the console
print(a) #prints always

#simple algebraic calculations
a<-2*3
a<-a/2
print(a) # and print again

#some vector / array functions
#vectors / arrays normally need to be defined that R can distinguish it
#from a scalar.

y<-vector() #Produces an empty vector...

#the size of vectors in R is dynamic... I can now assign y[1]...
#Assign elements of the vector [i], Vectors index are starting with 1
```

```r
y[1]<-1
y[2]<-2
print(y)
print(y[1])

#very often vectors filled with equidistant values are needed
x<-1:10    #x = vector (1,2,3,4,5,6,7,8,9,10)
print(x)
x<-(1:100)/5  # (0.2,0.4,0.6 ..........)
print(x)

#Control structures:
#for loop: for (variable in array) { }
for (i in 1:100)
{
  print(i)
}

#plotting function

#plot(y), plot y against equidistant steps
y<-1:100
plot(y)

#plot(x,y), plot y against x
x<-(1:100)*5
plot(x,y)  #note the changes in the x-axis

# more advanced functions:

#further parameters that can be used in plotting commands.
#type = "l" : as line type
#col = "color": plotting color
#ylim = c(minval,maxval):Set the limits of the y-axis
#main = "title": sets the title

plot(x,y,col="red",main="example",type="l",ylim=c(1,70))

# overplotting: line(x,y) or line(y) is the same as the plot command
# but plots a line on an already existing plot
# whereas plot is starting a new plot

z<-x^2 / 100
plot(x,y,col="red",main="example",type="l",ylim=c(1,70))
lines(x,z,col="blue")

#defining a function
multiply <- function(x,y)
{
  return(x*y)
}
```

```
print(multiply(3,4))
```

Reading and writing data

```
#Data Input from File, place file in dir. getwd()
#Store a table from a text file in an R-variable D
D<-read.table("test.txt",header=T)

# What read.table does is try to read data
# from the file named as the first argument.
# If header is specified as T (True),
# the first line will be read as the column names
# to which the values are assigned.  Header defaults to F (False).
# The function write.table() performs the opposite transformation.


#reading and writing data
x<-(1:100)*5
y<-x^2
write.table(y,file="xytdata.dat") # writing

#dev.print(pdf, MyPlot.pdf)
```

1. Load a R file into the R workspace

2. Save the file using another name

3. Keep the original version when modifying the file

4. Execute the whole file (CTRL-A to mark everything, CTRL-R to run it)

5. All functions are then in the memory

Exercise 10   –   Simple start of R

1. Download and install the R-Software. http://cran.r-project.org → Download CRAN → search a city near you Choose your system (Windows / Mac / Linux) Follow the instructions.

2. Create a vector t "t<-seq(-2*pi,2*pi,by=0.01)"

   plot several functions in one window $(\sin(t), \cos(t), \exp(\frac{t}{5}), (\frac{t}{5})^2, (\frac{t}{5})^3)$. Try some of the plot arguments: Set ylim, label the axes, set a different colour for each function, vary the line width. Save the plot as a figure.

   For help try "?plot" or "?plot.default"

3. Set up a vector of length 20 and create a vector b with a linear relationship to a (e.g. $a = 3b + 7$). Calculate the correlation("cor(a,b)").

4. Set up two random vectors a,b of length 20 and calculate the correlation. Repeat this procedure several times to get a feeling for the correlation coefficient. Than vary the length of vector a and b (vary the sample number) and discuss how the correlation coefficient changes (e.g. 10,50,100,1000).

5. Repeat the experiment from task 4 100 times by using a loop. Create before the loop an empty vector ("cor.val<-vector()") and save the correlation of a and b in this vector (e.g. "cor.val[i]<-cor(a,b)") for each realisation. Compute the mean value and plot the histogram of cor.val. What happens with the histogram when the length of a and b is varied (e.g. 10,50,100)? Save two different histograms as a figure and explain the difference between them.

6. Repeat the procedure of task 5. with partly linear dependent vectors: ("a<-rnorm(100); b<-r*a+rnorm(100)") Choose one value for r and shortly discuss the mean value and the histogram of cor.val compared to task 5. Save the histogram as a figure.

   ```
   # Important R-commands
   rnorm(N) # create vector with N normal distribution random numbers
   cor(a,b) # calculates the correlation coefficient
   hist(a)  # histogram of vector a
   mean(a)  # mean value of vector a
   ```

```
# Helpful introductions to R can be found in e.g.
```

link to Rintro.pdf

link to http://cran.r-project.org/doc/manuals/R-intro.pdf

Exercise 11   **–**   **Logistic equation**

As see in Fig. 2.10, the Lorenz system can exhibit chaotic behavior after a series of bifurca-tions. This concept is known as the Feigenbaum cascade Feigenbaum [1980]. In this scenario the solution undergoes a series of period-doublings, until the bifurcation parameter reaches a critical value where the system has an accumulation point of period-doublings. Feigenbaum also found the convergence behavior of the bifurcation points to the critical value. When the bifurcation parameter passes this point, chaos appears. A simple system which has such a behavior is the logistic equa-tion.. It is worth to analyze a one dimensional logistic equation (also known as Malthus-Verhulst model), which was originally proposed to describe the evolution of a biological population. Let x denote the number (or density) of individuals of a certain population. This number will change due to growth, death, and competition. In the simplest version, birth and death rates are assumed proportional to n,, but accounting for limited resources and competition it is modified by $(1 - x)$:

$$\frac{d}{dt}x(t) = a(1 - x)\,x \tag{2.1}$$

In climate, the logistic equation is also important for Lorenz's error growth model [Lorenz, 1982]: where $x(t)$ is the algebraic forecast error at time t and a is the linear growth rate. Here, will analyze a discrete version of the logistic equation.

1. Write a function which solves the logistic difference-equation $x_{n+1} = rx_n(1 - x_n)$ and returns the vector x_n. Use an initial value $x_0 \in [0, 1]$, and a parameter-value $r \in [1, 4]$.

2. Investigate the sensitivity of the solution on the parameter r (especially using $r \in [3, 4]$).

3. Now, investigate the solution dependence on r systematically: write a function which saves the local extrema of a vector (fixed points) and returns them in a vector.

4. For each value of r, iterate the logistic difference equation 500 times, discard the first 200
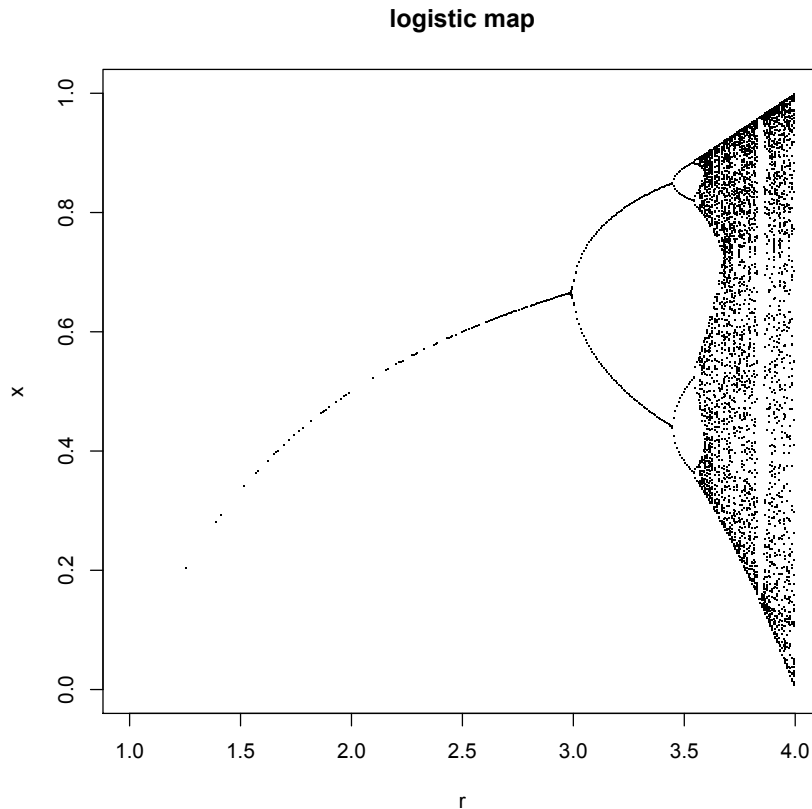
logistic map

Figure 2.2: Bifurcation diagram for the Logistic map by using r as the order parameter. The logistic map is an iterative function able to give chaotic dynamics in some of its parameter space. The parameter r is the responsible to cause the bifurcation scenario characterized by one of the most well-known route to chaos: the period doubling. The chaotic domain leaves a cloud of points in parameter space with a fractional dimensionality. Such structure is a Cantor set [Peitgen and Richter, 1986; Mandelbrot, 1983].

times, and plot the fix-points / local extrema against r. What do you observe? *Hint: use the zoom-in function of your plotting software of choice!*

5. Think of a climate analogy with x being the temperature. Describe the ice albedo feedback!

6. Calculate the map

$$z_{n+1} = z_n^2 + c \tag{2.2}$$

in the complex plane with c being a complex number. This set is called Mandelbrot set [Mandelbrot, 1967].

Solution

```r
# logistic difference equation
# with parameter r, N iterations, and initial value x0

f = function(r,N,x0)
{
  x <- vector()
  x[1]<-x0
  for (i in 2:N)  x[i]<-r*x[i-1]*(1-x[i-1])
  return(x)
}

#determine the values of the local extrema and give them back in a vector
local_extrema <- function(x)
{
  result <- vector()
  for (i in 1:(length(x)-2))
  {
  #save the local maximum in the result vector:
  if ((x[i]<x[i+1]) && (x[i+1] > x[i+2])) result<-c(result,x[i+1])
  #savel the local minimum in the result vector:
  if ((x[i]>x[i+1]) && (x[i+1] < x[i+2])) result<-c(result,x[i+1])
  }
  return(result)}


#test the functions:

#Test the logistic difference equation function
plot(f(3.9,100,0.4),type="l")

#Test the local extrema function
temp<-sin((1:300)/30)
plot(temp)
local_extrema(temp)

#test histo
hist(f(3.9,1000,0.4),1000)
hist(f(4,2000,0.4)[1000:2000],breaks=50)


#main program
resolution<-400   #number of r-parameter values to be scanned
rlim<-c(1,4)      #minimum and maximum r-value
xlim<-c(0,1)      #minimum and maximum x-value

#vector of all r-values we will scan
r<-rlim[1]+(1:resolution)*((rlim[2]-rlim[1])/resolution)

#empty plot with axes and title
plot(xlim=rlim,ylim=xlim,1,type="n",xlab="r",ylab="x",main="logistic map")
```

```r
for (i in 1:resolution)
{
  temp<-f(r[i],300,0.5)[200:300]
  save<-local_extrema(temp)
  points(rep(r[i],length(save)),save,pch=".")
}
```

Here is the solution for the Mandelbrot set.

```r
jet.colors = colorRampPalette(c("#00007F", "blue",
    "#007FFF", "cyan", "#7FFF7F", "yellow", "#FF7F00", "red",
    "#7F0000"))
m = 400
C = complex(real = rep(seq(-1.8, 0.6, length.out = m),
    each = m), imag = rep(seq(-1.2, 1.2, length.out = m), m))
C = matrix(C, m, m)
Z = 0
X = array(0, c(m, m, 20))
for (k in 1:20) {
    Z = Z^2 + C
    X[, , k] = exp(-abs(Z))
}
par(mar = rep(0, 4), ann = FALSE)
for (k in 1:20) {
    image(X[, , k])
    Sys.sleep(0.01)
}
```

This set is a mathematical set of points whose boundary is a distinctive and easily recognizable two-dimensional fractal shape, and is named after Mandelbrot [1967]. Images of the Mandelbrot set display an elaborate boundary that reveals progressively ever-finer recursive detail at increasing magnifications.
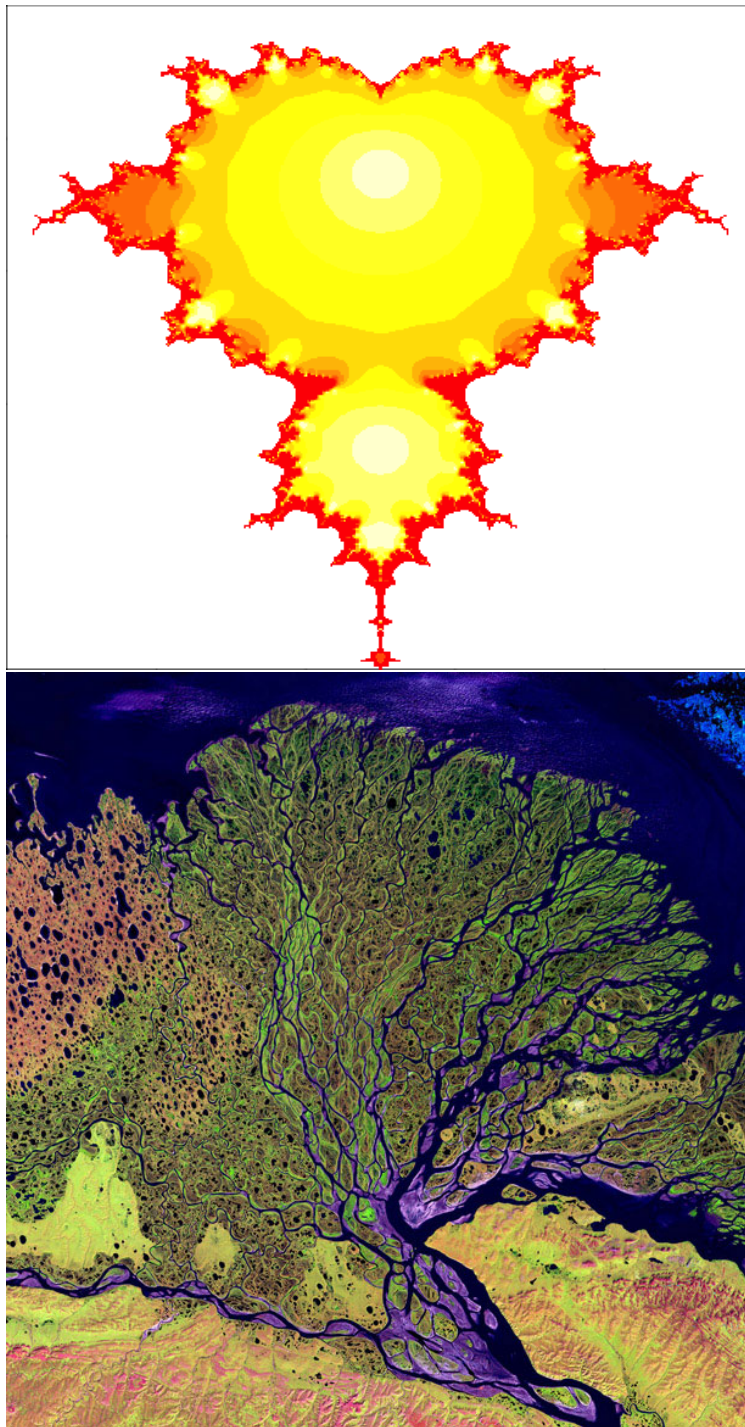
Figure 2.3: Upper panel: Mandelbrot set. The set's boundary also incorporates smaller versions of the main shape, so the fractal property of self-similarity applies to the entire set, and not just to its parts [Peitgen and Richter, 1986; Mandelbrot, 1983]. Lower panel: Lena Delta. The image is from the Landsat 7 satellite. Landsat satellites have taken specialized digital photographs of Earth's continents and surrounding coastal regions. The coastlines and morphometric subtypes may be characterized by a statistical self-similarity Mandelbrot [1967].

Exercise 12   –   Short programming questions

Write down the output for the following R-commands:

a) `0:10`

b) `a<-c(0,5,3,4); mean(a)`

c) `max(a)-min(a)`

d) `paste("The mean value of a is",mean(a),"for sure",sep="_")`

e) `a*2+c(1,1,1,0)`

f) `my.fun<-function(n){return(n*n+1)}`

   `my.fun(10)-my.fun(1)`

2.2   Netcdf and climate data operators

NetCDF (Network Common Data Form) is a set of software libraries and self-describing, machine-independent data formats that support the creation, access, and sharing of array-oriented scientific data. The project homepage is hosted by the Unidata program at the University Corporation for Atmospheric Research (UCAR). They are also the chief source of netCDF software, standards development, updates etc. The format is an open standard.

The software libraries supplied by UCAR provide read-write access to netCDF files, encoding and decoding the necessary arrays and metadata. The core library is written in C, and provides an API for C, C++ and Fortran applications. An independent implementation, also developed and maintained by Unidata, is written in 100% Java, which extends the core data model and adds additional functionality. Interfaces to netCDF based on the C library are also available in other languages including R (ncdf and ncvar packages), Perl, Python, Ruby, Matlab, IDL, and Octave.

A wide range of application software has been written which makes use of netCDF files. These range from command line utilities to graphical visualization packages.

- A commonly used set of Unix command line utilities for netCDF files is the NetCDF Operators (NCO) suite, which provide a range of commands for manipulation and analysis of

netCDF files including basic record concatenating, slicing and averaging.

- NcBrowse is a generic netCDF file viewer that includes Java graphics, animations and 3D visualizations for a wide range of netCDF file conventions.

- ncview is a visual browser for netCDF format files. Typically you would use ncview to get a quick and easy, push-button look at your netCDF files. You can view simple movies of the data, view along various dimensions, take a look at the actual data values, change color maps, invert the data, etc.

- Panoply is a netCDF file viewer developed at the NASA Goddard Institute for Space Studies which focuses on presentation of geo-gridded data. It is written in Java and thus platform independent. Although its feature set overlaps with ncBrowse and ncview, Panoply is distinguished by offering a wide variety of map projections and ability to work with different scale color tables.

- The NCAR Command Language is used to analyze and visualize data in netCDF files (among other formats).

- PyNIO is a Python programming language module that allows read and/or write access to a variety of data formats, including netCDF.

- Ferret is an interactive computer visualization and analysis environment designed to meet the needs of oceanographers and meteorologists analyzing large and complex gridded data sets. Ferret offers a Mathematica-like approach to analysis; new variables may be defined interactively as mathematical expressions involving data set variables. Calculations may be applied over arbitrarily shaped regions. Fully documented graphics are produced with a single command.

- nCDF-Browser is a visual nCDF browser, written in the IDL programming language. Variables, attributes, and dimensions can be immediately downloaded to the IDL command line

for further processing. All the Coyote Library files necessary to run nCDF-Browser are
available in the zip file.

- ArcGIS version 9.2 supports netCDF files. The Multidimension Tools toolbox can be used to
  create raster layers, feature layers, and table views from netCDF data in ArcMap, or convert
  feature, raster, and table data to netCDF.

- Origin 8 imports netCDF files as matrix books where each book can hold a 4D array. Users
  can select a subset of the imported data to make surface, controur or image plots.

- The Geospatial Data Abstraction Library provides support for read and write access to
  netCDF data.

Exercise netcdf

In this exercise we will learn to know various helpful tools and techniques that are used in the
analysis and processing of climate data. First, we will learn to know the NetCDF file format in
a few short practical demonstrations. Second, some data analysis will be performed on available
gridded climate data. In a third step we will transfer our knowledge to shell-programming, in that
we combine a number of common command-line tools in order to solve a simple scientific problem.
To this end, a Linux-shell, in combination with common shell-programs, is necessary. Since not
every attendee of this course has such a computing environment available, the shell-program will
be demonstrated by the tutor.

Topics that this exercise considers are:

- **Net**work **C**ommon **D**ata **F**orm (NetCDF): general information on the topic and how to use
  NetCDF files

- **C**limate **D**ata **O**perators (CDO): several examples of data reduction, analysis and transfor-
  mation of NetCDF files, including operator piping

- **B**ourne-**a**gain **sh**ell (Bash): definition of variables, if-then-else construct, checking for the existence of a specific file, for-counter-loops, integer-arithmetic, pipes, initializing of shell variables with program output, generation of strings via variable concatenation; these programming-methods are illustrated at the example of a simple scientific model of sea-level rise

- **S**tream **ED**itor (SED): removing spaces from a string

- **b**asic **c**alculator (bc): simple floating-point arithmetic

In the following some information regarding tools and methods of this exercise is collected in the form of a very general overview. This information collection is intended as a supplement to the lecture. Few further information sources are listed in the subsection "Further Reading". This list is obviously far from being complete, there is a vast amount of freely-accessible information available on-line.

A very precise definition on the characteristic and purpose of NetCDF is given in the NetCDF FAQ, "What is netCDF?" (for a link, see section "Further Reading"):

> "NetCDF (network Common Data Form) is a set of interfaces for array-oriented data access and a freely distributed collection of data access libraries for C, Fortran, C++, Java, and other languages. The netCDF libraries support a machine-independent format for representing scientific data. Together, the interfaces, libraries, and format support the creation, access, and sharing of scientific data."

NetCDF is a data-container that has been established as a widely used file-standard in science and engineering. It has been developed for storing array-oriented values in compact and inter-changeable files. The most important characteristics of NetCDF files can as well be found in the NetCDF FAQ, "What is netCDF?". An excerpt, that highlights the advantages of NetCDF with respect to this exercise, is listed here:

- Self-Describing. A NetCDF file includes a description of the data that it contains.

- Portable. A NetCDF file can be accessed by computers that apply different formats of storing integers, characters, and floating-point numbers.

- Scalable. A small subset of a large dataset may be accessed efficiently.

- Appendable. Data may be appended to a properly structured NetCDF file without copying the dataset or redefining its structure.

These characteristics make NetCDF a perfect choice for storing any kind of array-oriented data. The data form that we will work with in this exercise is as well array-oriented - therefore, we will use NetCDF as data container for both input and output of computations.

Since NetCDF is a binary format (in contrast to ASCII-text, which can be examined and edited by means of any common text editor), reading, writing and changing of NetCDF files necessitates the use of dedicated software. Fortunately, such software is freely available and can easily be installed on any UNIX system. The following tools are of particular importance:

- ncview (lightweight but mighty explorer for NetCDF files)

- ncbrowse (a Java-based alternative to ncview for Windows)

- Panoply (a flexible Java-based generator of geographic maps of NetCDF data)

- ncdump (tool for "dumping" the contents of a NetCDF file to human-readable ASCII-text; the complete description and structure of the NetCDF file is preserved)

- ncgen (complementary to ncdump, generates a binary NetCDF file from a NetCDF ASCII-dump)

- ncks (mighty toolbox for modification of NetCDF files)

- cdo (mighty toolbox for analysis and modification of NetCDF files, strong focus on climatological data)

2.2.1 The Bash, a popular UNIX-Shell

Shells with their scripting ability are probably the most powerful tool of UNIX-systems, which make UNIX-computers the choice for tackling complex scientific problems that involve the analysis and processing of large amounts of data. Particularly the bash-shell is a famous tool for scientists and commonly used during the daily work routine. Yet, giving a comprehensive overview on the use and ability of the bash clearly exceeds the scope of this course. Please refer to a bash-scripting guide referenced below if you would like to gain further insights into the topic and gain abilities in shell-scripting - and note: "... the only way to really learn scripting is to write scripts" (Advanced Bash-Scripting Guide).

Further Reading

The following resources provide an introduction to tools and methods considered in this exercise:

- NetCDF: The NetCDF FAQ
  (www.unidata.ucar.edu/software/netcdf/docs/faq.html)
  The NetCDF Fact Sheet
  (http://www.unidata.ucar.edu/publications/factsheets/current/netcdf_factsheet.pdf)

- CDO: The CDO User's Guide
  (https://code.zmaw.de/projects/cdo/embedded/1.6.3/cdo.html)
  The CDO Reference Card
  (http://www.iac.ethz.ch/edu/courses/master/modules/radiation_and_climate_change/download/cdo_refcard.pdf)

- Bash: Bash Guide for Beginners
  (http://www.tldp.org/LDP/Bash-Beginners-Guide/html/)
  Advanced Bash-Scripting Guide
  (http://www.tldp.org/LDP/abs/html/)

Practical exercises for UNIX, cdo, netcdf

For those of you who consider to work in a scientific field with a strong focus on programming or the analysis of large amounts of data: Consider to gain experience with a UNIX-environment, e.g. Ubuntu. In many scientific fields, definitely in climate sciences, UNIX-like operating-systems are the computing environments of choice. Many tools that are necessary for efficiently working in such scientific fields are not natively available on Windows-systems; even if ports are available, they may still suffer from limitations or incompatibilities. It definitely makes sense for you to setup an own partition on your laptop with a Linux-system, and to learn how to use and program the available software tools. For every free UNIX-environment (e.g. Ubuntu) and the included software tools, a vast amount of detailed, yet free, documentation, addressing both novices and experts, can be found on-line.

Visualizing the content of a NetCDF file

Among the provided data files you find a gridded global distribution of sea-ice concentration (variable seaice) and surface temperature (variable tsurf) retrieved from a simulation with a comprehensive climate model (file INIOM_PD_3901-4000_tsurf_seaice.nc). NetCDF is a binary data format that cannot be directly visualized with a simple text editor, special software exists for this purpose. Windows-users please use the program **ncbrowse**, linux-users may alternatively use the native Linux-tool **ncview**.

Please open the NetCDF file and visually inspect the content of variable tsurf. Try to generate an animation of the time evolution of tsurf and answer the following questions:

- What obvious time-dependent pattern is visible?

- How can this pattern be explained, considering that you see the time evolution of a global climatological field?

Creating a PDF that visualizes a regional selection of the content of a NetCDF file

Now please visualize a global distribution of seaice also found in the provided NetCDF file INIOM_PD_3901-4000_tsurf_seaice.nc. The software to be used for this task is **panoply**. Create two plots of Arctic sea-ice distribution, from 45°N to the North Pole, for March and September of model year 3911, 18:00:00. Apply a meaningful colorbar and export the figure to PDF. Answer the following questions:

- Where do you identify the southernmost extent of sea-ice in boreal winter?

- Where is the sea-ice retreat in summer, by visual inspection, most prominent?

Hint: You can create a map centered on the North Pole by applying a stereographic projection.

Identifying the spatial resolution and physical unit of a NetCDF data set

By use of the tool **ncdump** it is possible to generate a human-readable description of a NetCDF file, that may also include the full data record. Sometimes you are only interested in the header information, and do not want the data record to be extracted (extracting the data record to ASCII can lead to HUGE dumps). If you have ncdump available, extract only the NetCDF data header of file INIOM_PD_3901-4000_tsurf_seaice.nc. This can be done in a Linux shell-terminal via entering the following command:

```
ncdump -h INIOM_PD_3901-4000_tsurf_seaice.nc > headerdump.nc
#the parameter '-h' forces ncdump to omit the extensive data record
#the character '>' is an output-redirection command, that makes sure that
#  the output produced by ncdump is written to a file (here: headerdump.nc)
#  rather then to the computer screen
```

Open the resulting file, or alternatively the file INIOM_PD_39010131_tsurf_seaice.nc.dump that has been provided to you, in an arbitrary text-viewer or -editor, analyze the file content, and answer the following questions:

- What is the horizontal data resolution in units of degrees? Hint: Calculate the meridional (latitudinal) and zonal (longitudinal) resolution of the data set from the number of longitudes and latitudes. Assume that grid cells are equally spaced, and the data set has a global coverage.

- What is the physical unit of variable tsurf?

- If you analyze the header dump of the file INIOM_PD_3901-4000_tsurf_seaice.nc: What is the time resolution of the data? Assume that the complete data set covers a time span of 100 calendar years.

2.2.2   Reducing data sets with CDO

While NetCDF defines a file format (and supporting programs and routines) that can store climatological data in a practical way, the CDO are a collection of operators that allow analysis and modification of gridded binary climatological data. In climate sciences, the CDO have become a very common software tool due to the vast number of available operators and their flexibility:

- more than 400 designated operators are available

- operator-piping allows the application of complex methods on climatological data in a compact way

- the CDO are command-line programs; in combination with shell-scripts they can be automated, and enhanced complexity of data processing and analysis may be achieved

CDO operator piping in combination with shell-programming can be demonstrated shortly in the following very short bash-script, where two input files are interpolated to a common resolution, the resulting fields are added, and the sum is time-averaged, the result being stored in a new file. Shell-programming allows for the diagnostic output of additional information to the screen, here the spatial-average of the field resulting from the CDO-operator-chain. You may find this code in file fldmean.sh that is among the distributed files.

```bash
#!/bin/bash

#select level 6, interpolate to 1x1 degree, and convert from deg. C to Kelvin
cdo timmean -addc,273.15 -remapcon,r360x180 -sellevel,6 input.nc output.nc
#note: the rightmost command ist executed first

#compute global mean
spat_avg=$(cdo output -fldmean output.nc)

#print result to screen
echo
echo
echo "spatial average of global ocean surface temperature is ${spat_avg} K."

#clean up
rm output.nc
```

In the following tasks we will process NetCDF files using the CDO. In order to fulfill these tasks, you may refer to the documentation that is available online (https://code.zmaw.de/projects/cdo/embedded/index.html). For convenience, some useful CDO commands are summarized in the following listing. You may find this code in file cdo_examples.txt that is among the distributed files.

```bash
#extract a variable named "varname" from file input.nc
cdo selvar,varname input.nc output.nc

#extract the first month of all years in file input.nc
cdo selmon,1 input.nc output.nc

#calculate a time average over a time series input.nc
cdo timmean input.nc output.nc

#generate a seasonal mean from input.nc
cdo seasmean input.nc output.nc

#generate a year mean from input.nc
cdo yearmean input.nc output.nc

#calculate an average annual cycle from file input.nc
cdo ymonmean input.nc output.nc

#select a region from input.nc, from longitude "a" to "b", from latitude "c" to "d"
cdo sellonlatbox,a,b,c,d input.nc output.nc

#calculate a spatial average of field input.nc
```

```
cdo fldmean input.nc output.nc

#write the output of a CDO operator "a" to the screen (omits file output.nc)
cdo output -a input.nc

#calculate the difference between two NetCDF files input1.nc and input2.nc
cdo sub input1.nc input2.nc output.nc

#multiply two fields input1.nc and input2.nc
cdo mul input1.nc input2.nc output.nc

#add a constant "a" to field input.nc
cdo addc,a input.nc output.nc

#select only regions of input2.nc, for which mask input1.nc is true (i.e. 1)
#represents an if-then programming construct
cdo ifthen input1.nc input2.nc output.nc

#use input2.nc, where mask input1.nc is true - otherwise use input3.nc
#represents an if-then-else programming construct
cdo ifthenelse input1.nc input2.nc input3.nc output.nc

#reduce a data range (a,b) in input.nc to the constant value "c"
cdo setrtoc,a,b,c input.nc output.nc

#replace a data range (a,b) in input.nc by the missing value ("NaN")
cdo setrtomiss,a,b input.nc output.nc

#calculate the trend of a time series in input.nc;
#the trend is defined by offset "a" and slope "b" of the regression line;
#"a" is stored in a.nc, "b" is stored in b.nc
cdo trend input.nc a.nc b.nc

#calculate the horizontal area covered by each grid cell of input.nc
cdo gridarea input.nc output.nc
```

Reducing NetCDF data sets

The file INIOM_PD_3901-4000_tsurf_seaice.nc contains two time series of climatological fields.

Reduce the NetCDF file by performing the following tasks using the CDO:

- Task 1: Split the data set in two separate data sets, one for variable tsurf, one for variable seaice.

- Task 2: Calculate a time average over the full time period available in each of the separate data sets created in Task 1.

- Task 3: For each of the data sets created in Task 1 calculate an average annual cycle that is representative for the full time period of the time series.

  Hint: In order to create an average annual cycle (multi-year monthly mean), you have to generate a new data set that contains twelve months. The data stored in each time step (month) n of this new data set o must represent the average over all corresponding months contained in the full time series of the initial data set i, i.e. (see CDO documentation):

$$o\left(n, x\right) = \mathrm{mean}\left(i\left(t, x\right), \mathrm{month}(i(t)) == n\right); \qquad n \in (1, 12)$$

- Task 4: Calculate a seasonal mean from the average annual cycle of both variables retrieved in Task 3.

- Task 5: Select only the Northern Hemisphere of the time average retrieved in Task 2.

- Task 6: Calculate the global average temperature from the data set retrieved in Task 2.

- Task 7: Repeat Task 4 using CDO-pipes, i.e. pipe the output of the CDO operator that calculates the average annual cycle of a data set created in Task 3 into the operator for the seasonal mean. Is there any difference between the file generated in this task with respect to the result retrieved in Task 4?

In the following we will further analyze and transform files that were generated during previous tasks.

- Task 8: Transfer the time-average temperature field, retrieved in Task 2 above, to units of °C. What is the average temperature in °C over the Northern Hemisphere?

- Task 9: Calculate global average monthly temperatures for the average annual cycle retrieved in Task 3.

- Task 10: What is the global- and time-averaged temperature over land and ocean? As input you may use the file generated in Task 2. You need to supply to CDO a mask that defines distributions of land and ocean; use variable SLM of file T31GR30_jan_surf.nc.

- Task 11: What is the global average trend in the temperature time series created in Task 1?

- Task 12: Calculate the monthly average sea-ice cover in units of km^2 in the Northern Hemisphere. Use the average annual cycle of the sea-ice field generated in Task 3.

2.2.3   A simple model of sea level rise

In this task we will learn how shell-programming can help in solving scientific problems. We will, based on simple assumptions, write a shell-script that generates a NetCDF data set of a rising sea level, attributed to a loss of land ice, and produces a data set that illustrates the resulting continental flooding. This shell-script can be considered as a very simplified model of continental flooding due to sea-level rise. The only necessary input for this model is a global data set of surface elevation.

The world's water is unevenly distributed among four major climate subsystems and three different states of matter. Respective climate subsystems are the atmosphere (water vapor and droplets, ice crystals), the ocean (liquid water and frozen, solid water), and the land surface (liquid and solid water). The subsystem of the Earth composed of frozen water is commonly referred to as the Cryosphere. It includes permafrost, lake and river ice, sea ice, snow, glaciers, ice caps and ice sheets. Within the Cryosphere, ice sheets are the largest storage of frozen water. At present, there are two major ice-sheets, the Greenland Ice-Sheet (GIS) and the Antarctic Ice-Sheet (AIS), containing water volumes of **7.3** m and **56.6** m sea level equivalent, respectively (see Table 4.1 of the contribution of Working Group 1 to the Fourth Assessment Report by the IPCC, page 342 of http://www.ipcc.ch/pdf/assessment-report/ar4/wg1/ar4-wg1-chapter4.pdf). This means that, assumed both the GIS and the AIS melted completely, the global average sea level would rise by **63.9** m, leading to a flooding of large parts of coastal regions, where **44%** of humankind live.

The relative volume distribution of water between the different physical states of matter (solid, liquid, evaporated) depends on the average surface temperature of the Earth. There have been geologic time scales that were much warmer than present, where virtually no ice sheets were available (see e.g. Fig. 2 of Zachos et al., 2001, `http://science.sciencemag.org/content/292/5517/686.full`). Anthropogenic emissions of, particularly, carbon dioxide, are expected to increase the average surface temperature on Earth via a modification of the global radiative energy balance, since carbon dioxide in the atmosphere contributes to the so-called greenhouse effect (see the publication by Arrhenius, 1896, available at `http://www.rsc.org/images/Arrhenius1896_tcm18-173546.pdf`). Due to global warming, the volume of water stored in ice sheets will become smaller, and ultimately increase the volume of the global oceans, thus causing sea level to rise.

Outline of the exercise Here, we develop a simple model of sea level rise which illustrates regions of the earth that are flooded if the GIS and AIS melt. Our work employs command line tools that are freely available and should be present on the computer of any scientist that performs numerical modelling or substantial scientific data processing. If you run a linux system, these tools should either already be available, or should be installted with a minimum amount of work.

As it is the case for any scientific model, which is per definition only an idealization of a natural system, our methodology is based on several simplifications. First of all, it is assumed that the melt process of available ice sheets is linear and occurs over a time period of **1000** y. Furthermore, we do not apply a physical flow model that simulates water inflow from the coast, since such a model can hardly be designed and formulated within the framework of this exercise. Instead, we will calculate water height over land via subtracting the rising sea level from the global elevation field. Negative values indicate the presence of water over land.

Our work depends on the availability of a gridded land elevation data set and a land-sea-mask that separates the land surface into regions that belong to land and ocean. These data sets are also the only input files that are necessary for our analysis. The files are taken from a set of boundary

conditions for numerical models of the circulation of the atmosphere. These have been derived from global elevation data sets, that are commonly generated using modern satellite-based radar interferometry, e.g. by the satellite system ICESat (icesat.gsfc.nasa.gov/icesat/). For use in climate modelling, this data has been interpolated to the resolution of the climate model. The orography in a climate model itself is therefore also a model - a model of the land surface elevation, whose accuracy depends on the quality of the measurement and on the grid resolution to which it is interpolated to. Higher resolution of the model grid means better agreement of the gridded orography with the original high-resolution orography data set. The data set used here originates from the boundary conditions of the Atmosphere General Circulation Model ECHAM5 at T63-resolution, which corresponds to a horizontal resolution of $1.9°$ by $1.9°$. Close to the equator, this corresponds to roughly 200 km by 200 km per grid cell.

```bash
#!/bin/bash

#some definitions that control the script
n_time_steps=1000 #number of time steps over which sea level rises
time_unit='1year' #resolution of the time axis
time_reference='2000-01-01,00:00:00' #reference time (first date) of the time axis
sea_level_start=0 #sea level ramp starts with elevation of 0 m
sea_level_end=75 #sea level ramp ends with elevation of 75 m

#definition of input files
orography_file='data/orography.nc'
lsm_file='data/land_sea_mask.nc'

#definition of output files
sea_level_output_file="sea_level_time_series.nc"
flooding_file='flooding_due_to_ice_melt.nc'

#clean up from (potential) previous runs of this script
if [ -e ${sea_level_output_file} ]
then
  rm ${sea_level_output_file}
fi
if [ -e ${flooding_file} ]
then
  rm ${flooding_file}
fi

#find minimum and maximum values of orography (necessary for file processing)
min_val=$(cdo output -fldmin ${orography_file} | sed 's/ //g')
max_val=$(cdo output -fldmax ${orography_file} | sed 's/ //g')
```

```
#subtract / add small value from / to min_val and max_val in order to increase the
#data range
#(cdo setrtomiss, used below, does not include limits of the processed data range,
#but we need to include these)
min_val=$(echo "scale=3; ${min_val}-1" | bc)
max_val=$(echo "scale=3; ${max_val}-1" | bc)

#generate a NetCDF file that contains n_time_steps, all values NaN;
#this file will be used to generate a time-dependent sea level data set;
#our approach, taking an existing NetCDF file and modifying it, saves
#lots of tedious work in setting up a new NetCDF file from scratch,
#which involves definition of dimensions, variables, and the definition of
#links between these
for ((i=0; i<${n_time_steps}; i+=1))
do
  #generate n_time_steps copies of orography,
  #add a time axis, and replace all values by NaN
  echo "creating file timestep_${i}.nc ..."
  cdo setrtomiss,${min_val},${max_val} \
      -settaxis,${time_reference},${time_unit} \
      ${orography_file} timestep_${i}.nc 2>/dev/null
done

#merge the above generated files to one NetCDF file
cdo mergetime timestep_?.nc timestep_??.nc timestep_???.nc timeseries.nc

#set a proper time axis for the whole time series
cdo settaxis,${time_reference},${time_unit} timeseries.nc timeseries_timeaxis.nc

#clean up temporary files
rm timestep_*.nc timeseries.nc

#run a loop that generates a linear sea level ramp, starting a sea_level_start
#and ending at sea_level_end
current_sea_level=${sea_level_start}
for ((i=0; i<${n_time_steps}; i+=1))
do
  #redefine current_sea_level depending on the value of i
  current_sea_level=$(echo "scale=3; ${sea_level_start}+${sea_level_end}*${i}/(${n_

  #generate a three digit string of i containing a leading zero (for file names)
  if [ ${i} -le 9 ]
  then
    counter_str="00${i}"
  else
    if [ ${i} -le 99 ]
    then
      counter_str="0${i}"
    else
      counter_str="${i}"
    fi
```

```
  fi

  #generate from each time step of timeseries_timeaxis.nc a NetCDF file that
  #contains a global field corresponding to the respective sea level at that time
  echo "creating file sea_level_time_series_${counter_str}.nc ..."
  cdo setmisstoc,${current_sea_level} \
      -setrtomiss,${min_val},${max_val} \
      -seltimestep,$((i+1)) \
      timeseries_timeaxis.nc sea_level_time_series_${counter_str}.nc 2>/dev/null
done

#merge the single time steps of the sea level time series into one file
cdo mergetime sea_level_time_series_???.nc ${sea_level_output_file}

#clean up temporary files
rm sea_level_time_series_???.nc timeseries_timeaxis.nc

#subtract sea level from elevation in order to calculate water height over land
#negative values will depict water
cdo sub ${orography_file} ${sea_level_output_file} tmp.nc

#remove the elevation and sea level from the data set so that all values in file
#flooding_file depict the water level caused by flooding
cdo mulc,-1 \
    -mul ${lsm_file} \
    -setmisstoc,0 \
    -setrtomiss,0,${max_val} \
    tmp.nc ${flooding_file}

#clean up
rm tmp.nc
```

Exercise 13   –   Shell and netcdf

Modify the above source code with the purpose of simulating the flooding that results from
1010 years of sea-level rise due to ocean warming. To this end you need to:

1. Make a literature research with the purpose of finding an estimate of rates r of current or
   near-future annual sea-level rise due to ocean warming; commonly such estimates are in the
   order of a few millimeters per year. Convert your choice of r to units of m/y

2. Modify the above given shell-script so that it creates a 1010 y linear ramp that contains
   the time-dependent global sea level SSH, starting with zero and ending with the value
   $SSH_{\mathrm{end}} = r \cdot 1010$ y.

Run both the unmodified version of the model, simulating sea-level rise due to land-ice melt,

and the modified version of the model, simulating sea-level rise due to ocean warming.

Note: in order to execute a bash script "script.sh", open a Linux shell-session, navigate to the folder where the script is located, and execute the command "./script.sh". You might have to activate executable rights for this script by executing "chmod u+x script.sh" in advance.

Analyze and discuss your results:

1. Make a time-dependent analysis of the flooding that is predicted by this simple model of sea level rise; you could, for example, generate plots of the flooding over land for two or three interesting time steps that you select from the 1010 y time series.

2. Make a similar analysis for the sea level rise that is simulated by the unchanged original shell-script, which considers melting of land-ice.

3. Compare the results of both analyses and discuss differences. Where are regions located, where people should not live anymore in the future?

4. Make a critical analysis of the flooding simulated by this very simple model.

 For the last point, these are questions that you might consider:

1. What are weak points of the simulation? Consider in your discussion the term "volume conservation", the stationarity of the applied sea level rise, and the spatial resolution of the orographic data. How would you expect the simulated flooding to look like if the spatial resolution of orography was higher?

2. The model, in its current version, has at least one dramatic flaw. Where is the simulated flooding, that should only be attributed to water inflow from the ocean, obviously wrong? What is the reason for the erroneous result?

3. Could you imagine what additional physical processes a corrected version of the model would need to contain in order to avoid the identified problem? Give a short description of your ideas, maybe illustrate them with some sketches.

Exercise 14   –   Evaluate possible zones of wine production and climatic conditions

 For wine production there exists some empirical laws:

Northern Hemisphere: degree of latitude 40 – 50

Southern Hemisphere: degree of latitude 30 – 40

The climatic demands are:

1. Vegetation period 180 – 250 days (days with daily mean > 5 °C)

2. Mean temperature 9 °C– 21 °C

3. White wines optimal annual mean temperature 9.5–11.5 °C

4. Red Wines optimal annual mean temperature 10.5–13 °C

5. Sunshine hours per year: 1500–2000 h

6. Precipitation 480 mm–700 mm

Task: Please calculate the possible zones of red wine production using 4 and 6 using temperature and precipitation constraints. The resulting figure should be similar to Fig. 2.4. Here are the links for getting the CRU surface temperature:

http://climexp.knmi.nl//CRUData/cru_ts3.23.1901.2014.tmp.dat_1.nc
and precipitation data:

http://climexp.knmi.nl//CRUData/cru_ts3.23.1901.2014.pre.dat_1.nc
Choose two different reference periods: 1901-1920 and 1991-2010

Calculate furthermore the zones for future scenarios (RCP4.5 and RCP8) for the years 2081-2100. Data are accessible here:

http://cmip-pcmdi.llnl.gov/cmip5/data_getting_started.html.
For this, choose one particular model and compare the zones with a reference period using the same model (e.g., 1960-1980). Some people could speculate that the alcohol belts (Fig. 2.5) may be shifting under global warming. Can this be substantiated by your results?

Hint: use the cdo commands from the lecture!
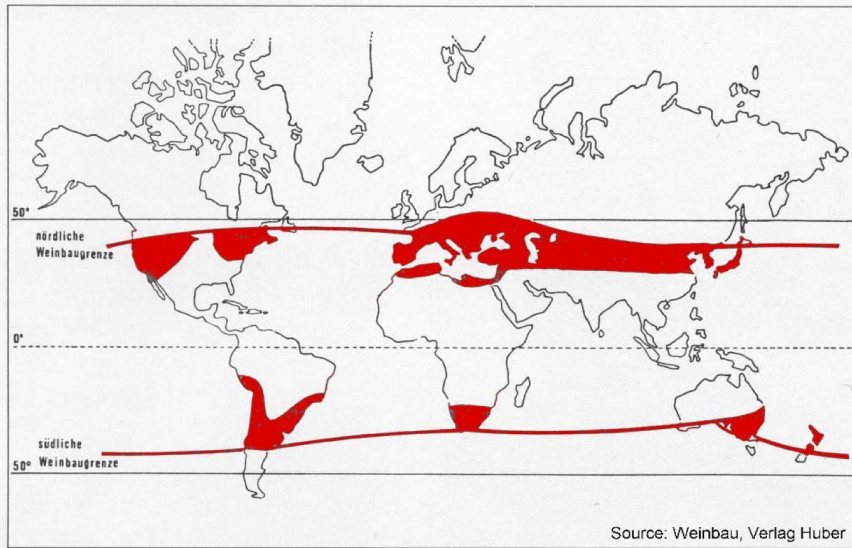
Figure 2.4: Zones of todays wine production.
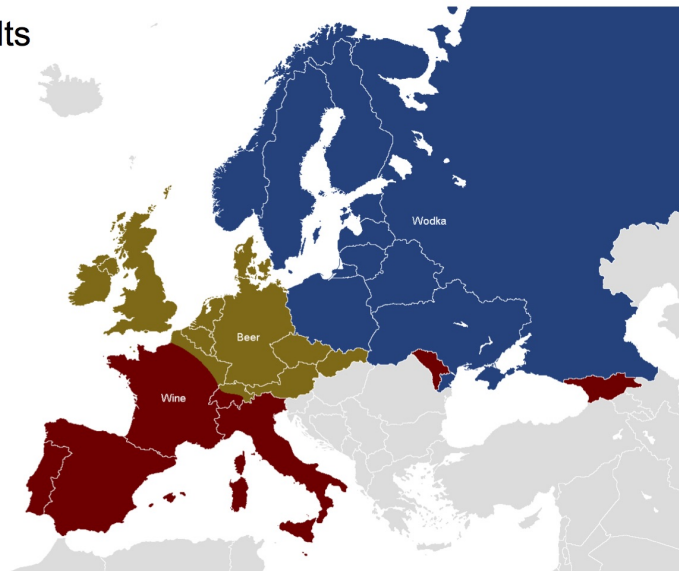
Figure 2.5: Alcohol belts in Europe.

2.3   Bifurcations

Before we start with some applications of fluid stability, I provide a framework to analyze the stability of dynamical systems. A bifurcation occurs when a parameter change causes the stability of an equilibrium (or fixed point) to change [Strogatz, 2000]. In continuous systems, this corresponds to the real part of an eigenvalue of an equilibrium passing through zero. In discrete systems (those described by maps rather than ordinary differential equations (ODEs)), this corresponds to a fixed point having a Floquet multiplier with modulus equal to one. In both cases, the equilibrium is "non-hyperbolic" at the bifurcation point (for a sketch: Fig. 2.6). The topological changes in the phase portrait of the system can be confined to arbitrarily small neighbourhoods of the bifurcating fixed points by moving the bifurcation parameter close to the bifurcation point. We will discuss as one particular example the Lorenz system (Rayleigh [1916], Saltzman [1962], Lorenz [1976]).
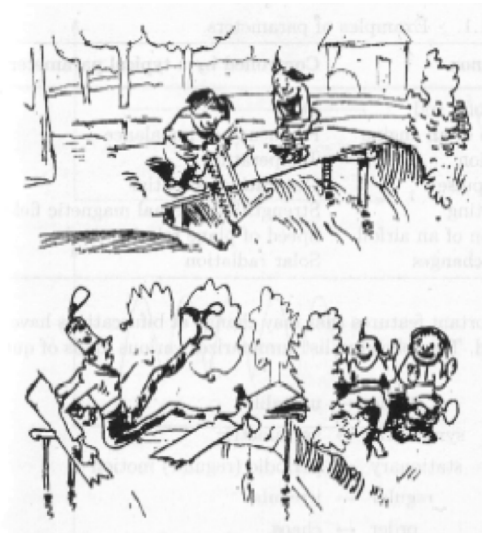
Figure 2.6: Bifurcation sketch. The boys Max and Moritz torment Schneider Böck, a well-liked tailor who has a fast stream flowing in front of his house. They saw through the planks of his wooden bridge, making a precarious gap, then taunt him by making goat noises, until he runs outside. The bridge breaks; the tailor is swept away and nearly drowns (but for two geese, which he grabs a hold of and which fly high to safety). Source: Busch [1865].

2.3.1 Linear stability analysis

Consider the continuous dynamical system described by the ODE

$$\dot{x} = f(x, \lambda) \quad f \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n. \tag{2.3}$$

A bifurcation occurs at (x_0, λ_0) if the Jacobian matrix $\mathrm{d}f_{x_0, \lambda_0}$ has an Eigenvalue with zero real part. If the eigenvalue is equal to zero, the bifurcation is a steady state bifurcation, but if the eigenvalue is non-zero but purely imaginary, this is a Hopf bifurcation.

For discrete dynamical systems, consider the system

$$x_{n+1} = f(x_n, \lambda). \tag{2.4}$$

Then a local bifurcation occurs at (x_0, λ_0) if the matrix $\mathrm{d}f_{x_0, \lambda_0}$ has an eigenvalue with modulus equal to one. If the eigenvalue is equal to one, the bifurcation is either a saddle-node (often called fold bifurcation in maps), transcritical or pitchfork bifurcation. If the eigenvalue is equal to -1, it is a period-doubling (or flip) bifurcation, and otherwise, it is a Hopf bifurcation.

Examples of bifurcations include [Strogatz, 2000]:

- A transcritical bifurcation is one in which a fixed point exists for all values of a parameter and is never destroyed. However, such a fixed point interchanges its stability with another fixed point as the parameter is varied. The normal form of a transcritical bifurcation is

$$\frac{dx}{dt} = rx - x^2. \tag{2.5}$$

  This equation is similar to logistic equation but in this case we allow r and x to be positive or negative. The two fixed points are at $x = 0$ and $x = r$. When the parameter r is negative, the fixed point at $x = 0$ is stable and the fixed point $x = r$ is unstable. But for $r > 0$, the point at $x = 0$ is unstable and the point at $x = r$ is stable. So the bifurcation occurs at $r = 0$.

- A '"saddle-node bifurcation'" is a bifurcation in which two fixed points collide and anni-
  hilate each other. If the phase space is one-dimensional, one of the equilibrium points is
  unstable (the saddle), while the other is stable (the node). The normal form of a saddle-node
  bifurcation is:

$$\frac{dx}{dt} = r + x^2 \tag{2.6}$$

  Here x is the state variable and r is the bifurcation parameter. If $r < 0$ there are two
  equilibrium points, a stable equilibrium point at $-\sqrt{-r}$ and an unstable one at $+\sqrt{-r}$.
  At $r = 0$ (the bifurcation point) there is exactly one equilibrium point. At this point the
  fixed point is no longer hyperbolic. In this case the fixed point is called a saddle-node fixed
  point. If $r > 0$ there are no equilibrium points. Saddle-node bifurcations may be associated
  with hysteresis loops. The term 'saddle-node bifurcation' is most often used in reference to
  continuous dynamical systems. In discrete dynamical systems, the same bifurcation is often
  instead called a '"fold bifurcation'".

- A Hopf is a bifurcation in which a fixed point of a dynamical system loses stability as a pair
  of complex conjugate eigenvalues of the linearization around the fixed point cross the imag-
  inary axis of the complex plane. In a bifurcation, a small-amplitude limit cycle branching
  from the fixed point. The normal form of a Hopf bifurcation is:

$$\frac{dz}{dt} = z((\lambda + i) + b|z|^2), \tag{2.7}$$

  where z, b are both complex and λ is a parameter. Write $b = \alpha + i\beta$. The number 'α is
  called the first Lyapunov coefficient. If α is negative then there is a stable limit cycle for
  $\lambda > 0$:

$$z(t) = re^{i\omega t} \tag{2.8}$$

where

$$r = \sqrt{-\lambda/\alpha} \quad \text{and} \quad \omega = 1 + \beta r^2. \tag{2.9}$$

The bifurcation is then called "'supercritical.'" If α is positive then there is an unstable limit cycle for $\lambda < 0$. The bifurcation is called "'subcritical.'"

- Pitchfork bifurcations occur generically in systems with symmetry. Pitchfork bifurcations, like Hopf bifurcations have two types - supercritical or subcritical. The normal form of the supercritical pitchfork bifurcation is

$$\frac{dx}{dt} = rx - x^3. \tag{2.10}$$

For negative values of r, there is one stable equilibrium at $x = 0$. For $r > 0$ there is an unstable equilibrium at $x = 0$, and two stable equilibria at $x = \pm\sqrt{r}$. The normal form for the subcritical case is

$$\frac{dx}{dt} = rx + x^3. \tag{2.11}$$

In this case, for $r < 0$ the equilibrium at $x = 0$ is stable, and there are two unstable equilbria at $x = \pm\sqrt{-r}$. For $r > 0$ the equilibrium at $x = 0$ is unstable.

For computational methods to obtain bifurcations: [Doedel et al., 1997; Kuznetsov, 1998].

Exercise 15 – Graphical method for bifurcations

We introduce a graphical method to obtain stability or instability. Consider the "'saddle-node bifurcation'", one of the equilibrium points is unstable (the saddle), while the other is stable (the node). In Fig. 2.7, we can plot $\frac{dx}{dt} = f(x)$ dependent on x (left panel) for

$$\frac{dx}{dt} = b + x^2 \tag{2.12}$$

with $b < 0$ in this particular case (For $b > 0$ we would have no equilibrium, and we have no

point x_e with $f(x_e) = 0$.). We just consister the slope $f'(x_e)$ and see that the filled circles with positive slope are unstable, the open circles with negative slopes are stable (right panel in Fig. 2.7).

1. Draw the bifurcations as in Fig. 2.7 for the pitchfork bifurcation.

2. Draw the bifurcations as in Fig. 2.7 for the transcritical bifurcation.
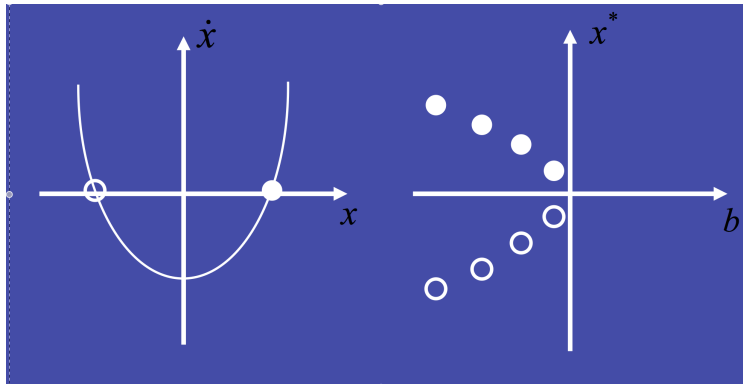
Figure 2.7: Saddle-node bifurcation diagram using the graphical method.

Exercise 16   –   Bifurcation example $rx\,(1-x)^2$

Consider the differential equation

$$\frac{d}{dt}x \;=\; rx\,(1-x)^2 \tag{2.13}$$

a) Calculate the bifurcation with respect to parameter r, consister the slope $f'(x_e)$. Draw the bifurcation diagram!

b) Discuss the stability in terms of the potential V(x) ! Remember that the potential can be calculated from the right hand side of equation (2.13): rhs of (2.13)) $= -\frac{dV(x)}{dx}$

Solution of Exercise 16

a) Equilibria solutions are $x_e = 0, 1$.

$$f'(x) = r\,(1-x)^2 - 2rx\,(1-x)$$

Check $f'(x_e)$: $f'(0) = r$ (indifferent stability); $f'(1) = 0$ (stability or instability depending on r);

b)
$$V(x) = -r/2x^2 + 2/3rx^3 - 1/4rx^4$$

Plotting of the potential using R:

```
y=-100:100
x=y/10
x=y/50
r=1
z=-r * x^2/2 +2/3 * x^3 -r/4 * x^4
plot(x,z,type='lines')
```

2.3.2   Population Dynamics

Exercise 17   –   Population Dynamics

Consider population dynamics with population $x > 0$ and reproduction (birth-death) r:

$$\frac{d}{dt}x = r(x)x \tag{2.14}$$

1. Solve the differential equation for $r = r_0 = const.$! What happens for $t \to \infty$ when $r_0 > 0$ or $r_0 < 0$?

2. Solve the differential equation for $r = r_0(1 - x)$! (limited growth)! What happens for $t \to \infty$?

3. Consider the case $r = r_0(1 - x/K)$ with $K > 0$! Give a physical interpretation for K!

Solution of Population Dynamics

Solve for $r(x) = r_0$ using separation of variables:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = r_0 x$$

$$\int \frac{\mathrm{d}x}{x} = \int r_0 \mathrm{d}t$$

$$\ln(x) = r_0 t + A'$$

$$\implies x = A e^{r_0 t} \quad \text{with} \quad A = e^{A'}$$

$$\text{with} \quad \lim_{t \to \infty} x = \begin{cases} \infty & , r_0 > 0 \\ 0 & , r_0 < 0 \end{cases}$$

Solve for $r(x) = r_0(1 - x)$ using separation of variables:

$$\frac{dx}{dt} = r_0(1 - x)x$$

$$\frac{dx}{x(1 - x)} = r_0 dt$$

$$\int \left(\frac{1}{x} + \frac{1}{1 - x} \right) dx = \int r_0 dt$$

$$\ln(x) - \ln(1 - x) = r_0 t + A'$$

$$\implies x = \frac{Ae^{r_0 t}}{1 + Ae^{r_0 t}} \quad \text{with} \quad A = e^{A'}$$

$$\text{and the limiting cases} \quad \lim_{t \to \infty} x = \begin{cases} 1 & , r_0 > 0 \\ 0 & , r_0 < 0 \end{cases}$$

Consider $r(x) = r_0\left(1 - \frac{x}{K}\right)$ with $K > 0$. Analogous procedure results then in a similar solution with an additional scaling factor K which provides an upper limit for any population.

$$\frac{dx}{dt} = r_0(1 - \frac{x}{K})x$$

$$\vdots$$

$$\implies x = \frac{KAe^{r_0 t}}{1 + Ae^{r_0 t}}$$

$$\text{with} \quad \lim_{t \to \infty} x = \begin{cases} K & , r_0 > 0 \\ 0 & , r_0 < 0 \end{cases}$$

Exercise 18 – Difference equations

Consider the discretised form of (2.14) with $r = r_0(1 - x)$. Using the Euler scheme

$$\frac{d}{dt}x \approx \frac{x_{n+1} - x_n}{\Delta t} \quad . \tag{2.15}$$

1. Write down the iteration x_{n+1} as a function of x_n for the case 1a!

2. What is the solution of x_{n+1} as a function of x_0? Consider the stability for the cases $r > 0$, $0 > \Delta t\, r > -1$, $-1 > \Delta t\, r > -2$, $-2 > \Delta t\, r$. Do you have a graphical interpretation of the oscillation/decay?

3. Write down the iteration x_{n+1} as a function of x_n for the case 1b!

Exercise 19   –   Bifurcation of the logistic equation

1. Consider the system (2.14) $r = r_0(1 - x)$. Calculate the bifurcation with respect to parameter r!

   Draw the bifurcation diagram!

2. as in 1., but for

$$\frac{d}{dt}x = r_0 + x^2 \tag{2.16}$$

3. as in 1., but for

$$\frac{d}{dt}x = x\sqrt{(r_0 + x)^2} \tag{2.17}$$

Solution of Bifurcation of the logistic equation

Given the logistic equation

$$f(x) \equiv \frac{dx}{dt} = r_0 x(1 - x)$$
$$\implies \quad f'(x) = r_0 - 2r_0 x$$

we calculate the corresponding equilibrium points x_i:

$$f(x) = r_0 x(1 - x) = 0$$
$$\implies \quad x_1 = 0, \quad x_2 = 1$$

Hence, both equilibrium points do not depend on the parameter r_0. To check whether we are dealing with stable or unstable equilibrium points, we need to calculate the second derivative at the equilibrium points.

$$f'(x_1) = r_0$$
$$f'(x_2) = r_0 - 2r_0 = -r_0$$

That is, the equilibrium points x_1 and x_2 are independent of r_0. x_1 is stable for $r_0 < 0$ and unstable for $r_0 > 0$, x_2 is stable for $r_0 > 0$ and unstable for $r_0 < 0$.

Given the equation

$$f(x) \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = r_0 + x^2$$
$$\implies \quad f'(x) = 2x$$

we calculate the corresponding equilibrium points x_i:

$$f(x) = r_0 + x^2 = 0$$
$$\implies \quad x_{1,2} = \begin{cases} \pm\sqrt{-r_0} & , r_0 \le 0 \\ \pm i\sqrt{r_0} & , r_0 > 0 \end{cases}$$

We just consider real solutions and neglect the imaginary ones. Then the stability conditions for

the equilibrium points are given by

$$f'(x_1) = 2\sqrt{-r_0} \quad \begin{cases} < 0 & \text{stable} \\ \geq 0 & \text{unstable} \end{cases}$$

$$f'(x_2) = -2\sqrt{-r_0} \quad \begin{cases} < 0 & \text{stable} \\ \geq 0 & \text{unstable} \end{cases}$$

From the condition $r_0 \leq 0$ follows that x_1 is always unstable and x_2 is always stable. For the special case $r_0 = 0$ there is just one equilibrium point $x_1 = 0$ which is unstable as well.

Given the equation

$$f(x) \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = x\sqrt{(r_0 + x)^2} = \begin{cases} x(r_0 + x) & , x \geq -r_0 \\ -x(r_0 + x) & , x < -r_0 \end{cases}$$

$$\implies \quad f'(x) = \begin{cases} r_0 + 2x & , x > -r_0 \\ -r_0 - 2x & , x < -r_0 \\ \text{not defined} & , x = -r_0 \end{cases}$$

we calculate the corresponding equilibrium points x_i:

$$f(x) = x\sqrt{(r_0 + x)^2} = 0$$

$$\implies \quad x_1 = 0, \quad x_2 = -r_0$$

Since for $x_2 = -r_0$ the derivative $f'(x)$ does not exist, we need to treat both cases of a small deviation $\delta > 0$ from the equilibrium point x_2 to each side separately. The stability conditions

then yield:

$$f'(x_1) = \begin{cases} r_0 & , x_1 = 0 > -r_0 \quad \Rightarrow \quad r_0 > 0 \quad \Rightarrow \quad \text{unstable} \\ -r_0 & , x_1 = 0 < -r_0 \quad \Rightarrow \quad r_0 < 0 \quad \Rightarrow \quad \text{unstable} \end{cases}$$

$$f'(x_2 + \delta) = -r_0 + \delta \quad \Rightarrow \quad \begin{cases} \text{stable} & , r_0 > 0 \\ \text{unstable} & , r_0 < 0 \end{cases}$$

$$f'(x_2 - \delta) = r_0 - \delta \quad \Rightarrow \quad \begin{cases} \text{unstable} & , r_0 > 0 \\ \text{stable} & , r_0 < 0 \end{cases}$$

Exercise 20   –   Bifurcation of the logistic map

1. Write a function which solves the logistic difference equation $x_{n+1} = ax_n(1 - x_n)$ and returns the vector $x(n)$. Use an initial value $x_0 \in [0, 1]$, and a parameter value $a \in [1, 4]$

2. Investigate the sensitivity of solution on the parameter a (especially using $a \in [3, 4]$)

3. Now investigate the solutions dependent on r systematically: write a function which saves the local extrema of a vector (fixed points) and returns them in a vector.

4. For each value of a, iterate the logistic difference equation 500 times, discard the first 200 times, and plot the fix-points/local extrema against a. What do you see? Zoom into the plot!

Solution: Bifurcation of the logistic map

```
#logistic difference equation,
#with parameter a, N iterations, and initial value x0
f = function(r,N,x0)
{
  x <- vector()
  x[1]<-x0
  for (i in 2:N)   x[i]<-a*x[i-1]*(1-x[i-1])
```

```r
  return(x)
}

#determine the values of the local extrema and give them back in a vector
local_extrema <- function(x)
{
  result <- vector()
  for (i in 1:(length(x)-2))
  {
  if ((x[i]<x[i+1]) && (x[i+1] > x[i+2])) result<-c(result,x[i+1])
#save the local maximum in the result vector
      if ((x[i]>x[i+1]) && (x[i+1] < x[i+2])) result<-c(result,x[i+1])
#savel the local minimum in the result vector
  }
  return(result)}


#test the functions:

#Test the logistic difference equation function
plot(f(3.9,100,0.4),type="l")

#Test the local extrema function
temp<-sin((1:300)/30)
plot(temp)
local_extrema(temp)

#main program
resolution<-400   #number of r-parameter values to be scanned
rlim<-c(2,4)      #minimum and maximum r-value
xlim<-c(0,1)      #minimum and maximum x-value
```

2.3.3   Lorenz system

This system is an idealization of the Rayleigh-Bénard problem (section 5.3) and provides an example for chaotic behavior in a dissipative system.

$$\dot{X} = -\sigma X + \sigma Y \tag{2.18}$$

$$\dot{Y} = rX - Y - XZ \tag{2.19}$$

$$\dot{Z} = -bZ + XY \tag{2.20}$$

Equations (2.18, 2.19, 2.20) are called *Lorenz model* in the literature [Lorenz, 1960, 1963, 1984; Maas, 1994; Olbers, 2001]. As we will see later in section 5.3, the system may give realistic results when the Rayleigh number is slightly supercritical, but their solutions cannot be expected to resemble those of the complete dynamics when strong convection occurs, in view of the extreme truncation. Figure 2.9 shows the numerical solution in the phase-space with the parameters $r = 28$, $\sigma = 10$, and $b = 8/3$.

For some experiments go to the Lorenz model, Lorenz model 2

The same equations as (2.18, 2.19, 2.20) appear in studies of lasers, batteries, and in a simple chaotic waterwheel that can be easily built. Lorenz found that the trajectories of this system, for certain settings, never settle down to a fixed point, never approach a stable limit cycle, yet never diverge to infinity. What Lorenz discovered was at the time unheard of in the mathematical community, and was largely ignored for many years. Now this beautiful attractor is the most well-known strange attractor that chaos has to offer.

Properties of the Lorenz equations

- Symmetry: The Lorenz equations have the following symmetry of ordinary differential equations: $(X, Y, Z) \rightarrow (-X, -Y, Z)$. This symmetry is present for all parameter-values of the Lorenz system.

- Invariance: The Z-axis is invariant, meaning that a solution that starts on the Z-axis (i.e. $X = Y = 0$) will remain on the z-axis. In addition, the solution will tend toward the origin if the initial conditions are on the z-axis.

- Equilibrium points: To solve for the equilibrium points we let $|f\rangle (X, Y, Z) = 0$, where we used the ket-notation to denote the vector $|f\rangle = (\dot{X}, \dot{Y}, \dot{Z})^T$. It is easy to notice that $(X, Y, Z) = (0, 0, 0)$ is a trivial equilibrium-point. The other equilibrium-points, when $X \neq 0$, are also easy to determine analytically. We leave this task as an exercise to the reader.

- Solutions stay close to the origin: If $\sigma, b, a > 0$, then all solutions of the Lorenz system

will enter an ellipsoid in finite time. In addition, the solution will remain inside the ellipsoid once it has entered. It follows that the ellipsoid is an attracting set. To quantify this, we define an ellipsoid centered at $(0, 0, 2r)$ in finite time, and the solution will remain inside the ellipsoid once it has entered. To observe this, we define a Lyapunov function

$$V(X, Y, Z) = \tau X^2 + \sigma Y^2 + \sigma(Z - 2r)^2 \quad .$$

It then follows that

$$\begin{aligned}
\dot{V} &= 2rX\dot{X} + 2\sigma Y\dot{Y} + 2\sigma(Z - 2r)\dot{Z} \\
&= 2rX\sigma(Y - X) + 2\sigma Y(X(r - Z) - Y) + 2\sigma(Z - 2r)(XY - bZ) \\
&= -2\sigma(rX^2 + Y^2 + b(Z - r)^2 - br^2).
\end{aligned}$$

We then choose an ellipsoid which all solutions will enter and remain inside. This is done by choosing a constant $C > 0$ such that the ellipsoid

$$rX^2 + Y^2 + b(Z - r)^2 = br^2$$

is strictly contained in the ellipsoid

$$rX^2 + \sigma Y^2 + \sigma(Z - 2r)^2 = C \quad .$$

Therefore all solutions will eventually enter and remain inside the above ellipsoid since $\dot{V} < 0$ when a solution is located at the exterior of the ellipsoid.

- The Lorenz system exhibit bifurcations. If $r < 1$ then there is only one equilibrium point, which is at the origin. This point corresponds to no convection. A saddle-node bifurcation

occurs at $r = 1$, and for $r > 1$ two additional critical points appear at

$$\left(\pm \sqrt{b(r-1)}, \pm \sqrt{b(r-1)}, r-1 \right). \tag{2.21}$$

These correspond to steady convection. This pair of equilibrium points is stable only if

$$r < r_c = \sigma \frac{\sigma + b + 3}{\sigma - b - 1} \quad (\approx 24.74) \quad, \tag{2.22}$$

which can hold only for positive r if $\sigma > b+1$. At the critical value, both equilibrium points lose stability through a (inverse) Hopf bifurcation. One normally assumes that the parameters σ, r, and b are positive. Lorenz used the values $\sigma = 10, b = 8/3$ and $r = 28$. At such large r the three mode approximation for the Rayleigh-Bénard system describing thermal convection has of course ceased to be physically realistic, but mathematically the model now starts to show its most fascinating properties, because the aperiodic strange attractor behavior becomes dominant for $r > r_c$. The system exhibits chaotic behavior[1] for these values (Fig. 2.9) and the state variables that can be represented in phase space[2]. Repeating, X is proportional to the circulatory fluid velocity, Y characterizes the temperature difference between ascending and descending fluid elements, and Z is proportional to the distortion of the vertical temperature profile from its equilibrium (which is linear with height). The Lorenz system has either stable or unstable fixed points, a globally attracting periodic or nonperiodic solutions, bistability and hysteresis, and a variety of cascading bifurcations (see Fig. 2.10).

---

[1]Lorenz's conclusions about weather forecasting stated: "When our results concerning the instability of nonperiodic flow are applied to the atmosphere, which is ostensibly nonperiodic, they indicate that prediction of the sufficiently distant future is impossible by any method, unless the present conditions are known exactly. In view of the inevitable inaccuracy and incompleteness of weather observations, precise very-long-range forecasting would seem to be non-existent".

[2]The set of chaotic solutions make up the Lorenz attractor with a Hausdorff dimension which is estimated to be 2.06 ± 0.01 and the correlation dimension estimated to be 2.05 ± 0.01. For other values of r, the system displays knotted periodic orbits. For example, with $r = 99.96$ it becomes a "T"(3,2) torus knot (Grassberger and Procaccia [1983]).

Exercise 21   –   **Bifurcation Lorenz and map**

1. Following Fig. 2.10, show the bifurcation diagram for the intervals $45 < \sigma < 55$ and $8.0 < \sigma < 9.5.$ Notice, that except for their different scales the pictures are much like mirror images of each other.

2. Show that in both cases the scenarios coincide in many aspects (though not completely) with the bifurcation scheme of the antisymmetric cubic map

$$x_{n+1} = (1 - c)x_n + cx_n^3 \quad , -1 \leq x \leq 1, \tag{2.23}$$

in the ranges $3.2 \leq c \leq 3.4$ and $0.25 \leq x \leq 0.8.$

3. Show that the reason for the good correspondence seems to be that (2.23) is the simplest polynomial 1-dim map that shares with the Lorenz model a reflection symmetry.
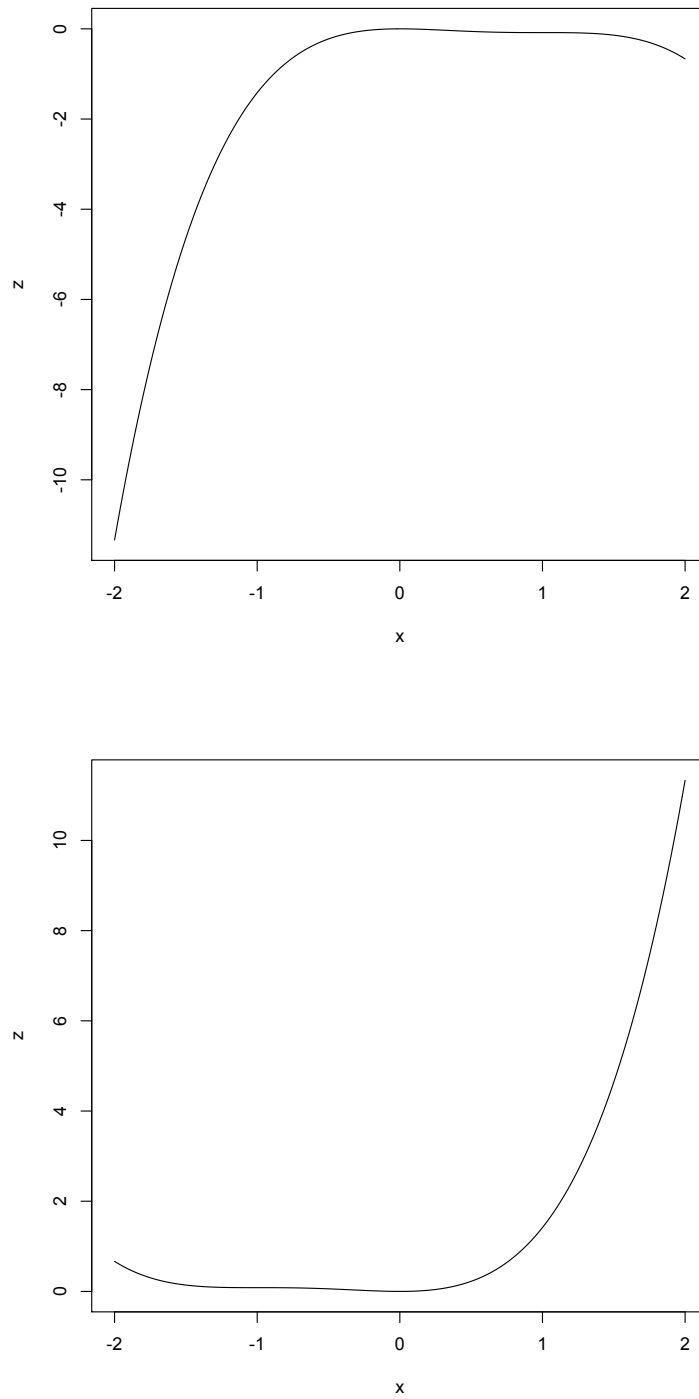
Figure 2.8: Potential $V(x) = -r/2x^2 + 2/3rx^3 - 1/4rx^4$ for $r = 1$ (upper panel) and $r = -1$ (lower panel).
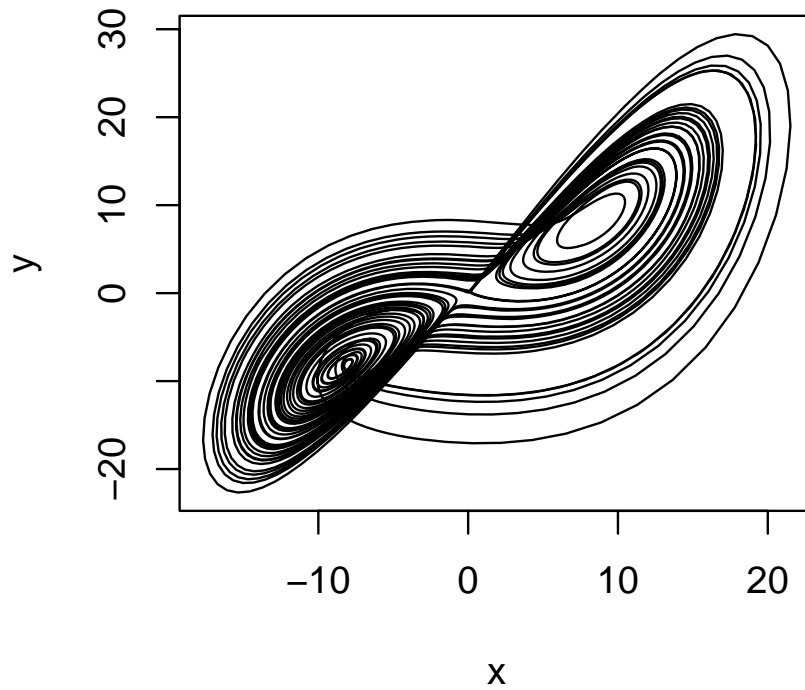
Figure 2.9: Numerical solution of the Lorenz model, in the $X-Y$ phase-space with the parameters $r = 28, \ \sigma = 10$, and $b = 8/3$. For the numerics, see Exercise 11.
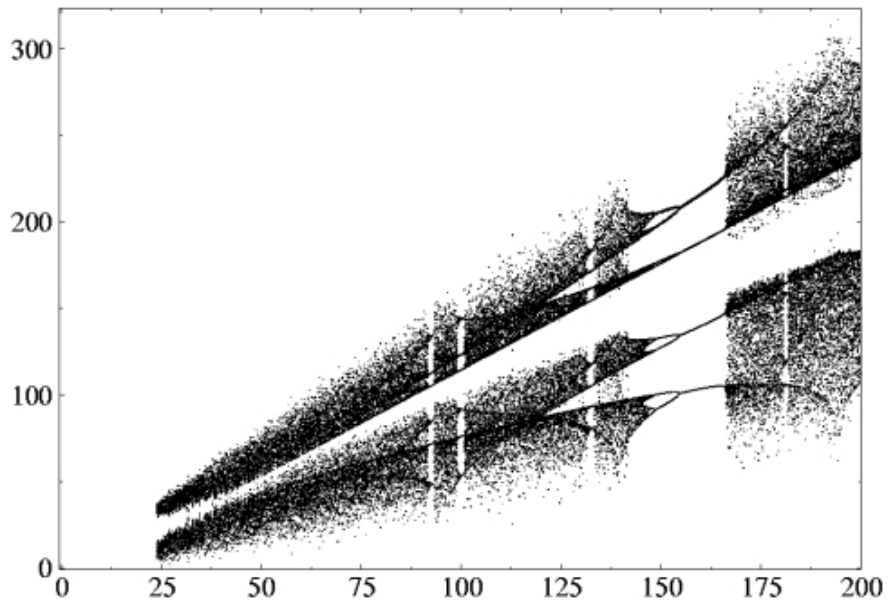
Figure 2.10: Bifurcation diagram for the Lorenz system by using r as the order parameter.

Exercise 22 – **Lorenz equations**

Consider the Lorenz equations (which were derived from the Rayleigh-Bernard system)

$$\dot{x} = \sigma(y - x) \tag{2.24}$$

$$\dot{y} = rx - xz - y \tag{2.25}$$

$$\dot{z} = xy - bz \tag{2.26}$$

with $\sigma, r, b > 0$. σ is the Prandtl number. Furthermore, Rayleigh number $R_a \sim \Delta T$, critical Rayleigh number R_c, and $r = R_a/R_c$.

1. Evaluate the equilibrium points.

2. Determine the stability of the $(0, 0, 0)$−equilibrium through linearization! Control parameter is r.

3. Show the symmetry: The Lorenz equation has the following symmetry $(x, y, z) \rightarrow (-x, -y, z)$ independent on the parameters σ, r, b.

4. Show the invariance: The z-axis is invariant, meaning that a solution that starts on the z-axis (i.e. $x = y = 0$) will remain on the z-axis. In addition the solution will tend toward the origin if the initial condition are on the z-axis.

5. Lorenz system has bounded solutions: Show that all solutions of the Lorenz equation will enter an ellipsoid centered at $(0, 0, 2r)$ in finite time, and the solution will remain inside the ellipsoid once it has entered. To observe this, define a Lyapunov function

$$V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \tag{2.27}$$

Exercise 23 – **Lorenz Problem**

1. Write the numerical solution for the Lorenz system.

2. Use an initial value $x_0 \in [0, 1]$, and a parameter value $r \in [0, 1]$

3. Investigate the sensitivity of the solution on the parameter r (especially using $r = 13, 14$

and $r \in [20, 30]$)

4. Display the function in the phase-space and time-dependence.

5. Now investigate the solution dependence on r systematically: write a function which saves
   the local extrema of a vector (fixed points) and returns them in a vector. This vector shall be
   displayed (use the experience you gained from exercise 11).

6. Nonlinear systems are often sensitive to initial conditions, and an error in the restart-file
   would lead the model to evolve on a completely different phase-space trajectory on the long
   term. Such a (seemingly trivial) technical problem was encountered by Lorenz himself (see
   e.g. Kambe [2007]), which led him to the notion of deterministic chaos in the first place.
   Please document the sensitivity with respect to the initial conditions.

Solution 1 of the Lorenz Problem

Here is the most simple way to get the Lorenz system (using R).

```
print("STRANGE ATTRACTOR: Lorenz system")
# parameters
r=24
s=10
b=8/3

dt=0.01  # time step

# initial conditions:
x=0.1
y=0.1
z=0.1

# provide the solution vector
vx<-c(0)
vy<-c(0)
vz<-c(0)

# time stepping:
for(i in 1:10000){
x1=x+s*(y-x)*dt
y1=y+(r*x-y-x*z)*dt
z1=z+(x*y-b*z)*dt
vx[i]=x1
vy[i]=y1
```

```
vz[i]=z1
x=x1
y=y1
z=z1
}
plot(vx,vy,type="l",xlab="x",ylab="y",main="LORENZ ATTRACTOR")
```

Solution 2 of the Lorenz Problem can be also used for Fig. 2.9.

```
LORENZ<-function(r,s,b,N.i=10000,dt=0.01,x0=0.1,y0=0.1,z0=0.1)
{

  vx<-vector()
  vy<-vector()
  vz<-vector()


  #Initial conditions
  vx[1]=x0
  vy[1]=y0
  vz[1]=z0

  for(i in 1:N.i)
  {
    vx[i+1]=vx[i]+s*(vy[i]-vx[i])*dt
    vy[i+1]=vy[i]+(r*vx[i]-vy[i]-vx[i]*vz[i])*dt
    vz[i+1]=vz[i]+(vx[i]*vy[i]-b*vz[i])*dt

  }

  return(list(x=vx,y=vy,z=vz))

}


# Lorenz bifurcation diagram
print("STRANGE ATTRACTORS-LORENZ SYSTEM")

result<-LORENZ(r=28,s=10,b=8/3,N.i<-5000)

#Plot the phase space
par(mfcol=c(2,2))
plot(result$x,result$y,type="l",xlab="x",ylab="y",main="LORENZ ATTRACTOR")
plot(result$x,result$z,type="l",xlab="x",ylab="z",main="LORENZ ATTRACTOR")
plot(result$y,result$z,type="l",xlab="y",ylab="z",main="LORENZ ATTRACTOR")
```

Here is the method how to obtain the bifurcation diagram. Try to understand the method and modify the code.

```
#Now calculate the bifurcation diagram


#User parameters
s<-10

#Parameter range of r
r1<-0
r2<-30
NR<-61

#Skip N.Skip iterations
N.Skip<-800

#Total number of iterations
N.i<-1000

#Vector of initial conditions
x<-(-3:2)

####################

bins<-(-50:50)
r<-r1+(0:(NR-1))*(r2-r1)/(NR-1)


m<-matrix(0,length(r),length(bins)-1)  #Matrix to save the densities


for(j in 1:length(r))  #Loop over the parameter r
{
  together<-vector()
  for (ix in 1:length(x))  #Loop over initial conditions
  {
    result<-LORENZ(r=r[j],s,b,N.i=N.i,dt=0.01,x0=x[ix],y0=0.1,z0=0.1)
    together<-c(together,result$x[-1*(1:N.Skip)])
  }

  m[j,]<-hist(together,breaks=bins,plot=F)$density
  print(j)
}


#Plot the results
filled.contour(x=r,y=bins[-1],log(m+0.001),col=rainbow(15),ylim=c(-30,30),
               main="Bifurcation diagram",xlab="r",ylab="x")
```

2.3.4 Conceptual model of the ocean circulation: Stommel's box model

The foundational paper on the analysis of the ocean circulation is by Stommel [1961] who proposes and analyzes simple "box models". This paper culminates in the analysis of the equilibrium solutions of a system in which two vessels connected to reservoirs are joined by a capillary that exchanges heat and salt (Fig. 2.11).

One reservoir is warm and salty, the other cold and fresh. The flow through the capillary is proportional to the difference in density of the two water masses, which is taken to be a linear function of temperature and salinity. Upon substituting the equation of state into the equations governing the evolution of the water masses, Stommel finds two coupled nonlinear equations. In some parameter regimes there are three steady state solutions, two of which are stable. These two stable modes have opposite directions of flow, which he interprets as a competition between temperature and salinity effects on density.
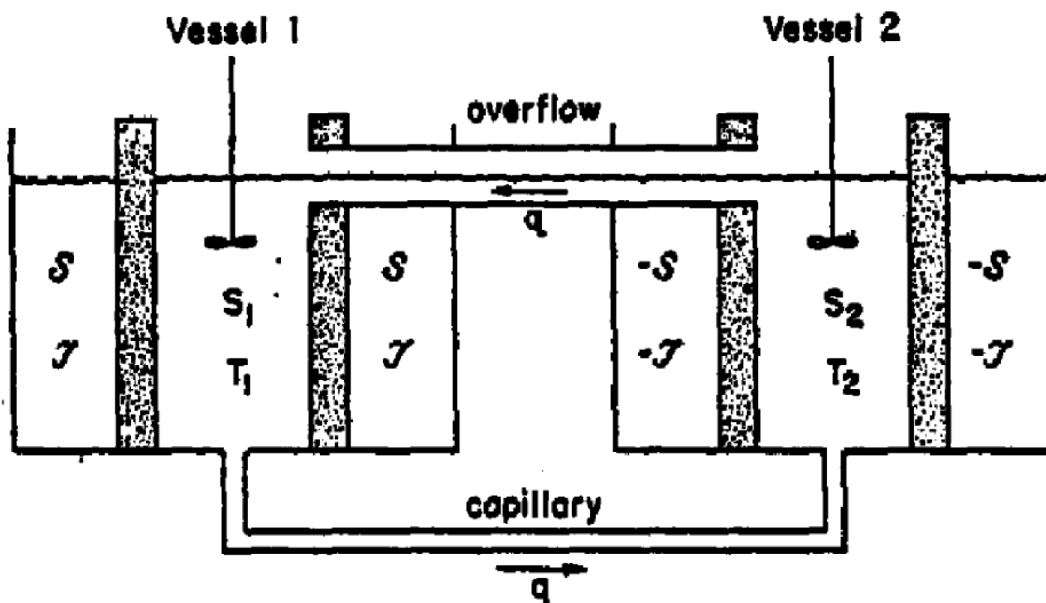
Figure 2.11: Schematic picture of the box model described by Stommel [1961].

As stated above Stommel Stommel [1961] considered a two-box ocean model where the boxes are connected by an overflow at the top and a capillary tube at the bottom (Fig. 2.11), such that

the capillary flow is directed from the high density vessel to the low density vessel following with a rate Φ. The common assumption of these box models is that the oceanic overturning rate Φ can be expressed by the meridional density difference:

$$\Phi = -c\left(\alpha \Delta T - \beta \Delta S\right) \qquad , \tag{2.28}$$

where α and β are the thermal and haline expansion coefficients, c is a tunable parameter, and Δ denotes the meridional difference operator applied to the variables temperature T and salinity S, respectively. $\Delta T = T_1 - T_2$ with T_1, T_2 are the high-latitude and the tropical boxes in Fig. 2.11. The equations for temperature T and salinity S are the heat and salt budgets using an upstream scheme for the advective transport and fluxes with the atmosphere:

$$\frac{d}{dt}T_1 = \frac{\Phi}{V}T_2 - \frac{F_1^{oa}}{\rho_0 c_p h} \tag{2.29}$$

$$\frac{d}{dt}S_1 = \frac{\Phi}{V}S_2 - \frac{S_0}{h}(P - E)_1 \qquad , \tag{2.30}$$

$$\frac{d}{dt}T_2 = \frac{\Phi}{V}T_1 - \frac{F_2^{oa}}{\rho_0 c_p h} \tag{2.31}$$

$$\frac{d}{dt}S_2 = \frac{\Phi}{V}S_1 - \frac{S_0}{h}(P - E)_2 \qquad , \tag{2.32}$$

where V is the volume of the box with depth h, and $(P - E)$ denotes the freshwater flux (precipitation minus evaporation plus runoff). Foa is the heat flux at the ocean-atmosphere interface, S_0 is a reference salinity, and $\rho_0 c_p$ denotes the heat capacity of the ocean. Subtraction leads to

$$\frac{d}{dt}\Delta T = -\frac{\Phi}{V}\Delta T - \Delta\frac{F^{oa}}{\rho_0 c_p h} \tag{2.33}$$

$$\frac{d}{dt}\Delta S = -\frac{\Phi}{V}\Delta S - \frac{S_0}{h}\Delta(P - E) \qquad . \tag{2.34}$$

The heat flux F^{oa} at the ocean-atmosphere interface can be replaced by a restoring term to the respective atmospheric temperatures, and to a first order approximation the temperatures are fixed.

We now make an approximation of (2.33, 2.34) and assume that $\Delta T, c,$ and $\Delta(P - E)$ are fixed parameters. The dynamics is then given by

$$\frac{d}{dt}\Delta S = \frac{c}{V}(\alpha\Delta T - \beta\Delta S)\,\Delta S \quad - \quad \frac{S_0}{h}\Delta(P - E) \quad . \tag{2.35}$$

The steady state solution of (2.35) for ΔS can be obtained as

$$0 \quad = \quad \frac{c}{V}\;(\alpha\Delta T - \beta\Delta S_{eq})\;\;\Delta S_{eq} \quad - \quad \frac{S_0}{h}\Delta(P - E) \quad , \tag{2.36}$$

which leads to a quadratic equation for

$$\Delta S_{eq} = \frac{\alpha\,\Delta T}{\beta}\left(\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{\beta S_0 V \Delta(P - E)}{ch(\alpha\Delta T)^2}}\right) \quad . \tag{2.37}$$

It can be shown (exercise 24) that the negative root leads to an unstable solution. Furthermore

$$ch(\alpha\Delta T)^2 > 4\,\beta S_0 V \Delta(P - E) \tag{2.38}$$

which means there exists a critical $\Delta(P - E)_{crit}$ above which the flow has no solution:

$$\Delta(P - E)_{crit} = ch\frac{(\alpha\Delta T)^2}{4\,\beta S_0 V} \quad . \tag{2.39}$$

What will hapen if $\Delta(P - E) > \Delta(P - E)_{crit}$? Stommel [1961] modified equation (6.158) to

$$\Phi = -c\,|\alpha\Delta T - \beta\Delta S| \tag{2.40}$$

Then the steady-state solutions are classified according to the sign of $q = \alpha\Delta T - \beta\Delta S$. When $q > 0$, the circulation is driven by the thermal contrast. When $q < 0$, the haline contrast is dominant in driving the current.

Exercise 24   –   Bifurcation of Stommel's model

Consider Fig. 2.11 where the ocean surface water is heated at the equatorial region and flows toward high latitudes. At the pole the water is cooled and sinks, upwelling is at the equator.

1. Starting from (2.35), calculate the linear stability of the equilibrium solution (2.37).

2. Investigate the sensitivity of the stability with respect to $(P - E)_{crit}$ and the other parameters in the model.

Solution for Bifurcation of Stommel's model

We rewrite (2.35) into

$$\frac{V}{c}\frac{d}{dt}\beta\Delta S \quad = \quad (\alpha\Delta T - \beta\Delta S)\,\beta\Delta S \quad - \quad \frac{\beta S_0 V}{ch}\Delta(P - E) \qquad . \qquad (2.41)$$

Denoting $x = \beta\Delta S, \quad a = \alpha\Delta T, \quad b = \dfrac{\beta S_0 V}{ch}\Delta(P - E),$ and a non-dimentional time $t_d = t\dfrac{c}{V},$ we have

$$\frac{d}{dt_d}x = (a - x)\cdot x - b \tag{2.42}$$

The equilibrium solutions are

$$x_{1,2} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b} \tag{2.43}$$

Therefore, (2.42) can be rewritten as

$$\frac{d}{dt_d}x = f(x) = -(x - x_1)\cdot(x - x_2) \tag{2.44}$$

The derivative is

$$f'(x) = -(x - x_1) - (x - x_2) \tag{2.45}$$

and

$$f'(x_1) \;=\; -(x_1 - x_2) = -2\sqrt{\frac{a^2}{4} - b} < 0 \qquad \textbf{stable} \tag{2.46}$$

$$f'(x_2) \;=\; -(x_2 - x_1) = +2\sqrt{\frac{a^2}{4} - b} > 0 \qquad \textbf{unstable} \tag{2.47}$$

Furthermore,

$$b \;<\; \frac{a^2}{4} \qquad \text{which means that} \tag{2.48}$$

$$\Delta(P - E) \;<\; \Delta(P - E)_{crit} = ch\frac{(\alpha\Delta T)^2}{4\,\beta S_0 V} \quad . \tag{2.49}$$

Reversed mode of the model: What happens if $\Delta(P - E) > \Delta(P - E)_{crit}$?

Then the direction of the circulation is anti-clockwise and the current is driven predominantly by haline contrast with higher density at low latitudes. The equation has to modified according to (2.40) and the equilibrium solutions are

$$x_{3,4} = \frac{a}{2} \pm \sqrt{\frac{a^2}{4} + b} \tag{2.50}$$

This solution has the requirement that $\frac{a^2}{4} + b > 0$. Let us now look on the linear stability of $x_{3,4}$

$$\frac{d}{dt_d}x = f(x) = (x - x_3) \cdot (x - x_4) \tag{2.51}$$

The derivative is

$$f'(x) = (x - x_3) + (x - x_4) \tag{2.52}$$

and

$$f'(x_3) = (x_3 - x_4) = +2\sqrt{\frac{a^2}{4} + b} > 0 \qquad \textbf{unstable} \tag{2.53}$$

$$f'(x_4) = (x_4 - x_3) = -2\sqrt{\frac{a^2}{4} + b} < 0 \qquad \textbf{stable} \tag{2.54}$$

This means tha tthere exists two stable equilibria (Fig. 2.12) for

$$-\frac{a^2}{4} < b = 2\frac{\beta S_0 V}{ch}(P - E) < \frac{a^2}{4} = \frac{(\alpha \Delta T)^2}{4} \qquad . \tag{2.55}$$
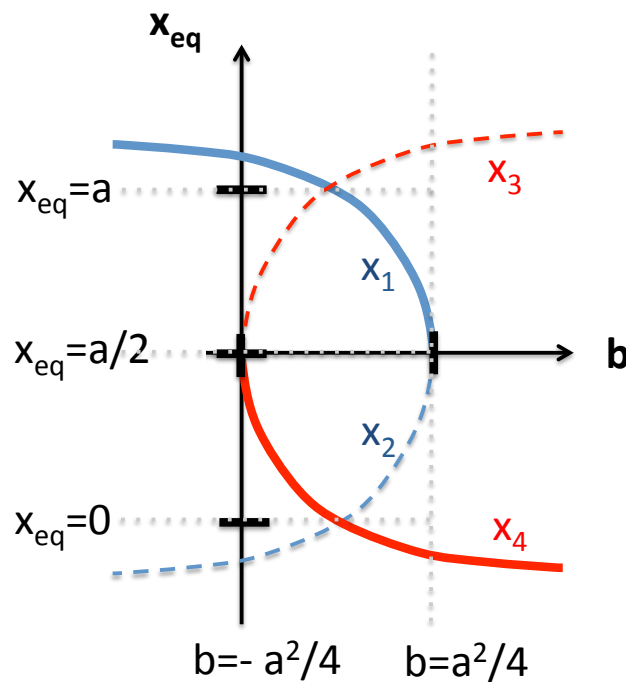
Figure 2.12: Schematic bifurcation of the Stommel box model. Dashed solutions are unstable, the solid red and blue lines represent the stable solutions, x_1 and x_4, respectively.

2.3.5   Non-normal dynamics of the ocean box model*

In this section, a category of the non-linear models following the simple thermohaline model of Stommel Stommel [1961] is analyzed. We start with (2.33, 2.34). Denoting furthermore $x \in \mathbf{R}^2$ for the anomalies of $(\Delta T, \Delta S)$, Lohmann and Schneider ? have shown that the evolution equation is of the following structure:

$$\frac{d}{dt}x = Ax + \langle b|x \rangle \; x. \tag{2.56}$$

The brackets $\langle \; | \; \rangle$ denote the Euclidean scalar product. This evolution equation (2.56) can be transferred to a

$$x(t) = \frac{1}{\gamma(t)} \; \exp(At) \, x_0 \;, \tag{2.57}$$

with a scaling function $\gamma(t, x_0)$. The models of Stommel ?, and many others are of this type, and their dynamics are therefore exactly known.

It is worth knowing that (2.33, 2.34) is equivalent to the multi-dimensional Malthus-Verhulst model (also known as logistic equation), which was originally proposed to describe the evolution of a biological population. Let x denote the number (or density) of individuals of a certain population. This number will change due to growth, death, and competition. In the simplest version, birth and death rates are assumed proportional to n,, but accounting for limited resources and competition it is modified by $(1 - x)$:

$$\frac{d}{dt}x(t) = a(1 - x) \, x \tag{2.58}$$

In climate, the logistic equation is important for Lorenz's Lorenz [1982] error growth model: where $x(t)$ is the algebraic forecast error at time t and a is the linear growth rate.

It is useful to analyze the dynamics in the phase space spanned by temperature and salinity anomalies and investigate the model sensitivity under anomalous high latitude forcing, induced

as an initial perturbation. The lines in Fig. 2.13 are phase space trajectories after perturbations of different magnitude have been injected into the North Atlantic. We notice that for most trajectories, the distances from zero $(0, 0)$ increase temporarily, where the maximal distance from zero is after a decade. After about 10 years the trajectories in Fig. 2.13 point into a "mixed temperature/salinity direction", denoted further as e_1.

Fig. 2.13 implies that the adjustment of the THC involves two phases: A fast thermal response and a slower response on the e_1-direction. The vector e_1 is identical with the most unstable mode in the system. Because the scaling function $\gamma(t)$ acts upon both temperature and salinity (2.57), the evolution of the non-linear model can be well characterized by the eigenvectors of the matrix A, which is discussed in the following.

In our system, the operator A of the box model is found to be non-normal, and the eigenvectors are not orthogonal. One eigenvalue (e_2) is closely related to temperature anomalies, whereas the other (e_1) is a "mixed temperature/salinity eigenvector" (Fig. 2.14). The eigenvectors of the adjoint matrix A^+ are denoted by e_1^* and e_2^*, respectively. For the non-normal matrix A, the eigenvectors of A and A^+ do not coincide, but fulfill the "biorthogonality condition":

$$e_1^* \perp e_2 \text{ and } e_2^* \perp e_1 . \tag{2.59}$$

Both eigenvalues $\lambda_{1,2}$ are real and negative. Because of $\lambda_2 < \lambda_1$, the first term dominates for long time scales and the second for short time scales. Using the biorthogonality condition, we get furthermore the coefficients

$$c_i = \frac{\langle e_i^* | x_0 \rangle}{\langle e_i^* | e_i \rangle} \qquad \text{for} \quad i = 1, 2 \tag{2.60}$$

A perturbation is called "optimal", if the initial error vector has minimal projection onto the subspace with the fastest decaying perturbations, or equivalently if the coefficient c_1 is maximal. This is according to (2.60) equivalent to x_0 pointing into the direction of e_1^*. This unit vector e_1^* is called the "biorthogonal" **?** to the most unstable eigenvector e_1 which we want to excite.
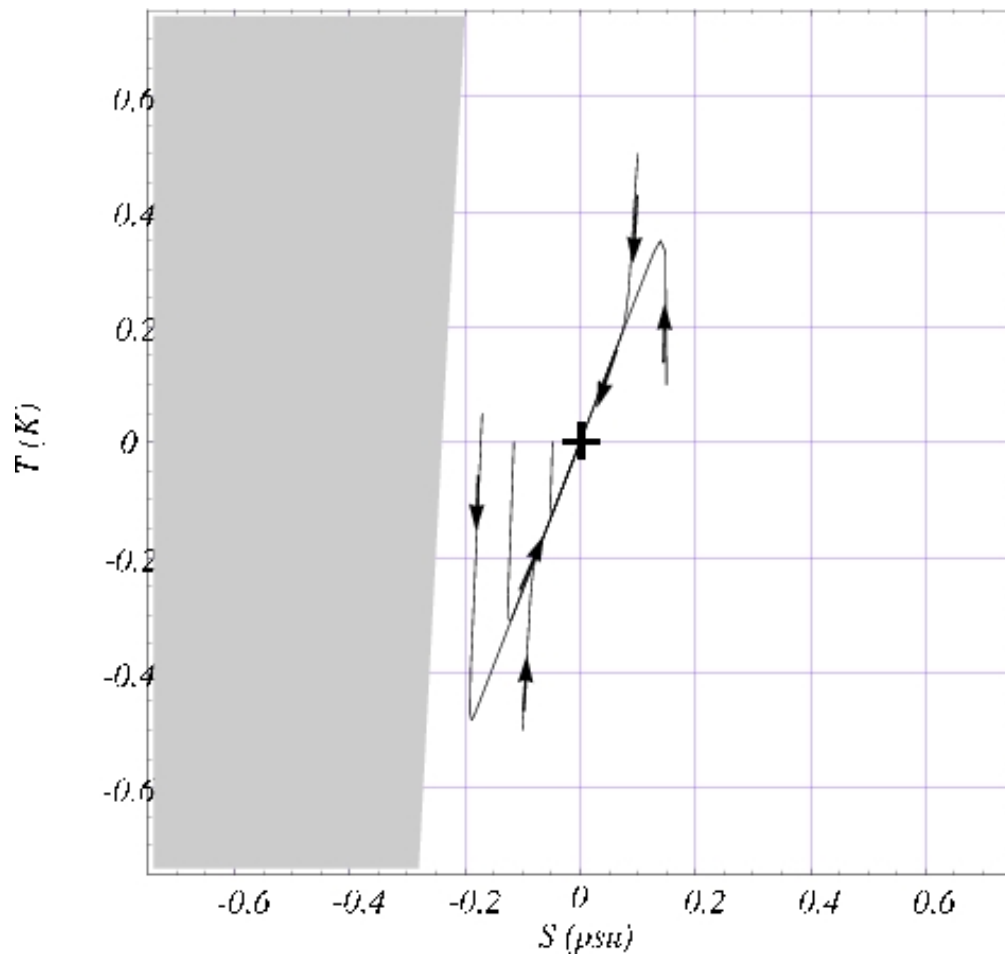
Figure 2.13: The basin of attraction (white area) and the dynamics in the thermohaline phase space. With initial conditions outside the gray area, the trajectories converge asymptotically to the origin corresponding to the thermally driven solution of the THC. Due to the fast thermal response during the first decade of relaxation, the distance of the trajectories from zero can increase temporarily.
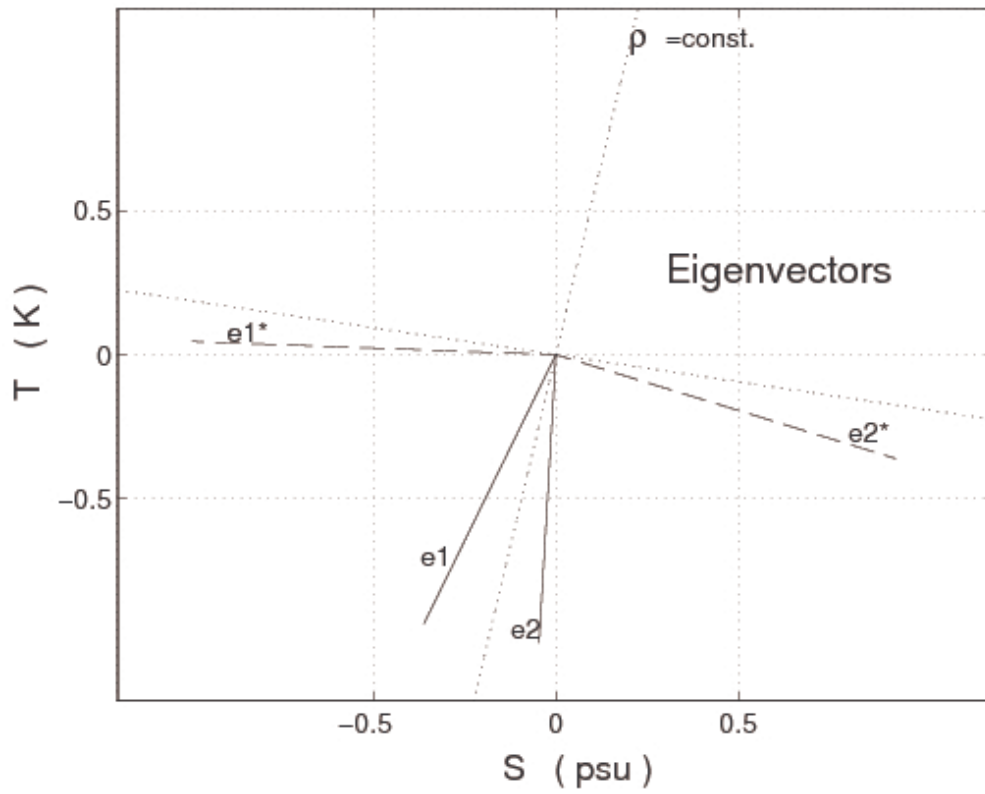
Figure 2.14: Eigenvectors e_1, e_2, and adjoint eigenvectors e_1^*, e_2^* of the tangent linear operator A^+. The dotted lines show the line of constant density and the perpendicular.

In order to make a geometrical picture for the mathematical considerations, assume that the tail of the vector x_0 is placed on the e_1−line and its tip on the e_2−line. This vector is stretched maximally because the tail decays to zero quickly, whereas the tip is hardly unchanged due to the larger eigenvalue λ_1. The most unstable mode e_1 and its biorthogonal e_1^* differ greatly from each other, and the perturbation that optimally excites the mode bears little resemblance to the mode itself.

It is remarkable that the optimal initial perturbation vector e_1^* does not coincide with a perturbation in sea surface density at high latitudes, which would reside on the dotted line perpendicular to $\rho = const.$ in Fig. 2.14. Even when using a space spanned by $(\alpha T, \beta S)$ instead of (T, S), to take into account the different values for the thermal and haline expansion coefficients, vector e_1^* is much more dominated by the scaled salinity anomalies than the temperature anomalies of the high latitudinal box.

Numerical simulations by Manabe and Stouffer Manabe and Stouffer [1993] showed, for the North Atlantic, that between two and four times the preindustrial CO_2 concentration, a threshold value is passed and the thermohaline circulation ceases completely. One other example of early Holocene rapid climate change is the '8200 yr BP' cooling event recorded in the North Atlantic region possibly induced by freshwater. One possible explanation for this dramatic regional cooling is a shutdown in the formation of deep water in the northern North Atlantic due to freshwater input caused by catastrophic drainage of Laurentide lakes Barber et al. [1999]; Lohmann [2003]. The theoretic considerations and these numerical experiments suggest that the formation of deep water in the North Atlantic is highly sensitive to the freshwater forcing.

Chapter 3

Statistical Mechanics and Fluid Dynamics*

There are two ways of changing the descrition of the dynamics: from the micro to the macro scales. This is a common problem since we are not able to describe the systems on all temporal and spatial scales. One straitforward approach is coarse graining where the underlying dynamics is projected onto the macroscopic dynamics (section 3.5), the other is the statistical physics theory of non-equilibrium statistical mechanics (section 3.2).

A general question within the micro-macro dynamic is that of integration between the two different levels. Two distinctly different levels emerge with different rules governing each, but they then need to be reconciled in some way to create an overall functioning system. Physical, chemical, biological, economic, social and cultural systems all exhibit this micro-macro dynamic and how the system comes to reconcile it forms a primary determinate in its identity and overall structure. This multi-dimensional nature to a system that results in the micro-macro dynamic is a product of synthesis and emergence. In many instances when we put elements together they do not simply remain discrete separate entities but they interact, co-evolve and they differentiate their states and function with respect to each other to become an interdependent whole, which comes to have properties and features that none of its parts possess. A whole new level of organization emerges that is different from the parts. This is made manifest in ecosystems; as they have co-evolved over millennia the parts are intricately interdependent forming a whole system that has

features and dynamics independent from any of its parts and thus a two-tier system and a resulting emergent micro-macro dynamic. The whole ecosystem goes through processes of change - such as ecological succession - that are not associated with any of the parts but condition what creatures can viably exist within that macro regime.

We start from the point of view of kinetic theory of fluids where a gas is composed of a set of interacting particles Boltzmann [1896]. We are then interested in the probability of finding a fluid particle at a certain point in space and with a certain velocity. The moments of this probablility are related to our macroscopic fluid-dynamical quantities like density or velocity.

3.1 Mesoscopic dynamics

3.1.1 Liouville equation

In the deterministic framework, the dynamics is characterized by

$$\frac{d}{dt}x(t) = f(x(t)) \quad , \tag{3.1}$$

and in the special case of classical mechanics can be descibed by a set of differential equations known as the Hamilton equations for that system. Hamiltonians can be used to describe such simple systems as a bouncing ball, a pendulum or an oscillating spring in which energy changes from kinetic to potential and back again over time. Hamiltonians can also be employed to model the energy of other more complex dynamic systems such as planetary orbits in celestial mechanics and also in quantum mechanics. The Hamilton equations are generally written as follows:

$$\dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \tag{3.2}$$

$$\dot{q} = \frac{\partial \mathcal{H}}{\partial p} \tag{3.3}$$

In the above equations, the dot denotes the ordinary derivative with respect to time of the functions $p = p(t)$ (called generalized momenta) and $q = q(t)$ (called generalized coordinates), taking values in some vector space, and $\mathcal{H} = \mathcal{H}(p, q, t)$ is the so-called Hamiltonian, or (scalar valued) Hamiltonian function. The associated probability distribution for the generalized dynamics (3.1) is given in the phase space

$$p(x, t) = \delta(x - x(t)) \tag{3.4}$$

yielding the Liouville equation

$$\partial_t p = -\frac{d}{dx(t)} \left[\delta(x - x(t)) \right] \frac{d}{dt} x(t) = -\frac{\partial p}{\partial x} f(x) \quad . \tag{3.5}$$

The Liouville equation is often used in the framework of the Hamiltonian dynamics (3.3). Since the phase space velocity $(\dot{p}_i, \dot{q}_i)$ has zero divergence, and probability is conserved. Its substantial derivative can be shown to be zero and so

$$\frac{\partial}{\partial t} \rho = -\{\rho, \mathcal{H}\}. \tag{3.6}$$

using the Poisson bracket

$$\{f, g\} = \sum_{i=1}^{N} \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]. \tag{3.7}$$

3.1.2 Master equation

The master equation is a phenomenological set of first-order differential equations describing the time evolution of the probability of a system to occupy each one of a discrete set of states:

$$\frac{dP_k}{dt} = \sum_{\ell} T_{k\ell} P_\ell, \tag{3.8}$$

where P_k is the probability for the system to be in the state k, while the matrix $T_{\ell k}$ is filled with a grid of transition-rate constants. In probability theory, this identifies the evolution as a continuous-time Markov process, with the integrated master equation obeying a Chapman-Kolmogorov equation. Note that

$$\sum_{\ell} T_{\ell k} = 0 \tag{3.9}$$

(i.e. probability is conserved), so the equation may also be written as

$$\frac{dP_k}{dt} = \sum_{\ell} (T_{k\ell} P_\ell - T_{\ell k} P_k). \tag{3.10}$$

allowing us to omit the term $\ell = k$ from the summation. Thus, in the latter form of the master equation there is no need to define the diagonal elements of T.

The master equation exhibits detailed balance if each of the terms of the summation disappears separately at equilibrium, i.e. if, for all states k and l having equilibrium probabilities pi_k and π_ℓ

$$T_{k\ell} \pi_\ell = T_{\ell k} \pi_k \tag{3.11}$$

Many physical problems in classical, quantum mechanics and problems in other sciences, can be reduced to the form of a master equation, thereby performing a great simplification of the problem. In the continous case, the Chapman-Kolmogorov equation has similarities with the Master equation. The Chapman-Kolmogorov equation is an identity relating the joint probability distributions of different sets of coordinates on a stochastic process. Suppose that $\{x_i\}$ is an indexed collection of random variables, that is, a stochastic process. Let

$$p_{i_1,\ldots,i_n}(x_1, \ldots, x_n) \tag{3.12}$$

be the joint probability density function of the values of the random variables $x1$ to x_n. Then, the

Chapman-Kolmogorov equation is

$$p_{i_1,\dots,i_{n-1}}(x_1,\dots,x_{n-1}) = \int_{-\infty}^{\infty} p_{i_1,\dots,i_n}(x_1,\dots,x_n)\, dx_n \tag{3.13}$$

i.e. a straightforward marginalization over the nuisance variable.

When the stochastic process under consideration is Markovian, the Chapman-Kolmogorov equation is equivalent to an identity on transition densities. In the Markov chain setting, one assumes that $i_1 < \dots < i_n$. Then, because of the Markov property,

$$p_{i_1,\dots,i_n}(x_1,\dots,x_n) = p_{i_1}(x_1)p_{i_2;i_1}(x_2 \mid x_1) \cdots p_{i_n;i_{n-1}}(x_n \mid x_{n-1}), \tag{3.14}$$

where the conditional probability $p_{i;j}(x_i \mid x_j)$ is the transition probability between the times $i > j$. So, the Chapman-Kolmogorov equation takes the form

$$p_{i_3;i_1}(x_3 \mid x_1) = \int_{-\infty}^{\infty} p_{i_3;i_2}(x_3 \mid x_2)p_{i_2;i_1}(x_2 \mid f_1)\, dx_2. \tag{3.15}$$

When the probability distribution on the state space of a Markov chain is discrete and the Markov chain is homogeneous, the Chapman-Kolmogorov equations can be expressed in terms of (possibly infinite-dimensional) matrix multiplication, thus:

$$P(t+s) = P(t)P(s) \tag{3.16}$$

where P(t) is the transition matrix, i.e., if Xt is the state of the process at time t, then for any two points i and j in the state space, we have

$$P_{ij}(t) = P(X_t = j \mid X_0 = i). \tag{3.17}$$

Example for the Chapman-Kolmogorov and Master equations in climate dynamics are related to transitions between different states.

3.1.3 Fokker-Planck dynamics

In the stochastic context, we make a Taylor expansion up to order two in $dx = x(t+dt) - x(t)$ from the Master equation:

$$
\begin{aligned}
dp &= p(x, t+dt) - p(x, t) \\
&= \; <\delta(x - x(t+dt))> - <\delta(x - x(t))> \\
&= \; - <\frac{d}{dx(t)}\,[\delta(x - x(t))]\,dx> + \frac{1}{2} <\frac{d^2}{dx^2}\,[\delta(x - x(t))]\,dx^2> \\
&= \; -\frac{\partial p}{\partial x}<dx> + \frac{1}{2}\frac{\partial^2}{\partial x^2}p <dx^2> \\
&= \; -\frac{\partial p}{\partial x}f(x)dt + \frac{1}{2}\frac{\partial^2}{\partial x^2}p\,g^2 dt
\end{aligned}
\tag{3.18}
$$

The probability density $p(x, t)$ for the variable $x(t)$ in (7.53) obeys therefore the Fokker-Planck equation

$$
\partial_t p = -\frac{\partial}{\partial x}\,[f(x)p] + \frac{\partial}{\partial x}\left[g(x)\frac{\partial}{\partial x}\,\{g(x)p\}\right] \quad .
\tag{3.19}
$$

Its stationary probability density of (7.53) is given by

$$
p_{st}(x) = \; \aleph \exp\left(-2\int_{x_0}^{x} \frac{f(y) - g(y)g'(y)}{g(y)^2}\,dy\right) \quad .
\tag{3.20}
$$

where $\aleph$ is a normalization constant. $g'(y)$ stands for the derivative of g with respect to its argument. The extrema x_m of the steady state density obey the equation

$$
f(x_m) - g(x_m)g'(x_m) = 0
\tag{3.21}
$$

for $g(x_m)n_e 0$. Here is the crux of the noise-induced transition phenomenon: one notes that this equation is not the same as the equation $f(x_m) = 0$ that determines the steady states of the system in the absence of multiplicative noise. As a result, the most probable states of the noisy system

need not to coincide with the deterministic stationary states. More importantly, new solutions may appear or existing solutions may be destabilized by the noise. These are the changes in the asymptotic behavior of the system caused by the presence of the noise, e.g. **?**.

3.2   The Boltzmann Equation

One of the most significant theoretical breakthroughs in statistical physics was due to Ludwig Boltzmann (Boltzmann [1896], Boltzmann [1995] for a recent reprint of his famous lectures on kinetic theory), who pioneered non-equilibrium statistical mechanics. Boltzmann postulated that a gas was composed of a set of interacting particles, whose dynamics could be (at least in principle) modelled by classical dynamics. Due to the very large number of particles in such a system, a statistical approach was adopted, based on simplified physics composed of particle streaming in space and billiard-like inter-particle collisions (which are assumed elastic). Instrumental to the theory is the single-particle distribution function (hereafter SPDF), $f(\vec{x}, \vec{e}, t)$ which represents the probability density of having a particle at the point $(\vec{x}, \vec{e})$ in the phase space. Hence, the quantity

$$f(\vec{x}, \vec{e}, t)d\vec{x}d\vec{e} \tag{3.22}$$

represents the probability of finding a particle inside an infinitesimal space cubelet centered around $\vec{x}$, and inside an infinitesimal momentum-space cubelet around $\vec{e}$ at any given time t. In the presence of a body-force $\vec{F}$, the SPDF will evolve according to

$$f(\vec{x} + d\vec{x}, \vec{e} + d\vec{e}, t + dt)d\vec{x}d\vec{e} = f(\vec{x}, \vec{e}, t)d\vec{x}d\vec{e} \,, \tag{3.23}$$

where $d\vec{x} = \vec{e}dt$ and $d\vec{e} = \vec{F}dt/m$. If we also include the effect of the collisions, and denote by $\Gamma_+ d\vec{x}d\vec{e}dt$ the probability for a particle to start from outside the $d\vec{x} \times d\vec{e}$ domain and to enter this phase-space region during the infinitesimal time dt and by $\Gamma_- d\vec{x}d\vec{e}dt$ the probability for a particle to start from the $d\vec{x} \times d\vec{e}$ domain and leave this phase-space region during the infinitesimal

time dt, the evolution of the SPDF becomes

$$f(\vec{x}+d\vec{x},\vec{e}+d\vec{e},t+dt)d\vec{x}d\vec{e} = f(\vec{x},\vec{e},t)d\vec{x}d\vec{e} + (\Gamma_+ - \Gamma_-)\,d\vec{x}d\vec{e}dt \qquad (3.24)$$

Expanding the LHS into a Taylor series around the phase-space point $(\vec{x},\vec{u},t)$, we obtain:

$$f(\vec{x}+d\vec{x},\vec{e}+d\vec{e},t+dt)d\vec{x}d\vec{e} = f(\vec{x},\vec{e},t)d\vec{x}d\vec{e} + \left(\frac{\partial f}{\partial t}\right)dt + (\nabla_{\vec{x}}f)\cdot d\vec{x} + (\nabla_{\vec{e}}f)\cdot d\vec{e} + \dots$$
$$(3.25)$$

Inserting Eq. (3.25) into Eq. (3.24) and cancelling terms, we easily obtain Boltzmann's Equation:

$$\frac{\partial f}{\partial t} + \vec{e}\cdot\nabla_{\vec{x}}f + \vec{F}/m\cdot\nabla_{\vec{e}}f = \Gamma_+ - \Gamma_- \qquad (3.26)$$

where $\nabla_{\vec{x}}$ is the gradient operator in physical space and $\nabla_{\vec{e}}$ the same in momentum space.[1]

For the sake of clarity, we have not written the collision operator explicitly yet. The important point is that the separation of the dynamics into collisions and streaming is already apparent from Eq. (3.26). The collision operator, which is in itself a complex integro-differential expression, reads

$$\Gamma_+ - \Gamma_- \equiv \int d\vec{e}_1 \int d\Omega\, \sigma(\Omega)\,|\vec{e}-\vec{e}_1|\,\left[f(\vec{e}')f(\vec{e}_1') - f(\vec{e})f(\vec{e}_1)\right] \qquad (3.27)$$

where σ is the differential cross-section in the case of the 2-particle collisions (which is a function of the solid angle Ω only), unprimed velocities are incoming (before collision) and primed

---

[1]The collisionless Boltzmann equation is often mistakenly called the Liouville equation (the Liouville Equation is an N-particle equation being N the number of microscopic particles). The Boltzmann equation is a mesoscopic dynamics with degrees of freedom $<< N$.

velocities are outgoing (after collision).[2] In another notation

$$\Gamma_+(\vec{x}, \vec{e}, t) \;=\; \int d\vec{e}_1 \int d\vec{e}'' \int d\vec{e}_1' \quad P_{(e', e_1') \to (e, e_1)} \quad f(\vec{e}'')f(\vec{e}_1'') \qquad (3.28)$$

$$\Gamma_-(\vec{x}, \vec{e}, t) \;=\; \int d\vec{e}_1 \int d\vec{e}'' \int d\vec{e}_1' \quad P_{(e, e_1) \to (e', e_1')} \quad f(\vec{e})f(\vec{e}_1) \qquad (3.29)$$

where $P_{(e', e_1') \to (e, e_1)}$ is the probability density to go from initial state (e', e_1') to final state (e, e_1) in time dt. It follows from symmetry considerations that $P_{(e', e_1') \to (e, e_1)} = P_{(e, e_1) \to (e', e_1')}$ and

$$d\vec{e}' d\vec{e}_1' P_{(e', e_1') \to (e, e_1)} = d\Omega \, \sigma(\Omega) \, |\vec{e} - \vec{e}_1| \qquad (3.30)$$

A fundamental property of the collision operator [Cercignani, 1987] is that it conserves mass, momentum and kinetic energy (hence also a linear combination thereof). Also, it can be shown that the local Maxwell-Boltzmann distribution pertains to a certain class of positive SPDFs for which the collision integral vanishes (variational principle, Lagrange parameters). It can be shown that this equilibrium distribution is given by

$$f_0(\vec{x}, \vec{e}) = \rho(\vec{x}) \left[\frac{m}{2\pi k T(\vec{x})} \right]^{3/2} \exp\{-m \left[\vec{e} - \vec{u}(\vec{x})\right]^2 / 2kT(\vec{x})\} \qquad (3.31)$$

where $\rho(\vec{x}), \vec{u}(\vec{x})$ and $T(\vec{x})$ are the local density, macroscopic velocity, and temperature, respectively.[3] If there are no external forces such as gravity or electrostatic interactions we have $\rho(\vec{x}) = \rho_0 = N/V$. In case the temperature is also independent of position, and if the gas as a

---

[2]Of course, finding or modeling the collision term is the biggest challenge in the kinetic theory. In the simplest model one only takes into account binary collisions and assumes that the colliding particles are uncorrelated (i.e. molecular chaos assumption). The collisions are proportianal to the velocity difference between the particles $|\vec{u} - \vec{u}_1|$. Consider an elastic collision of two spherically symmetric (spin-less) molecules with mass m and velocities $\vec{e}$ and $\vec{e}_1$. After collision their respective velocities are $\vec{e}''$ and $\vec{e}_1''$. Then the following conservation laws apply:
Momentum conservation: $m(\vec{e} + \vec{e}_1) = m(\vec{e}'' + \vec{e}_1'')$.
Energy conservation: $m/2 \;\; \vec{e} \cdot \vec{e} + m_1/2 \;\; \vec{e}_1 \cdot \vec{e}_1 = m/2 \;\; \vec{e}'' \cdot \vec{e}'' + m_1/2 \;\; \vec{e}_1'' \cdot \vec{e}_1''$.
[3]This experssion of the SPDF can be approximated through a Taylor series of the exponential: $\exp(y) = 1 + y$.
Task: Show that

$$f_a^{eq}(\vec{x}, \vec{e}) = \rho(\vec{x}) \left[1 + 3\frac{\vec{u} \cdot \vec{e}}{c_s^2} + \frac{9}{2}\frac{(\vec{u} \cdot \vec{e})^2}{c_s^4} - \frac{3}{2}\frac{\vec{e}^2}{c_s^2} \right], \qquad (3.32)$$

whole is not moving ($\vec{u} = 0$), then $f(\vec{x}, \vec{e}) = \rho_0 f_0(\vec{e})$, with

$$f_0(\vec{e}) = \left[\frac{m}{2\pi kT}\right]^{3/2} e^{-m\vec{e}^2/2kT},$$

This implies that, if this distribution is attained, we also have a state where incoming SPDFs exactly balance the outgoing ones, maintaining a local dynamic equilibrium. This observation is of paramount importance for our method, which uses the (discretized) Maxwell-Boltzmann distribution as the equilibrium distribution functions.

3.3   H-Theorem and approximation of the Boltzmann equation

The other important feature of this equation is that the integral

$$H = \int \int d\vec{x} d\vec{e} \; f(\vec{x}, \vec{e}, t) \ln f(\vec{x}, \vec{e}, t) \tag{3.34}$$

can only decrease. This can be seen by using the following:

$$\frac{dH}{dt} = \int d\vec{e}_1 \int d\Omega \, \sigma(\Omega) \, |\vec{e} - \vec{e}_1| \; \left[f(\vec{e}')f(\vec{e}_1') - f(\vec{e})f(\vec{e}_1)\right] \left[1 + \ln f(\vec{e}_1)\right] \tag{3.35}$$

and the same term for

$$\frac{dH}{dt} = \int d\vec{e}_1 \int d\Omega \, \sigma(\Omega) \, |\vec{e} - \vec{e}_1| \; \left[f(\vec{e}')f(\vec{e}_1') - f(\vec{e})f(\vec{e}_1)\right] \left[1 + \ln f(\vec{e}_1')\right] \tag{3.36}$$

---

with the speed of sound c_s and

$$\frac{1}{c_s^2} = \frac{1}{\gamma}\frac{m}{kT} \tag{3.33}$$

and γ the adiabatic factor.

The term is also invariant with respect to the notation ('), i.e.

$$\frac{dH}{dt} = \int d\vec{e}_1 \int d\Omega \, \sigma(\Omega) \, |\vec{e} - \vec{e}_1| \, \left[f(\vec{e}) f(\vec{e}_1) - f(\vec{e}') f(\vec{e}_1') \right] \left[1 + \ln f(\vec{e}_1') \right] \quad (3.37)$$

and

$$\frac{dH}{dt} = \int d\vec{e}_1 \int d\Omega \, \sigma(\Omega) \, |\vec{e} - \vec{e}_1| \, \left[f(\vec{e}) f(\vec{e}_1) - f(\vec{e}') f(\vec{e}_1') \right] \left[1 + \ln f(\vec{e}_2') \right] \quad (3.38)$$

Use furthermore

$$\eta' = f(\vec{e}') f(\vec{e}_1') \quad \text{and} \quad \eta = f(\vec{e}) f(\vec{e}_1) \qquad (3.39)$$

$$E = (\eta' - \eta) \left[\ln \eta - \ln \eta' \right] \qquad (3.40)$$

and recognize that E is negative. $\frac{d}{dt} H$ is equal zero for

$$f(\vec{e}') f(\vec{e}_1') = f(\vec{e}) f(\vec{e}_1) \quad . \qquad (3.41)$$

For a system of N statistically independent particles, H is related to the thermodynamic entropy S through:

$$S \stackrel{\text{def}}{=} -NkH \qquad (3.42)$$

Therefore, according to the H-theorem, S can only increase.[4] The same function H is also used as "information function":

$$I = -\sum_i f_i \ln f_i = < -\ln f > . \qquad (3.43)$$

where the f_i can be interpreted as probablity and not only as a measure of the breadth of the spread of states available to a single particle in a gas of like particles, where f_i represented the relative

---

[4]Please see the link to the Lyapunov function for the Lorenz system in Chapter 5.3.

frequency distribution of each possible state. When all the probabilities f_i are equal, I is maximal, and we have minimal information about the system. When our information is maximal (i.e., one f_i is equal to one and the rest to zero, such that we know what state the system is in), the function is minimal. This information function is also called "reduced entropic function" in thermodynamics [Shannon, 1948]. Gibbs proposed a general formula for statistical-mechanical entropy, no longer requiring identical and non-interacting particles, but instead based on the probability distribution p_i for the complete microstate i of the total system:

$$S \;=\; -k \sum_i p_i \ln p_i \tag{3.44}$$

$$\frac{dS}{dt} \;=\; -k \sum_i \left(\frac{dp_i}{dt} \ln p_i + \frac{dp_i}{dt} \right) = -k \sum_i \frac{dp_i}{dt} \ln p_i \tag{3.45}$$

because $\sum_i \frac{dp_i}{dt} = \frac{d}{dt} \sum_i p_i = \frac{d}{dt}(1) = 0$. Now, formulate a master equation [van Kampen, 1981] for the average rate of jumps[5] from state α to β, and from state β to α:

$$\frac{dp_\alpha}{dt} \;=\; \sum_\beta \nu_{\alpha\beta}(p_\beta - p_\alpha) \tag{3.46}$$

$$\frac{dp_\beta}{dt} \;=\; \sum_\alpha \nu_{\alpha\beta}(p_\alpha - p_\beta) \tag{3.47}$$

where the reversibility of the dynamics ensures that the same transition constant $\nu_{\alpha\beta}$ appears in both expressions. So

$$\frac{dS}{dt} = \frac{1}{2}k \sum_{\alpha,\beta} \nu_{\alpha\beta}(\ln p_\beta - \ln p_\alpha)(p_\beta - p_\alpha). \tag{3.48}$$

---

[5]The master equation is quite often written as: $\frac{d\vec{P}}{dt} = \mathbf{A}\vec{P}$, where $\vec{P}$ is a column vector (where element i represents state i), and $\mathbf{A}$ is the matrix of connections. The way connections among states are made determines the dimension of the problem. When the connections are time-independent rate constants, the master equation represents a kinetic scheme and the process is Markovian (any jumping time probability density function for state i is an exponential, with a rate equal to the value of the connection). When the connections depend on the actual time (i.e. matrix $\mathbf{A}$ depends on the time, $\mathbf{A} \rightarrow \mathbf{A}(t)$), and the process is not stationary. For an application in meteorgy, e.g. Egger [2001].

But the two brackets will have the same sign (the same argument as in equation 3.40), so each contribution to dS/dt cannot be negative and therefore, $\frac{dS}{dt} \geq 0$ for an isolated system. Due to the complex expression for the collision operator, it became clear that approximations were desirable. It was also proven (see Cercignani [1990]) that such approximations were also reasonable, since the details of the two-body interaction are not likely to influence significantly experimentally-measured quantities. Hence, approximate collision operators were proposed, all of which had to $\boxed{1}$ conserve local mass, momentum and energy and $\boxed{2}$ develop a collisional contribution in Boltzmann's equation (3.26) which tends to a local Maxwellian distribution. It was soon realized that a model developed at the middle of last century Bhatnagar et al. [1954] (also known as Bhatnagar-Gross-Krook; hereafter BGK) satisfied both of these conditions. Chapman and Enskog developed a general procedure for the approximate solution of Boltzmann's equation. For certain simple model systems such as hard spheres their method produces predictions for (or its moments) which may be tested in computer simulations. Another more modern approach to the numerical solution of the transport equation is the "Lattice Boltzmann" method in which the continuous variables and are restricted to a set of discrete values; the time change of these values is then described by a modified transport equation which lends itself to fast computation. The moments of the distribution function represent macroscopic variables density and velocity fields:

$$\rho(\vec{x}, t) \;=\; m \int d\vec{e}\, f(\vec{x}, \vec{e}, t) \tag{3.49}$$

$$\rho(\vec{x}, t)\vec{u}(\vec{x}, t) \;=\; m \int d\vec{e}\, \vec{e}\, f(\vec{x}, \vec{e}, t) \tag{3.50}$$

Note that the molecular velocities $\vec{e}$ is different from the macroscopic veleocity field $\vec{u}(\vec{x}, t)$. The basic idea was that each collision changes the SPDF by an amount which is proportional to the departure from the local Maxwellian distribution:

$$\Gamma_+ - \Gamma_- = -\,\frac{f(\vec{x}, \vec{e}, t) - f_0(\vec{x}, \vec{e})}{\tau} \tag{3.51}$$

with relaxation constant τ. In dimensionless units, τ is replaced by the dimensionless Knudsen number $Kn = l/L$ with l is the mean-free-path. It is the small parameter in the kinetics - fluid dynamics transition. If the $Kn \gg 1$ then the continuum assumption of fluid mechanics is no longer a good approximation and kinetic equations must be used.

3.4   Application: Lattice Boltzmann Dynamics

3.4.1   Lattice Boltzmann Methods

LBMs recently proved to be viable alternatives to traditional computational fluid dynamics (CFD). The latter adopts a strategy consisting of: writing the macroscopic flow equations; discretizing the macroscopic equations using finite differences, finite volumes or finite elements; solving the discretized equations on a computer. In contrast, LBM takes a different route towards the same results. The LBM approach is composed of formulating a mesoscopic model for the evolution of the PDF such that the desired macroscopic flow equations are obtained. The end result of both approaches are similar. However, the algorithms differ due to the different perspective on the physics of the flow. There are in principle an infinite set of possible mesoscopic models. However, we focus on the most common ones, which consist of a streaming and a collision process. These LBMs use a simplified collision operator Bhatnagar et al. [1954], hence they are also referred to as LBM-BGK models.

There are several possible choices for the underlying lattice. These are usually classified in the literature using the $D\alpha Q\beta$-notation, where α is an integer number denoting the space dimensionality and β is another integer indicating the number of discrete velocities (including the particle at rest) within the momentum discretization. Some restrictions have to be fulfilled (especially Galilean and rotational invariance)[6] to ensure that a particular discretization can simulate the Navier-Stokes equations. Among the lattices in common use there are the $D2Q9$ and $D3Q19$-models (see for example discussion in He and Luo [1997]). Our focus here is the $2D$ case, hence we have chosen the $D2Q9$ momentum discretization. The discrete velocity directions for the $D2Q9$ lattice are shown in Fig 3.1. The macroscopic variables are defined as functions of the

---

[6]A lattice with reduced symmetry can be (and has been) used, see d'Humieres et al. [2001], where a $D3Q13$-lattice is used. However, this approach also departs from the classical BGK-LBM dynamics.
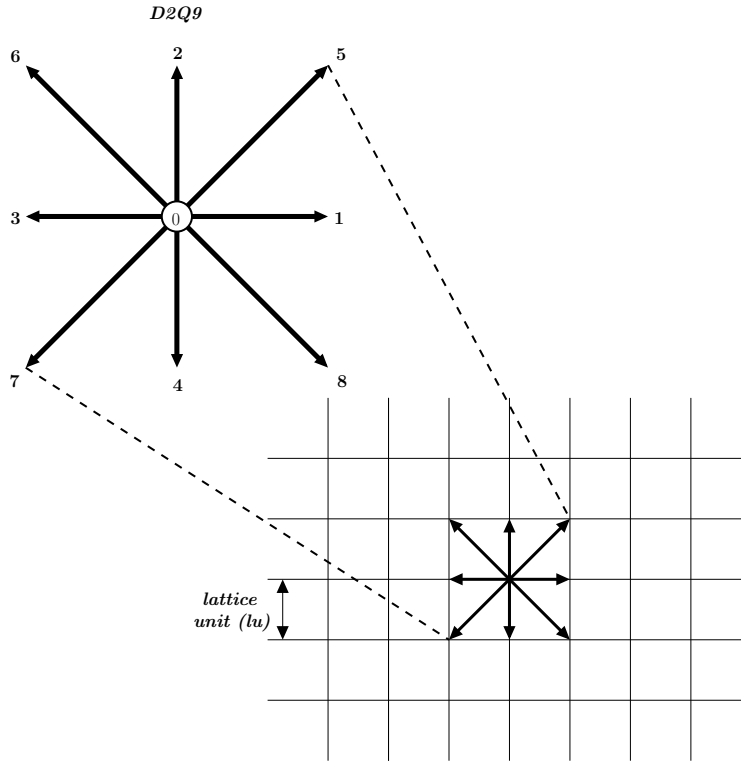
Figure 3.1: Discrete lattice velocities for the $D2Q9$ model.

particle distribution functions (hereafter DFs) according to:

$$\rho = \sum_{a=0}^{\beta-1} f_a \qquad \text{(macroscopic fluid density)} \qquad (3.52)$$

$$\text{and} \quad \vec{u} = \frac{1}{\rho} \sum_{a=0}^{\beta-1} f_a \vec{e}_a \qquad \text{(macroscopic velocity)}. \qquad (3.53)$$

The DFs at each lattice point are updated using the equation:

$$\underbrace{f_a(\vec{x} + \vec{e}_a \delta_t, t + \delta_t) = f_a(\vec{x}, t)}_{\text{Streaming}} - \underbrace{\frac{[f_a(\vec{x}, t) - f_a^{eq}(\vec{x}, t)]}{\tau}}_{\text{Collision}}, \qquad (3.54)$$

where $a \in [0, \beta - 1]$ is an index spanning the (discretized) momentum space and τ is a relaxation parameter, which is related to the fluid viscosity. The streaming step, where the DFs are translated

to the neighbouring sites according to the respective discrete velocity direction, is illustrated in Fig. 3.2, in the $D2Q9$ model. The collision step (illustrated in Fig. 3.3) consists of a re-distribution of the DFs towards the local discretized Maxwellian equilibrium DFs, in such a way that local mass and momentum are invariant. The equilibrium DFs can be obtained from the local Maxwell-

Figure 3.2: Illustration of the streaming process on a $D2Q9$ lattice. Note that the magnitude of the DFs remain unchanged, but they move to a neighbouring node according to their direction.

Figure 3.3: Illustration of the collision process on a $D2Q9$ lattice. Note that the local density ρ and velocity $\vec{v}$ are conserved, but the DFs change according to the relaxation-to-local-Maxwellian rule.

Boltzmann SPDF (see for example He and Luo [1997]); they are
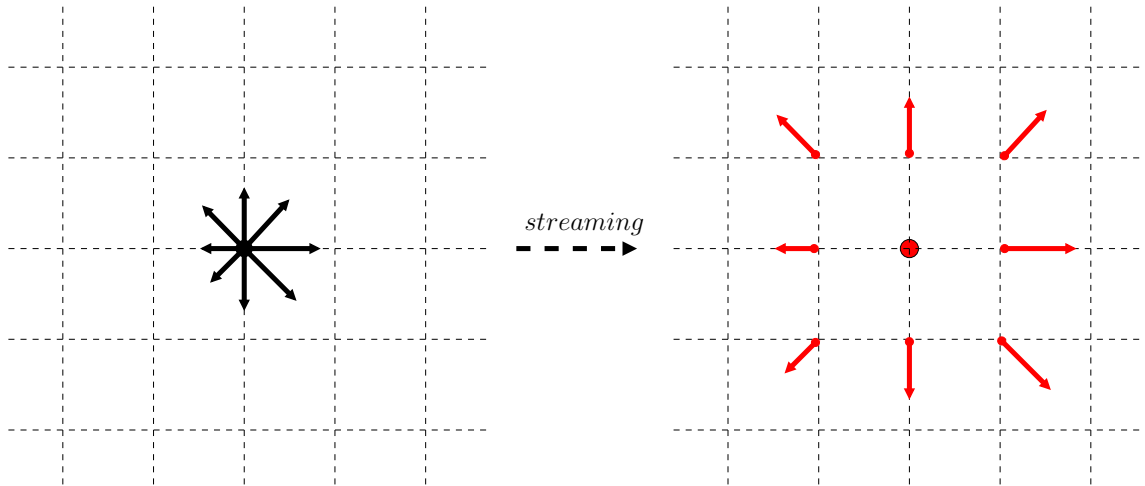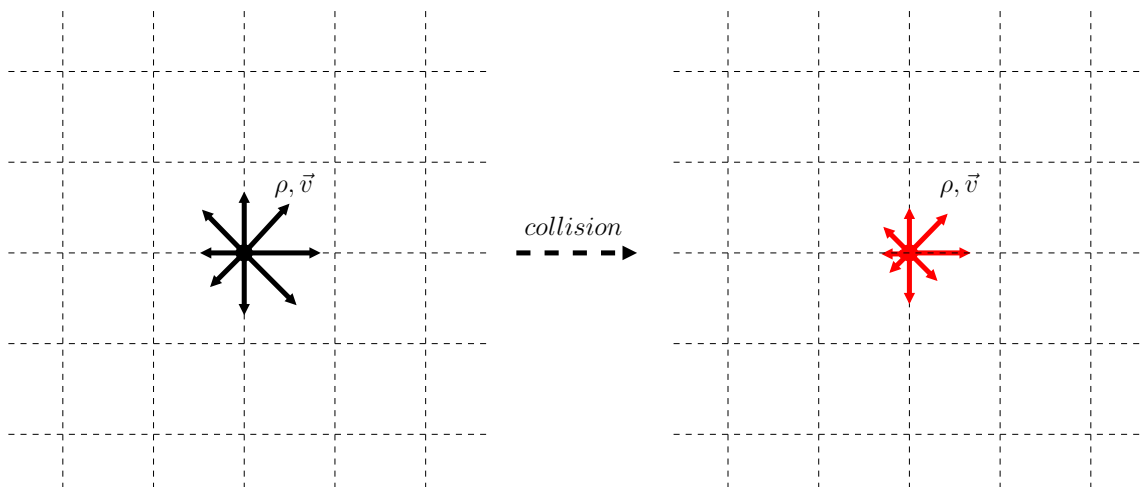
$$f_a^{eq}(\vec{x}) = w_a \rho(\vec{x}) \left[1 + 3\frac{\vec{e}_a \cdot \vec{u}}{c^2} + \frac{9}{2}\frac{(\vec{e}_a \cdot \vec{u})^2}{c^4} - \frac{3}{2}\frac{\vec{u}^2}{c^2} \right], \qquad (3.55)$$

where for the $D2Q9$ model the weights are $w_{a=0} = 4/9, w_{a=\{1..4\}} = 1/9, w_{a=\{5..8\}} = 1/36$ and c is the propagation speed on the lattice, $c = \delta_x/\delta_t$. Under the afore-mentioned assumption of a low Mach number, and further taking $\{Kn^7, \delta_t, \delta_x\} \to 0$, this model recovers the incompressible Navier-Stokes equations:

$$\nabla \cdot \vec{u} = 0, \qquad (3.56)$$

$$\rho \partial_t \vec{u} + \rho \vec{u} \cdot \nabla \vec{u} = -\nabla P + \rho \nu \nabla^2 \vec{u} \qquad (3.57)$$

with an isothermal equation of state:

$$P = c_s^2 \rho, \qquad (3.58)$$

where P is the pressure. The viscosity of the fluid is related to the relaxation parameter τ by the equation

$$\nu = c_s^2 \left(\tau - 1/2\right) \frac{\delta_x^2}{\delta_t} \Rightarrow \tau = \frac{\nu}{c_s^2}\frac{\delta_t}{\delta_x^2} + \frac{1}{2} \xrightarrow{c_s^2|_{D2Q9}=1/3} \tau_{D2Q9} = 3\nu\frac{\delta_t}{\delta_x^2} + \frac{1}{2} \qquad (3.59)$$

The proof of these results follows from the Chapman-Enskog analysis. Eq. (3.59) provides a straightforward method for adjusting the fluid viscosity in the model. It is obvious that $\tau \geq 0.5$ is required in order to ensure a positive viscosity. The limit $\tau \to 0.5$ corresponds to the inviscid flow, while the $\tau \to \infty$ limit represents the Stokes (creeping) flow. The model described so far is only applicable to athermal liquids. While there are many flow situations which can be attributed to this class, thermal effects are often essential to many natural phenomena. A suitable approach consists of solving the passive scalar equation for temperature on a separate lattice. The temperature field is

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[7]The assumption of $Kn \equiv \frac{\lambda}{L} \to 0$ is a requirement for continuum models to apply, hence it is not specific to LBM.

influenced by the fluid advection, and influences the fluid through a buoyancy term. This approach is only valid in the Boussinesq approximation, which is a reasonable assumption for many flows (for example, in ocean flows). The LB evolution algorithm is the same on the temperature lattice, but with different equilibrium DFs. Also, because the macroscopic quantity is a scalar (in contrast to the LBM for the velocity field, which is a vector), a lattice with fewer velocity directions is sufficient (D2Q5). The evolution equation on the temperature lattice is described by the same type of LB equation:

$$\underbrace{g_a(\vec{x} + \vec{e}_a\delta_t, t + \delta_t) = g_a(\vec{x}, t)}_{\text{Streaming}} \underbrace{- \frac{[g_a(\vec{x}, t) - g_a^{eq}(\vec{x}, t)]}{\tau_T}}_{\text{Collision}}, \qquad (3.60)$$

The macroscopic temperature is recovered by summation:

$$T = \sum_{i=0}^{4} g_i \qquad (3.61)$$

The main difference however lies in modified equilibrium distributions:

$$g_i^{eq} = T w_{T,i} \left[1 + 3e\vec{r}_{,i} \cdot \vec{u} \right] \qquad (3.62)$$

where the weights on the thermal lattice read $w_{i=0} = 1/3$, $w_{i=\{1..4\}} = 1/6$, and the thermal diffusivity is related to the thermal relaxation time τ_T through:

$$\tau_T = 3\kappa \frac{\delta_t}{\delta_x^2} + 1/2 \qquad (3.63)$$

The back-coupling to the velocity field is accomplished through an additional force term in the RHS Eq. (3.54):

$$dF_i = -3w_i\rho\beta g(T - T_0)(\vec{e}_i \cdot \hat{j}) \qquad (3.64)$$

3.4.2 Simulation set-up of the Rayleigh-Bénard convection

An important part of any numerical simulation is relating the simulation input parameters and output results to the exact flow we intend to model. The key concept during these procedures is *dynamic similarity*, which tells us that two flows with different physical parameters are effectively equivalent as long as several dimensionless numbers are the same. This idea is of special importance in experimental and numerical fluid dynamics (e.g., sections 4.5, 8.5). Similarily, in CFD, the fluid solver usually works in a different lengthscale than the original, physical system that is to be simulated. We can distinguish **3** different frames of reference in a simulation, described below. The dimensionless system may seem like an unnecessary complication in the beginning, but it reflects the fact that flows are often given in the literature in this form.

1. **Physical system:** is the actual system that we intend to simulate. Here, we measure things in the usual meters, seconds and kilograms. A problem with this system is that it is very dependent on the units, which are not important to the mathematics behind the PDEs governing the flow. However, any practical application of fluid mechanics has to start from this system and return to it when results are to be reported.

2. **Dimensionless system:** by choosing typical length- and time-scales for our flows, we can non-dimensionalize the equations, which then become more amenable to numerical simulation. Note that, sometimes, it is necessary to choose also a typical mass and/or temperature, depending on the form we take for the macroscopic equations.

3. **Discrete system:** is the coordinate system in which our numerical simulation lives. The input parameters for our simulation propagate from the physical system, through the non-dimensional system until here. Due to reasons of numerical stability, several restrictions are in place at this level, as will be discussed during the practical examples below.

The application we are looking at is the two-dimensional convection driven by a temperature gradient (Rayleigh-Bénard convection). The geometry consists of a rectangular channel, with periodic BCs at the sides and no-slip and constant temperature BCs on the top and bottom walls (section 5.3). Now we can non-dimensionalize the equations by choosing some typical values

for lengthscale L and timescale T of the system. As a reference length L, we take the distance between the two walls. We also need a value for scaling our temperature. Since we are imposing a specific temperature difference throughout our fluid domain, the temperature values will be within this range everywhere, and it makes sense to scale temperature by this value (ΔT). The presence of the gravitational constant in the equations provides us a natural timeframe. The first guess would be to take $g = L/T^2$, but we can make a better choice which also allows us to cancel-out the thermal expansion coefficient in the dimensionless system, namely:

$$g = \frac{L}{\alpha(\Delta T)T^2} \Rightarrow T = \sqrt{\frac{L}{g\alpha(\Delta T)}} \tag{3.65}$$

The physical quantities can then be written in terms of the dimensionless ones as $p = \rho_0 \frac{L^2}{T^2} p_d$ and for temperature $T = T_d(\Delta T) + T_0$. Plugging-in these expressions into the eqs. in section 5.3 we eventually obtain:

$$\boldsymbol{\nabla}_d \cdot \vec{u_d} = 0 \tag{3.66}$$

$$\partial_{t_d}\vec{u_d} + (\vec{u_d} \cdot \boldsymbol{\nabla}_d)\vec{u_d} = -\boldsymbol{\nabla}_d p_d + \sqrt{\frac{Pr}{Ra}}\boldsymbol{\nabla}_d^2\vec{u_d} + T_d\hat{j} \tag{3.67}$$

$$\partial_{t_d}T_d + \boldsymbol{\nabla}_d \cdot (\vec{u_d}T_d) = \sqrt{\frac{1}{RaPr}}\boldsymbol{\nabla}_d^2 T_d \tag{3.68}$$

Where Ra and Pr are the characteristic Rayleigh and Prandtl numbers of the system, defined as

$$Pr \equiv \frac{\nu}{\kappa} \tag{3.69}$$

$$Ra \equiv \frac{g\alpha(\Delta T)L^3}{\nu\kappa} = Pr \cdot \frac{g\alpha(\Delta T)L^3}{\nu^2} \tag{3.70}$$

$$\xrightarrow{eq.3.65} \nu = \frac{T}{L^2}\nu = \sqrt{\frac{Pr}{Ra}}; \quad \kappa = \sqrt{\frac{1}{RaPr}} \tag{3.71}$$

The temperature BCs become in the dimensionless system:

$$T_{d,hot} = 1$$

$$T_{d,cold} = 0$$

Discretization of the dimensionless system Let us denote by N the number of gridpoints we use to discretize and by N_{iter} the number of time iterations which will resolve our unit timescale T_d. We then have the following discrete space- and time-step in the dimensionless system:

$$\delta_x = \frac{1}{N-2}; \quad \delta_t = \frac{1}{N_{iter}-1} \tag{3.72}$$

Note that for computing the space-step we need to subtract 1 because p points always delimitate $p-1$ segments, and $(2 \times 0.5) = 1$ due to the interpretation of the horizontal walls half-way between 1st and 2nd (respectively half-way between $N-1^{th}$ and N^{th}) lattice rows. For time-steps, we obviously do not have the second issue, thus we only subtract 1.

In a sense, we repeat the procedure we applied to non-dimensionalize the original equations, except that we use δ_x and δ_t instead of the previous L and T. There is no need to rewrite the equations, since we are interested at this stage only on the parameters that we need to provide to our simulation to get the desired flow. We can easily write expressions for the most relevant quantities in the discrete (LB) system:

$$\vec{u_{lb}} = \frac{\delta_t}{\delta_x}\vec{u_d}; \quad g_{lb} = \frac{\delta_t^2}{\delta_x}g_d;$$

$$\nu_{lb} = \frac{\delta_t}{\delta_x^2}\nu_d = \frac{\delta_t}{\delta_x^2}\sqrt{\frac{Pr}{Ra}}; \quad \kappa_{lb} = \frac{\delta_t}{\delta_x^2}\kappa_d = \frac{\delta_t}{\delta_x^2}\sqrt{\frac{1}{RaPr}}; \tag{3.73}$$

In order to ensure that the compressibility effects do not become significant, a general rule is to keep $\delta_t \sim \delta_x^2$. Let us denote by β the proportionality factor (i.e. $\delta_t = \beta\delta_x^2$). The choice of β is not very obvious. If it is chosen too big, the timesteps get too large and the accuracy of the simulation decreases. However, if β is too small, the simulation takes a long time. This means

a compromise for β has to be found (Here, we choose $\beta = 11.18$ for $\delta_x = 0.02$). Once the number of gridpoints is given, this relation gives number of timesteps to resolve t_{0p}.

We also need to choose a representative value for the temperature, but we can simply pick a one-to-one mapping from the dimensionless system.

$$T_{lb} = T_d \tag{3.74}$$

We could write the formulae for converting the results back to the dimensionless and/or physical system.

3.4.3   System preparations and running a simulation

There are only a few parameters that define the behavior of the system: The number of gridpoints (*lx*, *ly*) defines the size of the lattice and thereby directly affects the accuracy of the results. On the one hand we get better results with a finer grid but on the other hand the computational cost increases dramatically. The parameter *N_t0* descibes the maximal simulation time in units of t_{0d}. For a given physical system t_{0d} can be calculated using eq. 3.65. *N_t0* should be high enough to overcome the inital conditions.

Remember that the Rayleigh (*Ra*) and Prandtl (*Pr*) numbers are dimensionless numbers that define the character of the flow. *Pr* is the ratio of the viscosity ν and the thermal conductivity k. *Ra* describes the heat transfer of a buoyancy driven flow. Some results for different sets of parameters can be seen in figure 3.4. As seen in section 3.4.2 *beta* is the factor that couples the spatial and temporal step sizes of the lattice. As for the grid resolution, a compromise between accuracy of the results and computing time has to be found!

Run simulation   After installation run R and change the working directory of R to the path where the *.r-files of the model are located:

```
setwd('Path/of/Rayleigh_Benard_model')
```

If all parameters are set properly, the model is loaded and executed by the command
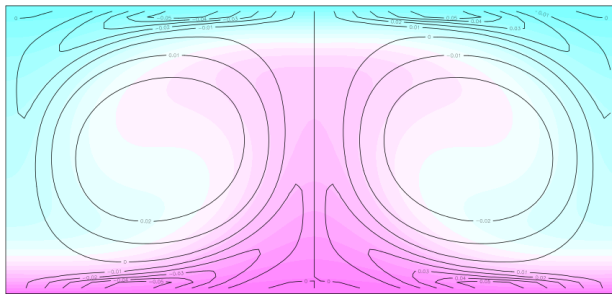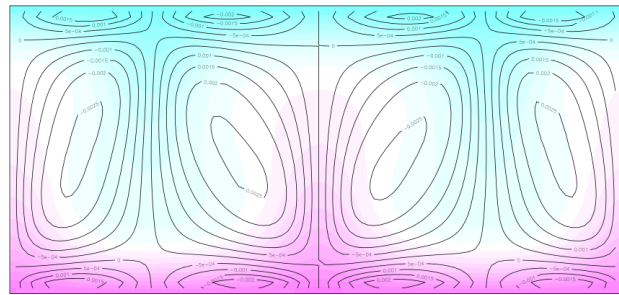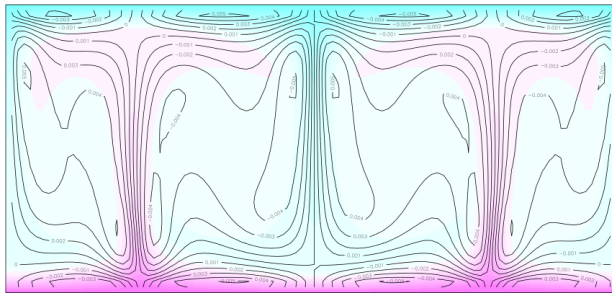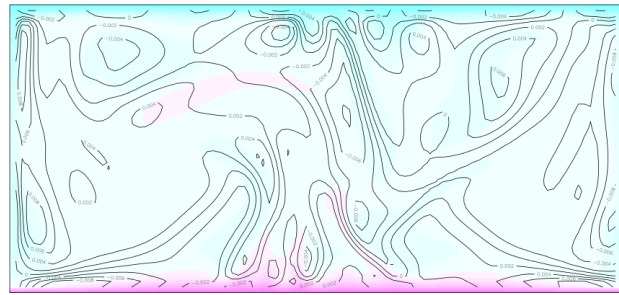
(a) $Ra = 2 \cdot 10^4, Pr = 0.1$

(b) $Ra = 2 \cdot 10^4, Pr = 10$

(c) $Ra = 5 \cdot 10^5, Pr = 10$

(d) $Ra = 1 \cdot 10^7, Pr = 10$

Figure 3.4: Four examples of the flow for different sets of Ra and Pr. The contours show lines of constant vorticity; the colors in the background display the temperatures (purple - warm, blue - cold).

```
source('rayleigh-benard.R')
```

After the simulation has completed the results can be found in the folder defined by the parameter *out_dir*. For a new run, the old directory has to be removed or renamed. All necessary files and parameters are shortly described here. Your application should come with the following files:

- rayleigh-benard.R | The R source code
- rb_functions.R | Some extra R functions needed by the model
- rb_plot_functions.R | Some R functions for plotting the results

There are two different types of parameters that can be edited: the 'model parameters' (which define the 'physical' values needed for the simulation), and the 'output parameters' (which define the frequency and kind of output).

Here is some R-code of the code calculating the macroscopic moments rho, ux, uy, T:

```
#Compute macroscopic values
  rho = colSums(flIn, dims=1);
  T = colSums(TIn, dims=1);
  ux = colSums( cx_fl*flIn, dims=1 ) / rho;
  uy = colSums( cy_fl*flIn, dims=1 ) / rho;
```

which is related to (3.52, 3.53) and (3.61), respectively. *cx_fl* and *cy_fl* denote the 9-dimensional momentum component ($\vec{e}_a$ in 3.53) and are related to the microscopic velocities $\vec{e}$ in the distribution function $f(\vec{x}, \vec{e}, t)$ of the Boltzmann dynamics (3.26). The main part of the code is the collision step for momentum and temperature:

```
#Collision Step
  #Fluid momentum
  for (i in idxRangeFluid){
    cu_fl = 3* (cx_fl[i] * ux + cy_fl[i] * uy);
    flEq= rho * w_fl[i] *
        (1 + cu_fl + 0.5 * cu_fl^2 - 1.5 * (ux^2 + uy^2));
    force = 3* w_fl[i]* rho * (T-T0)*
        (cx_fl[i] * g[1] + cy_fl[i] * g[2])/(T_bot - T_top);
    flOut[i,,] = (1.-omega_fl)*flIn[i,,] + omega_fl*flEq + force;
  }

  #Temperature
  for (i in idxRangeTemp){
    cu_T = 3* (cx_T[i] * ux + cy_T[i] * uy);
    TEq = T * w_T[i] * (1 + cu_T);
    TOut[i,,] = (1.-omega_T)*TIn[i,,] + omega_T*TEq;
  }
```

where *fl_Eq* and *T_Eq* denote the local Maxwell-Boltzmann single-particle distribution function.

Exercise 25   –   Investigations with the LB-model

1. Vary the Rayleigh and the Prandtl number by $R_a = 20000, 40000, 60000$ and $Pr = 0.5, 1, 1.5, 5, 10$ and describe the dynamics (words, figures) ! For high values of R_a the spatial resolution might be chosen higher (to the double). Here are the standart values:

```
lx  = 100;    #Number of horizontal cells
ly  = 52;    #Number of vertical cells
```

2. Vary the initial perturbation and obtain the reversed circulation! Look at the line

```
#Set small trigger to break symmetry
T[lx/2+1, 1] = 1.1 * T_bot;
```

Here, some remarks related to the boundary conditions are in order. When using a Dirichlet boundary condition, one prescribes the value of a variable at the boundary, e.g. temperature or density in our case. When using a Neumann boundary condition, one prescribes the gradient normal to the boundary of a variable at the boundary, e.g. the heat flux or density flux. When using a mixed boundary condition, different types of boundary conditions can be used for different variables (e.g. for temperature and salinity).

In viscous flows, no-slip condition enforced at walls:

-Tangential fluid velocity equal to wall velocity.

-Normal velocity component is set to be zero.

This is realized through a bounce back condition: a particle travelling in the e_1-direction is bounced back into the opposite e_5-direction. A modified version of the previous problem is an ocean box with solid walls and free slip at the surface (no friction). This is implemented by mirroring (relative to horizontal-axis) the distribution functions in the fluid-lattice:

```
#"Bounce Back" Boundary Conditions for Fluid
  for (i in idxRangeFluid){
    flOut[i,,1] = flIn[opp_fl[i],,1];
    flOut[i,,ly] = flIn[opp_fl[i],,ly];
  }
```

Used when physical geometry of interest and expected flow pattern and the thermal solution are

of a periodically repeating nature (as in the Rayleigh-Bénard problem). This reduces computational effort in problem.

Exercise 26   –   Ocean-like circulation

1. Evaluate the effect of different external temperatures (hemispheric, double hemispheric).

   The R code is

   `ocean_rb.R`

   Here are two options:

   ```
   # Pre-compute imposed temperature-profile on top (linear)
   tempTop = array(0, c(lx));
   for (x in 2:lx-1) {
     tempTop[x] = THot - (THot-TCold)*(x-2)/(lx-3);
    }
   ```

   for a single hemisphere, and for a double hemisphere version:

   ```
   # Pre-compute imposed temperature-profile on top (linear+sinus)
   tempTop = array(0, c(lx));
   bett= 0.2 # right boundary
   alph= (0.1-bett)/lx ;
   gamma =1.-alph * lx/2 -bett;
   for (x in 2:lx-1) {
     tempTop[x] = alph *x + bett + gamma * sin( 3.1416* x/lx);
   }
   ```

   Describe the dynamics with respect to the temperature at the top layer tempTop !

2. In lattice Boltzmann models, it is relatively easy to insert obstacles. The R code is

   `ocean_rb_ridge.R`

   Discuss the influence of the ridge on the ocean circulation!

3. Manage to change the Rayleigh-Bénard convection from a no-slip to free slip bounday conditions at the top. The upper plate is just removed and we have an air-water interface. What are the differences? Make a plot!

4. Provide a model for the atmospheric cells (the atmosphere is mainly heated from below).

5. Calculate the ocean heat transport in the model and compare it with the estimate in exercise 55! Use dimensionless parameters!

(a) Two hemisphere temperature

(b) Ridge and a two hemisphere temperature

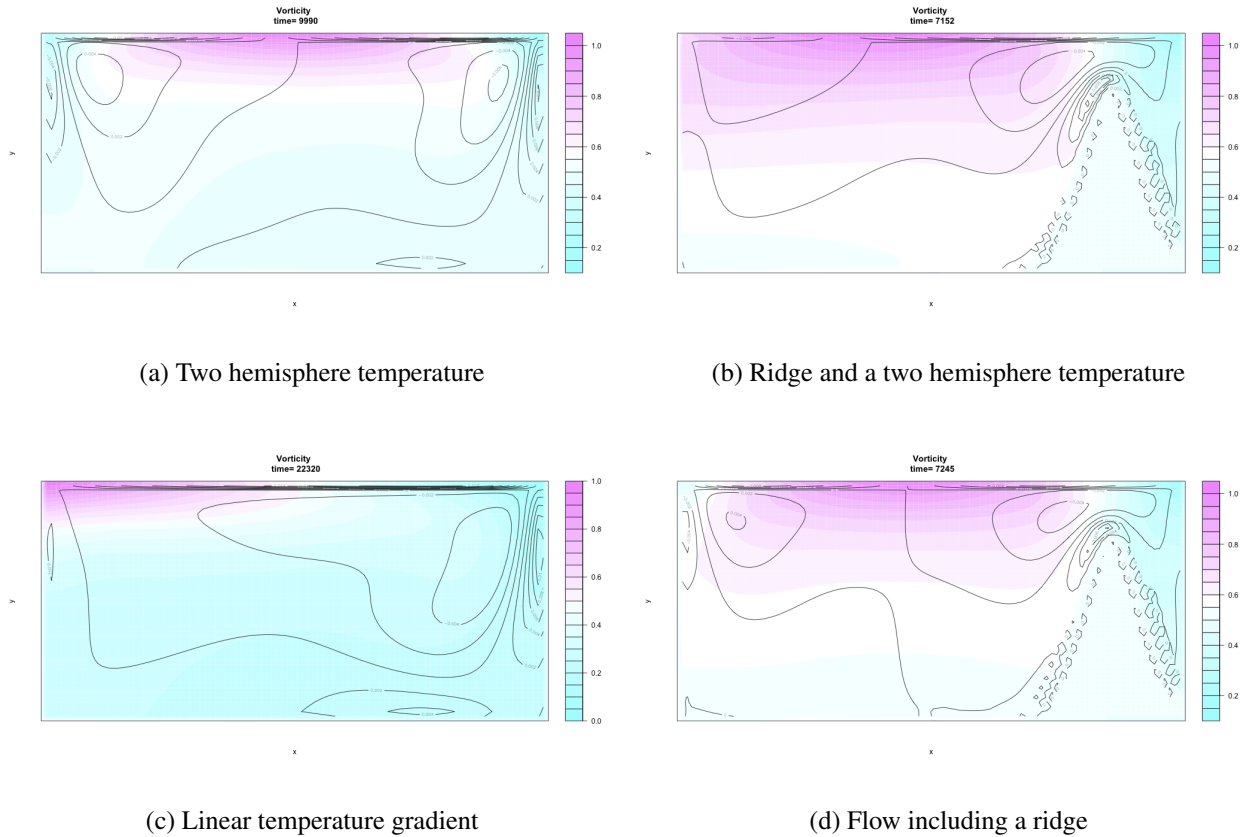(c) Linear temperature gradient

(d) Flow including a ridge

Figure 3.5: Four examples of the ocean flow for different boundary conditions, and fixed Prandtl number=1 and Rayleigh number=45000. The contours show lines of constant vorticity; the colors in the background display the temperatures (purple - warm, blue - cold). For the right scenarios, an obstacle representing an oceanic sill is implemented.

3.5   Projection methods: coarse graining

In order to get a first idea of coarse graining, one one may think of the transition from Rayleigh-Bénard convection to the Lorenz system (section 5.3). In our formula, the Galerkin approximation (5.80,5.80) provided a suitable projector to simply truncate the series at some specified wave number cut-off into a low-order system (such as in equations (5.81, 5.82).

The Mori-Zwanzig formalism [Mori, 1965; Zwanzig, 1960] provides a conceptual framework for the study of dimension reduction and the parametrisation of uninteresting variables by a stochastic process. It includes a generalized Langevin [1908] theory. Langevin [1908] studied Brownian motion from a different perspective to Einstein's seminal 1905 paper [Einstein, 1905], describing the motion of a single Brownian particle as a dynamic process via a stochastic differential equation, as an Ornstein-Uhlenbeck process [Uhlenbeck and Ornstein, 1930].

Ehrenfest introduced a special operation, the coarse-graining. This operation transforms a probability density in phase space into a "coarse-grained" density, that is a piecewise constant function, a result of density averaging in cells. The size of cells is assumed to be small, but finite, and does not tend to zero. The coarse-graining models uncontrollable impact of surrounding (of a thermostat, for example) onto ensemble of mechanical systems. To understand reasons for introduction of this new notion, let us take a phase drop, that is, an ensemble of mechanical systems with constant probability density localized in a small domain of phase space. Let us watch evolution of this drop in time according to the Liouville equation. After a long time, the shape of the drop may be very complicated, but the density value remains the same, and this drop remains "oil in water." The ensemble can tend to the equilibrium in the weak sense only: average value of any continuous function tends to its equilibrium value, but the entropy of the distribution remains constant. Nevertheless, if we divide the phase space into cells and supplement the mechanical motion by the periodical averaging in cells (this is the Ehrenfests' idea of coarse-graining), then the entropy increases, and the distribution density tends uniformly to the equilibrium. This periodical coarse-graining is illustrated by Fig. 3.6 in a two-dimensional phase space.

Mechanical motion of
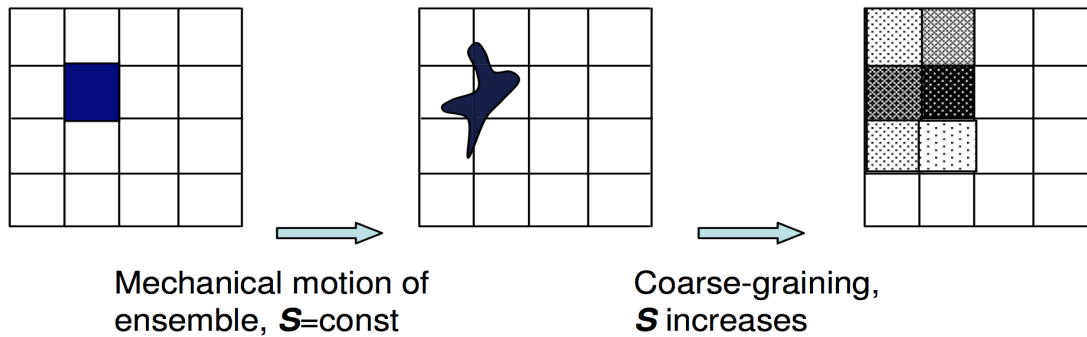ensemble, **S**=const

Coarse-graining,
S increases

Figure 3.6: The Ehrenfests coarse-graining: two motion - coarse-graining cycles in 2D (values of probability density are presented by hatching density).

Applications of the Ehrenfests' coarse-graining[8] outside statistical physics include simple, but effective filtering. The Gaussian filtering of hydrodynamic equations that leads to the Smagorinsky equations[9] is, in its essence, again a version of the Ehrenfests' coarse-graining. The central idea of the Ehrenfests' coarse-graining remains the same in most generalizations: we combine the genuine motion with the periodic partial equlibration. The result is the Ehrenfests' chain. After that, we can find the macroscopic equation that does not depend on an initial distribution and describes the Ehrenfests' chains as results of continuous autonomous motion. Alternatively, we can just create a computational procedure without explicit equations. In the sense of entropy production, the resulting macroscopic motion is "more dissipative" than initial (microscopic) one. It is the theorem about entropy overproduction. In practice, kinetic models in the form of lattice Boltzmann models are in use (section 3.2). The coarse-graining provides theoretical basis for kinetic models. First of all, it is possible to replace projecting (partial equilibration) by involution (i.e. reflection with respect to the partial equilibrium). This entropic involution was developed for the lattice Boltzmann methods. In the original Ehrenfests' chains, "motion-partial equilibration-motion-...," dissipation is coupled with time step, but the chains "motion-involution-motion-..." are conservative. The family of chains between conservative (with entropic involution) and maximally dissipative (with

---

[8]P. Ehrenfest, T. Ehrenfest-Afanasyeva, The Conceptual Foundations of the Statistical Approach in Mechanics, In: Mechanics Enziklopädie der Mathematischen Wissenschaften, Vol. 4., Leipzig, 1911. Reprinted: P. Ehrenfest, T. Ehrenfest-Afanasyeva, The Conceptual Foundations of the Statistical Approach in Mechanics, Dover Phoneix, 2002.
[9]J. Smagorinsky, General Circulation Experiments with the Primitive Equations: I. The Basic Equations, Mon. Weather Rev. 91 (1963), 99–164.

projection) ones give us a possibility to model hydrodynamic systems with various dissipation (viscosity) coefficients that are decoupled with time steps.

Of particular interest is the work of Mori [Mori, 1965] and Zwanzig [Zwanzig, 1960] which relates the evolution of macroscopic variables to microscopic dynamics. The standard Mori-Zwanzig theory has been given a nonlinear generalization by Zwanzig [Zwanzig, 1980], and is furthermore not limited to Hamiltonian dynamics [Chorin et al., 1999; Gottwald, 2010]. This approach of modelling fast small-scale processes by a stochastic process is intuitive: provided the fast processes decorrelate rapidly enough, the slow variables experience the sum of uncorrelated events of the fast dynamics, which according to the (weak) central limit theorem corresponds to approximate Gaussian noise. A method whereby many fast degrees of freedom are replaced by a stochastic process is called stochastic model reduction.

Consider the very simple coupled linear system[10]

$$\dot{x} \;=\; L_{11}x + L_{12}y \qquad \text{the "climate" equation} \tag{3.75}$$

$$\dot{y} \;=\; L_{21}x + L_{22}y \qquad \text{the "whether" equation.} \tag{3.76}$$

Suppose we are only interested in the dynamics of x, and have only some climatic knowledge of the initial conditions of the variables y, that is the mean and variance. The whether differential equation (3.76) can be solved by the ansatz

$$y(t) = e^{L_{22}t}y(0) \cdot C(t) \tag{3.77}$$

Inserting this, we can then solve the inhomogenous problem to obtain

$$y(t) = e^{L_{22}t}y(0) + \int_0^t e^{L_{22}(t-s)}L_{21}x(s)ds,$$

---

[10]We follow the notation of [Hasselmann, 1976; Chorin et al., 1999; Gottwald, 2010].

which we may use to express the dynamics of the climate variable as

$$\dot{x} = L_{11}x + L_{12}\int_0^t e^{L_{22}(t-s)}L_{21}x(s)ds + L_{12}e^{L_{22}t}y(0). \tag{3.78}$$

This is of the form of a generalised Langevin equation, where the first term is Markovian (no dependence on the history of the process), the second is a memory term, and the last can be interpreted as a noise term, provided that the initial conditions $y(0)$ are randomly distributed. A similar reduction of the dynamics can be described by fast and slow variables applying the center manifold theory [Arnold, 1995] or slaving principle [Haken, 1996].

For the more general non-linear case, the instantaneous state of the Earth System, comprising the components 'atmosphere-ocean-cryosphere-land', can be expressed by a set of variables $\mathbf{z} = (z_1, z_2, ...)$, representing the density, velocity, temperature, etc. of the various media. The evolution of this system will be given by a series of prognostic equations of the form

$$\dot{z} = f(z), \tag{3.79}$$

with initial condition $z(0) = z_0$ and $z \in R^d$, suppose we are not interested in the full solution $z(t)$, but rather only in a few $n \leq d$ observables $\Phi(z) = (\Phi_1(z), \Phi_2(z), ..., \Phi_n(z))$. This includes the case $\Phi(z) = (z_1, ..., z_n)$, when the state space is decomposed as $z = (x, y)$ into 'interesting' variables, $x = (z_1, ..., z_n) \in \mathbb{R}^n$, and 'uninteresting' variables, $y = (z_{n+1}, ..., z_d) \in \mathbb{R}^{d-n}$. In the Earth System, a separation may be into a fast 'weather subsystem' (y) and a slow 'climate subsystem' (x) with different order of magnitude in the correlation times (or, the response/relaxation times) for the slow variable is much larger than that of the fast variable, i.e.

$$\tau_y \ll \tau_x. \tag{3.80}$$

Now let us ask the following question: what are the effective dynamics of the interesting observables for an ensemble of initial conditions $z(0)$, where $\Phi(z(0))$ is known and the uninteresting subspace is equipped with a known distribution?

Rather than investigating the dynamical system (3.79) directly, one may choose to look at how observables $V(z(t))$ evolve in time. Applying the chain rule, one can naturally define the generator

$$\mathcal{L} = f(z) \cdot \nabla,$$

and write

$$\frac{d}{dt}V(z(t)) = \mathcal{L}V(z(t)).$$

Note that $\mathcal{L}$ is the adjoint operator of the Liouville operator $\mathcal{L}^*$ with $\mathcal{L}^*\rho = -\nabla \cdot (f(z)\rho)$ controlling the evolution of densities of ensembles propagated according to (3.79). We seek for the solution $v(z, t)$ of

$$\frac{\partial v}{\partial t} = \mathcal{L}v \ \text{ with } \ v(z, 0) = \phi(z), \tag{3.81}$$

where z is an independent variable and denotes initial conditions. The solution of (3.81) can be formally written as

$$v(z, t) = e^{\mathcal{L}t}\phi(z), \tag{3.82}$$

To filter out the dynamics of the interesting variables we require a projection operator $\mathbf{P}$ that maps functions of z to functions of $\Phi(z)$. If the manifold consists for example of a product of submanifolds of relevant and irrelevant variables, one can take a conditional expectation

$$(\mathbf{P}v)(\mathbf{x}) = \frac{\int_{\mathbb{R}^{d-n}} v(z)\rho(\mathbf{x}, \mathbf{y})d\mathbf{y}}{\int_{\mathbb{R}^{d-n}} \rho(\mathbf{x}, \mathbf{y})d\mathbf{y}} \tag{3.83}$$

where $\rho(\mathbf{x}, \mathbf{y})$ denotes the joint probability function of the initial conditions for the full system (3.79). It is easy to show that this a projection ($\mathbf{P^2} = \mathbf{P}$). In the context of PDEs one may use Galerkin approximations, a perfectly valid projector would be to simply truncate the Galerkin series at some specified high wave number cut-off. We also define the orthogonal projector $\mathbf{Q}$ that projects onto y, with $\mathbf{Q} = \mathbf{1} - \mathbf{P}$. Now, the derivation of the Mori-Zwanzig equation is a two-linear: given the Cauchy problem (3.81) and its formal solution (3.82) we write, using

$P + Q = 1$,

$$\frac{\partial v}{\partial t}(z, t) = \mathcal{L}e^{\mathcal{L}t}\Phi(z) = e^{\mathcal{L}t}P\mathcal{L}\Phi(z) + e^{\mathcal{L}t}Q\mathcal{L}\Phi(z)$$

which, upon using the Duhamel-Dyson formula [Evans and Morriss, 2008] for operators A and B, yields

$$e^{t(A+B)} = e^{tA} + \int_0^t e^{(t-s)(A+B)}Be^{sA}ds.$$

By differentiation, this becomes the celebrated Mori-Zwanzig equation [Mori et al., 1974; Zwanzig, 1960]

$$\frac{\partial v}{\partial t}(z, t) = e^{\mathcal{L}t}P\mathcal{L}\Phi(z) + \int_0^t e^{(t-s)\mathcal{L}}P\mathcal{L}e^{sQ\mathcal{L}}Q\mathcal{L}\Phi(z)ds + e^{tQ\mathcal{L}}Q\mathcal{L}\Phi(z). \quad (3.84)$$

Note that the Mori-Zwanzig equation (3.84) is not an approximation but is exact and constitutes an equivalent formulation of the full problem (3.79). The Mori-Zwanzig equation (3.84) is in the form of a generalised Langevin equation. The first term on the right-hand side is Markovian, the second term is a memory term, and the last term lives in the uninteresting orthogonal subspace and can be called noise. Ideally one would like to approximate the noise term by white noise. Heuristically this should be possible in the case of time-scale separation or of weak coupling. The advantage of looking at this limit is however that the noise autocorrelation function and memory kernel can now be written as simple correlation and response functions of the unresolved dynamics. The reader is referred to [Chorin and Hald, 2006; Chorin et al., 2000; Zwanzig, 2001; Evans and Morriss, 2008; Givon et al., 2004; Lucarini et al., 2014] for more details.

The projection method includes the procedure to parameterize the turbulent energy dissipation in turbulent flows, where the larger eddies extract energy from the mean flow and ultimately transfer some of it to the smaller eddies which, in turn, pass the energy to even smaller eddies, and so on up to the smallest scales, where the eddies convert the kinetic energy into internal energy of the fluid. At this scales (also known as Kolmogorov scale), the viscous friction dominates the flow [Frisch, 1996].

The theory of scientific reduction is important for different theories: the microscopic informa-

tion in the brain with enormous amount of possible solutions is reduced to macroscopic actions and human behaviour. This implies that the actions are not deterministic, but stochastic in the sense of the standard Mori-Zwanzig theory or Brownian motion. Without being a specialist, this seems to be important for neuroscience and for the philosophy of science in general. The activity of neurons in the brain can be modelled statistically (e.g., https://en.wikipedia.org/wiki/Ising_model).

Part II

Second part: Fluid Dynamics

Chapter 4

Basics of Fluid Dynamics

Our starting point is a mathematical model for the system of interest. In physics a model typically describes the state variables, plus fundamental laws and equations of state. These variables evolve in space and time. For the ocean circulation, we proceed as follows:

- State variables: Velocity (in each of three directions), pressure, temperature, salinity, density
- Fundamental laws: Conservation of momentum, conservation of mass, conservation of temperature and salinity
- Equations of state: Relationship of density to temperature, salinity and pressure, and perhaps also a model for the formation of sea-ice

The state variables for the ocean model are expressed as a continuum in space and time, and the fundamental laws as partial differential equations[1]. Even at this stage, though, simplifications may be made. For example, it is common to treat seawater as incompressible. Furthermore, equations of state are often specified by empirical relationships or laboratory experiments.

In the following, the general structure of ocean circulation, atmospheric energy balance as well as ice sheet models are described. The dynamics of flow are based on the Navier-Stokes equations. The derivation of the Navier-Stokes equations begins with an application of Newton's second law: conservation of momentum (often alongside mass and energy conservation) being written for an

---

[1]If the atmosphere is becoming too thin in the upper levels, a more molecular, statistical description is appropiate (section 3)

arbitrary control volume. In an inertial frame of reference, the general form of the equations of fluid motion is:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \mathbb{T} + \mathbf{F}, \tag{4.1}$$

where $\mathbf{u}$ is the flow velocity (a vector), ρ is the fluid density, p is the pressure, $\mathbb{T}$ is the $\mathbf{3 \times 3}$ (deviatoric) stress tensor, and $\mathbf{F}$ represents body forces (per unit volume) acting on the fluid and ∇ is the nabla operator. This is a statement of the conservation of momentum in a fluid and it is an application of Newton's second law to a continuum; in fact this equation is applicable to any non-relativistic continuum and is known as the Cauchy momentum equation (e.g., Landau and Lifshitz [1959]).

This equation is often written using the substantive derivative, making it more apparent that this is a statement of Newton's second law:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla \cdot \mathbb{T} + \mathbf{F}. \tag{4.2}$$

The left side of the equation describes acceleration, and may be composed of time dependent or advective effects (also the effects of non-inertial coordinates if present). The right side of the equation is in effect a summation of body forces (such as gravity) and divergence of stress (pressure and stress). A very significant feature of the Navier-Stokes equations is the presence of advective acceleration: the effect of time independent acceleration of a fluid with respect to space, represented by the nonlinear quantity $\mathbf{u} \cdot \nabla \mathbf{u}$. A general framework can be generally formulated as a transport phenomenon, see section 1.4.

4.1 Material laws

The effect of stress in the fluid is represented by the ∇p and $\nabla \cdot \mathbb{T}$ terms, these are gradients of surface forces, analogous to stresses in a solid. ∇p is called the pressure gradient and arises from

the isotropic part of the stress tensor. This part is given by normal stresses that turn up in almost all situations, dynamic or not. The anisotropic part of the stress tensor gives rise to $\nabla \cdot \mathbb{T}$, which conventionally describes viscous forces. For incompressible flow, this is only a shear effect. Thus, $\mathbb{T}$ is the deviatoric stress tensor, and the stress tensor is equal to:

$$\boldsymbol{\sigma} = -p\,\mathbb{I} + \mathbb{T} \tag{4.3}$$

where $\mathbb{I}$ is the 3×3 identity matrix. Interestingly, only the gradient of pressure matters, not the pressure itself. The effect of the pressure gradient is that fluid flows from high pressure to low pressure.

The stress terms p and $\mathbb{T}$ are yet unknown, so the general form of the equations of motion is not usable to solve problems. Besides the equations of motion -Newton's second law- a force model is needed relating the stresses to the fluid motion. For this reason, assumptions on the specific behavior of a fluid are made (based on observations) and applied in order to specify the stresses in terms of the other flow variables, such as velocity and density.

The Cauchy stress tensor can be also written in matrix form:

$$\mathbb{T} = \begin{pmatrix} \mathbf{T}^{(\mathbf{e}_1)} \\ \mathbf{T}^{(\mathbf{e}_2)} \\ \mathbf{T}^{(\mathbf{e}_3)} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \equiv \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \equiv \begin{pmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{pmatrix} \tag{4.4}$$

where $\boldsymbol{\sigma}$ are the normal stresses and τ are the shear stresses. From the Newton's third law (actio est reactio) the stress vectors $\mathbf{T}^{(\mathbf{e}_i)} = \frac{d\mathbf{F}}{dA}$ with $\mathbf{e}_i$ as normalvector acting on opposite sides of the same surface are equal in amount and opposite in direction $(-\mathbf{T}^{(\mathbf{e}_i)} = \mathbf{T}^{(-\mathbf{e}_i)})$. According to conservation of angular momentum, summation of moments is zero. Thus the stress tensor is symmetrical: $\mathbb{T} = \mathbb{T}^T$. In Fig. 4.1 the stress vectors $\mathbf{T}^{(\mathbf{e}_i)}$ can be decomposed in one normal stress and two shear stress components.
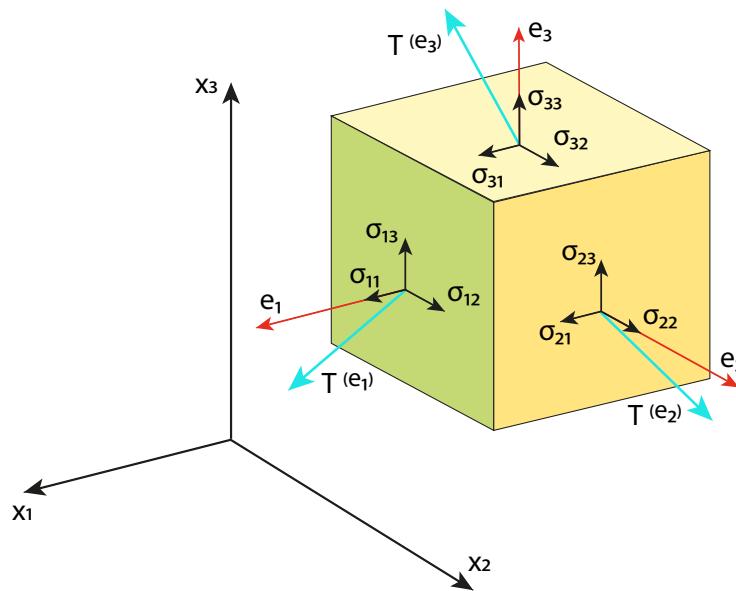
Figure 4.1: Components of stress in three dimensions.

4.2 Navier-Stokes equations

The so-called Navier-Stokes equations result from the following assumptions on the deviatoric stress tensor $\mathbb{T}$:

- the deviatoric stress vanishes for a fluid at rest, and by Galilean invariance also does not depend directly on the flow velocity itself, but only on spatial derivatives of the flow velocity

- in the Navier-Stokes equations, the deviatoric stress is expressed as the product of the tensor gradient $\nabla \mathbf{v}$ of the flow velocity with a viscosity tensor $\mathbb{A}$, i.e. $\mathbb{T} = \mathbb{A}(\nabla \mathbf{v})$

- the fluid is assumed to be isotropic, as valid for gases and simple liquids, and consequently $\mathbb{A}$ is an isotropic tensor; furthermore, since the deviatoric stress tensor is symmetric, it turns out that it can be expressed in terms of two scalar dynamic viscosities μ and μ'' : $\mathbb{T} = 2\mu\mathbb{E} + \mu''(\nabla \cdot \mathbf{v})\mathbb{I}$, where $\mathbb{E} = \frac{1}{2}(\nabla \mathbf{v}) + \frac{1}{2}(\nabla \mathbf{v})^{\mathrm{T}}$ is the rate-of-strain tensor and $\nabla \cdot \mathbf{v}$

is the rate of expansion of the flow

- the deviatoric stress tensor has zero trace, so for a three-dimensional flow $2\mu + 3\mu" = 0$

As a result, in the Navier-Stokes equations the deviatoric stress tensor has the following form:

$$\mathbb{T} = 2\mu \left(\mathbb{E} - \tfrac{1}{3}(\boldsymbol{\nabla} \cdot \mathbf{u})\mathbb{I} \right), \tag{4.5}$$

with the quantity between brackets the non-isotropic part of the rate-of-strain tensor $\mathbb{E}$. The dynamic viscosity μ does not need to be constant - in general it depends on conditions like temperature and pressure, and in turbulence modelling the concept of eddy viscosity is used to approximate the average deviatoric stress.

The Navier-Stokes equations are strictly a statement of the conservation of momentum. In order to fully describe fluid flow, more information is needed (how much depends on the assumptions made), this may include boundary data (no-slip, capillary surface, etc), the conservation of mass, the conservation of energy, and/or an equation of state. Regardless of the flow assumptions, a statement of the conservation of mass is generally necessary. This is achieved through the mass continuity equation, given in its most general form as:

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \mathbf{u}) = 0 \tag{4.6}$$

or, using the substantive derivative:

$$\frac{D\rho}{Dt} + \rho(\boldsymbol{\nabla} \cdot \mathbf{u}) = 0. \tag{4.7}$$

A simplification of the resulting flow equations is obtained when considering an incompressible flow of a Newtonian fluid. The assumption of incompressibility rules out the possibility of sound or shock waves to occur; so this simplification is invalid if these phenomena are important. The incompressible flow assumption typically holds well even when dealing with a "compressible" fluid -such as air at room temperature- at low Mach numbers (even when flowing up to about Mach

0.3).[2] Taking this into account and assuming constant viscosity, the Navier-Stokes equations will read, in vector form:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} \right) = -\boldsymbol{\nabla} p + \mu \boldsymbol{\nabla}^2 \mathbf{u} + \mathbf{F}. \tag{4.8}$$

The vector field $\mathbf{F}$ represents "other" (body force) forces. Typically this is only gravity, but may include other fields (such as electromagnetic). In a non-inertial coordinate system, other "forces" such as that associated with rotating coordinates may be inserted[3]. Often, these forces may be represented as the gradient of some scalar quantity. Gravity in the z direction, for example, is the gradient of $-\rho g z$. Since pressure shows up only as a gradient, this implies that solving a problem without any such body force can be mended to include the body force by modifying pressure. The shear stress term $\boldsymbol{\nabla}\mathbb{T}$ becomes the useful quantity $\mu \boldsymbol{\nabla}^2 \mathbf{u}$ when the fluid is assumed incompressible and Newtonian, where μ is the dynamic viscosity.

It's well worth observing the meaning of each term (compare to the Cauchy momentum equation):

$$\rho \big(\underbrace{\overbrace{\frac{\partial \mathbf{u}}{\partial t}}^{\text{Inertia (per volume)}} + \underbrace{\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}}_{\substack{\text{Advective} \\ \text{acceleration}}} \big) = \overbrace{\underbrace{-\boldsymbol{\nabla} p}_{\substack{\text{Pressure} \\ \text{gradient}}} + \underbrace{\mu \boldsymbol{\nabla}^2 \mathbf{u}}_{\text{Viscosity}}}^{\text{Divergence of stress}} + \underbrace{\mathbf{F}}_{\substack{\text{Other} \\ \text{body} \\ \text{forces}}}. \tag{4.9}$$

where the $\frac{\partial \mathbf{u}}{\partial t}$ term is labeled *Unsteady acceleration*.

Note that only the advection terms are nonlinear for incompressible Newtonian flow. This acceleration is an acceleration caused by a (possibly steady) change in velocity over position, for example the speeding up of fluid entering a converging nozzle. Though individual fluid particles are being accelerated and thus are under unsteady motion, the flow field (a velocity distribution) will not

---

[2] The density and pressure fields can be expressed as a perturbation from a hydrostatically balanced state around a reference density $\rho_r(z)$ (e.g. a horizontal mean of density in the area of interest) and associated pressure $p_r(z)$ which are linked through $dp_r/dz = -g\rho_r$ and $p_r(z = 0) = 0$. Sound waves are filtered by realizing that the time rate of change of density due to diabatic effects and compressibility is much smaller than that due to change of volume.

[3] We will see later that the Coriolis force will be one of the main contributions in the rotating Earth system (section 6.1)

necessarily be time dependent.

Another important observation is that the viscosity is represented by the vector Laplacian of the velocity field. This implies that Newtonian viscosity is diffusion of momentum, this works in much the same way as the diffusion of heat seen in the heat equation (which also involves the Laplacian).

If temperature effects are also neglected, the only "other" equation (apart from initial/boundary conditions) needed is the mass continuity equation. Under the incompressible assumption, density is a constant and it follows that the equation will simplify to:

$$\nabla \cdot \mathbf{u} = \mathbf{0} \qquad . \tag{4.10}$$

This is more specifically a statement of the conservation of volume (see divergence). These equations are commonly used in 3 coordinates systems: Cartesian, cylindrical, and spherical. While the Cartesian equations seem to follow directly from the vector equation above, the vector form of the Navier-Stokes equation involves some tensor calculus which means that writing it in other coordinate systems is not as simple as doing so for scalar equations (such as the heat equation).

4.3   Integral and differential formulation

On a volume work two types of forces:

1. The force $\vec{F}$ on each volume element. For gravity holds: $\vec{F} = \varrho\vec{g}$.

2. Surface forces working only on the margins: $\vec{t}$. For these holds: $\vec{t} = \vec{n}\,\sigma$, where σ is the *stress tensor*.

σ can be split in a part $p\,\mathsf{I}$ representing the normal tensions and a part $\mathbb{T}$ representing the shear stresses: $\sigma = \mathbb{T} + p\,\mathsf{I}$, where I is the unit tensor or identity matrix. When viscous aspects can be

ignored holds:

$$div \, \sigma = -\nabla p \quad . \tag{4.11}$$

When the flow velocity is $\vec{v}$ at position $\vec{r}$ holds on position $\vec{r} + d\vec{r}$:

$$\vec{v}(\vec{r} + d\vec{r}) = \underbrace{\vec{v}(\vec{r})}_{\text{translation}} + \underbrace{d\vec{r} \cdot (\nabla \vec{v})}_{\text{rotation, deformation, dilatation}}$$

The quantity $\mathsf{L} := \nabla \vec{v}$ can be split in a symmetric part D and an antisymmetric part W. $\mathsf{L} = \mathsf{D} + \mathsf{W}$ with

$$D_{ij} := \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) \quad , \quad W_{ij} := \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i} \right)$$

When the rotation or *vorticity* $\vec{\omega} = \mathrm{rot}\vec{v}$ is introduced holds: $W_{ij} = \frac{1}{2}\varepsilon_{ijk}\omega_k$. $\vec{\omega}$ represents the local rotation velocity: $\vec{dr} \cdot \mathsf{W} = \frac{1}{2}\omega \times \vec{dr}$.

For a *Newtonian liquid* holds: $\mathbb{T} = 2\eta\mathsf{D}$. Here, η is the dynamical viscosity. This is related to the shear stress τ by:

$$\tau_{ij} = \eta \frac{\partial v_i}{\partial x_j}$$

For compressible media can be stated: $\mathbb{T} = (\eta' \mathrm{div}\vec{v})\mathsf{I} + 2\eta\mathsf{D}$. From equating the thermodynamical and mechanical pressure it follows: $3\eta' + 2\eta = 0$. If the viscosity is constant holds: $\mathrm{div}(2\mathsf{D}) = \nabla^2\vec{v} + \mathrm{grad} \, \mathrm{div}\vec{v}$.

The conservation laws for mass, momentum and energy for continuous media can be written in both integral and differential form. They are:

Integral notation:

1. Conservation of mass: $\dfrac{\partial}{\partial t} \iiint \varrho d^3V + \oiint \varrho(\vec{v} \cdot \vec{n}) d^2A = 0$

2. Conservation of momentum: $\dfrac{\partial}{\partial t} \iiint \varrho\vec{v} d^3V + \oiint \varrho\vec{v}(\vec{v} \cdot \vec{n}) d^2A = \iiint f_0 d^3V + \oiint \vec{n} \cdot \mathsf{T} d^2A$

3. Conservation of energy: $\dfrac{\partial}{\partial t}\iiint (\tfrac{1}{2}v^2 + e)\varrho\, d^3V + \oiint (\tfrac{1}{2}v^2 + e)\varrho(\vec{v}\cdot\vec{n}\,)d^2A =$

$$-\oiint (\vec{q}\cdot\vec{n}\,)d^2A + \iiint (\vec{v}\cdot\vec{f_0})d^3V + \oiint (\vec{v}\cdot\vec{n}\,\mathsf{T})d^2A$$

Differential notation:

1. Conservation of mass: $\dfrac{\partial\varrho}{\partial t} + \mathrm{div}\cdot(\varrho\vec{v}\,) = 0$

2. Conservation of momentum: $\varrho\dfrac{\partial\vec{v}}{\partial t} + (\varrho\vec{v}\cdot\nabla)\vec{v} = \vec{f_0} + \mathrm{div}\mathsf{T} = \vec{f_0} - \mathrm{grad}p + \mathrm{div}\mathsf{T}'$

3. Conservation of energy: $\varrho T\dfrac{ds}{dt} = \varrho\dfrac{de}{dt} - \dfrac{p}{\varrho}\dfrac{d\varrho}{dt} = -\mathrm{div}\vec{q} + \mathbb{T}:\mathsf{D}$

Here, e is the internal energy per unit of mass E/m and s is the entropy per unit of mass S/m. $\vec{q} = -\kappa\vec{\nabla}T$ is the heat flow. Further holds:

$$p = -\frac{\partial E}{\partial V} = -\frac{\partial e}{\partial 1/\varrho}\ ,\quad T = \frac{\partial E}{\partial S} = \frac{\partial e}{\partial s}$$

so

$$C_V = \left(\frac{\partial e}{\partial T}\right)_V\quad\text{and}\quad C_p = \left(\frac{\partial h}{\partial T}\right)_p$$

with $h = H/m$ the enthalpy per unit of mass.

From this one can derive the *Navier-Stokes* equations for an incompressible, viscous and heat-conducting medium:

$$\begin{aligned}
\mathrm{div}\vec{v} &= 0\\
\varrho\frac{\partial\vec{v}}{\partial t} + \varrho(\vec{v}\cdot\nabla)\vec{v} &= \varrho\vec{g} - \mathrm{grad}p + \eta\nabla^2\vec{v}\\
\varrho C\frac{\partial T}{\partial t} + \varrho C(\vec{v}\cdot\nabla)T &= \kappa\nabla^2 T + 2\eta\mathsf{D}:\mathsf{D}
\end{aligned}$$

with C the thermal heat capacity. The force $\vec{F}$ on an object within a flow, when viscous effects are limited to the boundary layer, can be obtained using the momentum law. If a surface A surrounds

the object outside the boundary layer holds:

$$\vec{F} = - \oiint [p\vec{n} + \varrho\vec{v}(\vec{v} \cdot \vec{n}\,)]d^2A$$

Exercise 27   **–**   **Questions about advection**

1. A ship is steaming northward at a rate of 10 km/h. The surface pressure increases toward the northwest at a rate of 5 Pa/km. What is the pressure tendency recorded at a nearby island station if the pressure aboard the ship decreases at a rate of 100Pa/3h?

2. The temperature at a point 50 km north of a station is 3°C cooler than at the station. If the wind is blowing from the northeast at 20m/s and the air is being heated by radiation at a rate of 1°C/h, what is the local temperature change at the station?

3. The following data were received from 50 km to the east, north, west and south of a station, respectively: 90 degree, 10m/s; 120 degree,4m/s; 90degree,8m/s; 60 degree, 4m/s. Given are the angle and absolute value of the wind speed. Calculate the approximate horizontal divergence at the station.

4. Let the $x = (x_1, x_2, x_3)$ coordinates be inertial. What are the necessary and sufficient conditions that the coordinates $y_i = A_{ij}x_j + v_j(x, t)t$ be inertial for constant matrix $A = (A_{ij})$?

5. How can the movement of fluid particel be descibed in accordance with Newton's first law? Which forces can create accelerations or decelerations? Please use the definition of specific forces, that is, the force per unit mass: $f = F/m$.

6. The potential temperature in the atmosphere is defined as

$$\Theta = T(p_0/p)^{R/c_P} \tag{4.12}$$

With $p_0 = $ const. Calculate the vertical temperature gradient

$$\gamma = -\frac{dT}{dz} \tag{4.13}$$

What is the result when assuming the hydrostatic equillibrium

$$\frac{dp}{dz} = -g\rho$$

with $g = 9.81 m/s^2$? What is the condition for which the the potential temperature is constant in the vertical?

Solution of 2. Temperature Advection

The total change of temperature is given by

$$\frac{\mathrm{d}T}{\mathrm{d}t} = \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \dot{q}$$
$$\Leftrightarrow \quad \frac{\partial T}{\partial t} = -\mathbf{u} \cdot \nabla T + \dot{q}$$

Here we use the velocity

$$\mathbf{u} = -20\frac{m}{s} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \nabla T = \frac{3°C}{50km}\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \qquad \dot{q} = 1\frac{°C}{h}$$

	Horizontal Length L	Velocity V	Time T
Microturbulence	1-10 cm	1-10 cm/s	seconds
Thunderstorms	1-10 km	10 m/s	hours
Weather patterns	100-1000 km	1-10 m/s	days to weeks
Climatic variations	global	1-10 m/s	decades and beyond

Table 4.1: Table shows the typical scales in the environmental, atmosphere, ocean and climate system. Using these orders of magnitude, one can derive estimates of the timescales.

Then we calculate

$$
\frac{\partial T}{\partial t} = - \mathbf{u} \cdot \boldsymbol{\nabla} T + \dot{q}
$$

$$
= 20 \frac{m}{s} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \frac{3°C}{50km} + 1 \frac{°C}{h}
$$

$$
\approx -2.1 \frac{°C}{h}
$$

Exercise 28   –   Typical scales

Table 28 lists typical velocity, length and time scales of some fluid processes and systems. Not surprisingly, larger systems evolve on longer time scales. Depending on the size of the system under consideration, the spatial scale can be regional, continental or even global. Using the length and velocity scales (L and V), determine a typical time scale (T=L/V)! (Rough estimates are given in the last column in Table 28.)

Exercise 29   –   Weather chart

From the weather chart (Figure 4.2), identify the horizontal extent of a major atmospheric sea level pressure and the associated wind speed. Determine a typical time scale T !
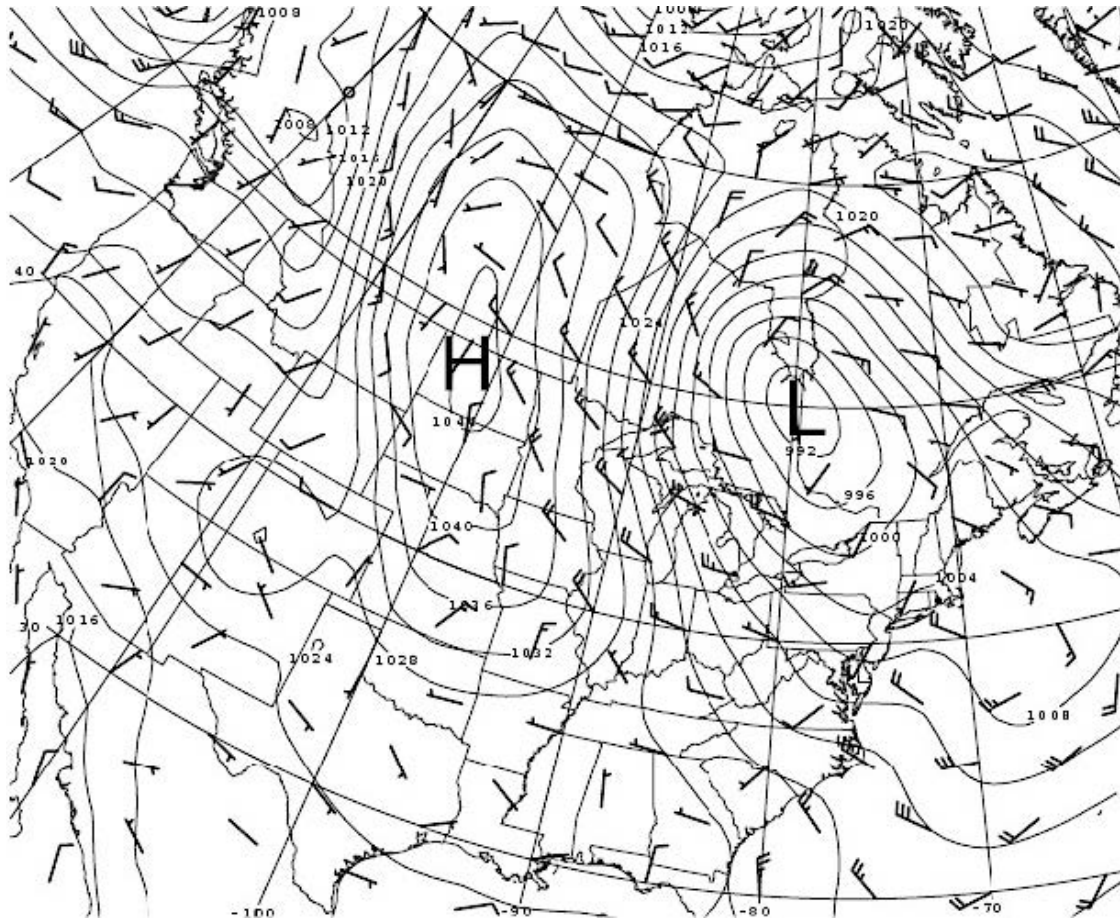
Figure 4.2:   Surface pressure field and surface wind at 12GMT on 10th Feb, 2008. The contour interval is 4mbar. High and low pressure systems are marked as H and L. The dark segments represent wind arrows, whose arrowhead is not drawn in meteorological plots, by convention. The reader should imagine arrowhead at the end of segment that has no quivers. The quivers are drawn at only one side, at the tail end. The wind blows in the direction of the quiver base to the arrowhead. One full quiver represents a wind of 5m/s. We see air circling in a generally anticlockwise direction around the low but spiraling inwards, and air circling in a generally clockwise direction around the high but spiraling outwards.

4.4 Elimination of the pressure term

Taking the curl of the Navier-Stokes equation results in the elimination of pressure. This is especially easy to see if 2D Cartesian flow is assumed ($w = 0$ and no dependence of anything on z), where the equations reduce to:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{4.14}$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \qquad . \tag{4.15}$$

Differentiating the first with respect to y, the second with respect to x and subtracting the resulting equations will eliminate pressure and any potential force. Defining the stream function ψ through

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \tag{4.16}$$

results in mass continuity being unconditionally satisfied (given the stream function is continuous), and then incompressible Newtonian 2D momentum and mass conservation degrade into one equation:

$$\frac{\partial}{\partial t} \left(\nabla^2 \psi \right) + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} \left(\nabla^2 \psi \right) - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y} \left(\nabla^2 \psi \right) = \nu \nabla^4 \psi \tag{4.17}$$

or using the total derivative

$$D_t \left(\nabla^2 \psi \right) = \nu \nabla^4 \psi \tag{4.18}$$

where ∇^4 is the (2D) biharmonic operator and ν is the kinematic viscosity $\nu = \frac{\mu}{\rho}$. This single equation together with appropriate boundary conditions describes 2D fluid flow, taking only kinematic viscosity as a parameter. Note that the equation for creeping flow results when the left

side is assumed zero. In axisymmetric flow another stream function formulation, called the Stokes stream function, can be used to describe the velocity components of an incompressible flow with one scalar function. The concept of taking the curl of the flow will become very important in ocean dynamics (section 6.5). The term $\zeta = \nabla^2 \psi$ is called relative vorticity, its dynamics can be described as

$$D_t \zeta = \nu \nabla^2 \zeta \quad . \tag{4.19}$$

4.5   Non-dimensional parameters: The Reynolds number

For the case of an incompressible flow in the Navier-Stokes equations, assuming the temperature effects are negligible and external forces are neglected, they consist of conservation of mass

$$\nabla \cdot \mathbf{u} = 0 \tag{4.20}$$

and conservation of momentum (4.8).

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0} \nabla p + \nu \nabla^2 \mathbf{u} \tag{4.21}$$

where $\mathbf{u}$ is the velocity vector and p is the pressure, ν denotes the kinematic viscosity. The equations can be made dimensionless by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. For inter-comparison of analytical solutions, numerical results, and of experimental measurements, it is useful to report the results in a dimensionless system. This is justified by the important concept of dynamic similarity (Buckingham [1914]). The main goal for using this system is to replace physical or numerical parameters with some dimensionless numbers, which completely determine the dynamical behavior of the system[4]. The procedure for converting

---

[4]It is this fact that allows engineers to make solid predictions of how a large-scale system would perform based on a miniature model. The dimensionless quantities can often be kept constant when the size of the system is changed by using a fluid with a different viscosity during the tests. The miniature and the "real" flows are then equivalent. The Buckingham π theorem is a key theorem in dimensional analysis. It is a formalization of Rayleigh's method of

to this system first implies, first of all, the selection of some representative values for the physical quantities involved in the original equations (in the physical system). For our current problem, we need to provide representative values for velocity (U), time (T), distances (L). From these, we can derive scaling parameters for the time-derivatives and spatial-gradients also. Using these values, the values in the dimensionless-system (written with subscript d) can be defined:

$$u = U \cdot u_d \tag{4.22}$$

$$t = T \cdot t_d \tag{4.23}$$

$$x = L \cdot x_d \tag{4.24}$$

with $U = L/T$. From these scalings, we can also derive

$$\partial_t = \frac{\partial}{\partial t} = \frac{1}{T} \cdot \frac{\partial}{\partial t_d} \tag{4.25}$$

$$\partial_x = \frac{\partial}{\partial x} = \frac{1}{L} \cdot \frac{\partial}{\partial x_d} \tag{4.26}$$

Note furthermore the units of $[\rho_0] = kg/m^3$, $[p] = kg/(ms^2)$, and $[p]/[\rho_0] = m^2/s^2$. Therefore the pressure gradient term in (4.8) has the scaling U^2/L. Furthermore, devide the equation (4.8) by U^2/L and the scalings vanish completely in front of the terms except for the $\nabla_d^2 u_d$-term! This procedure yields therefore for (4.20,4.21):

$$\nabla_d \cdot u_d = 0 \tag{4.27}$$

---

dimensional analysis. Loosely, the theorem states that if there is a physically meaningful equation involving a certain number n of physical variables, then the original equation can be rewritten in terms of a set of $p = n-k$ dimensionless parameters constructed from the original variables where k is the number of physical dimensions involved. For the system (4.20,4.21), $n = 4$ for velocity, density, pressure, ν; $k = 3$ for mass, length and time; $p = 4 - 3 = 1$ one dimensionless parameter, the Reynolds number.

and conservation of momentum

$$\frac{\partial}{\partial t_d}\mathbf{u}_\mathrm{d} + (\mathbf{u}_\mathrm{d} \cdot \boldsymbol{\nabla}_d)\mathbf{u}_\mathrm{d} = -\boldsymbol{\nabla}_d p_d + \frac{1}{Re}\boldsymbol{\nabla}_d^2\mathbf{u}_\mathrm{d} \tag{4.28}$$

The dimensionless parameter $\boldsymbol{Re} = \boldsymbol{UL}/\boldsymbol{\nu}$ is the Reynolds number and the only parameter left! For large Reynolds numbers, the flow is turbulent. In most practical flows $\boldsymbol{Re}$ is rather large $(10^4 - 10^8)$, large enough for the flow to be turbulent. A large Reynolds number allows the flow to develop steep gradients locally. The typical length-scale corresponding to these steep gradients can become so small that viscousity is not negligible. So the dissipation takes place at small scales. In this way different lengthscales are present in a turbulent flow, which range from L to the Kolmogorov length scale. This length scale is the typical length of the smallest eddy present in a turbulent flow. In the climate system, this dissipation by turbulence is modeled via eddy terms.

In the literature, the term "equations have been made dimensionless", means that this procedure is applied and the subscripts d are dropped.

Remark: For inter-comparison of analytical solutions, numerical results, and of experimental measurements, it is useful to report the results in a dimensionless system. The main goal for using this system is to replace physical or numerical parameters with some dimensionless numbers, which completely determine the dynamical behavior of the system.

Exercise 30   –   Repeat: Concept of dynamic similarity
1. Show: The equations (4.20,4.21) can be made dimensionless by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U.
2. What is the charateristic number? Discuss that it is $\frac{\text{Convective Inertial Force}}{\text{Shear Force}}$. When the number is large, it shows that the flow is dominated by convective inertial effects. When the number is small, it shows that the flow is dominated by shear effects.
3. Please start from the potential vorticity dynamics (4.19) instead of (4.20,4.21). Derive the non-dimensionalized potential vorticity dynamics.

Remark: Later we will include the Coriolis effect (exercise 39).

4.6 Characterising flows by dimensionless numbers

The advantage of dimensionless numbers is that they make model experiments possible: one has to make the dimensionless numbers which are important for the specific experiment equal for both model and the real situation. One can also deduce functional equalities without solving the differential equations. Some dimensionless numbers are given by:

$$\text{Strouhal:} \quad \mathbf{Sr} = \frac{\omega L}{v} \qquad \text{Froude:} \quad \mathbf{Fr} = \frac{v^2}{gL} \qquad \text{Mach:} \qquad \mathbf{Ma} = \frac{v}{c}$$

$$\text{Fourier:} \quad \mathbf{Fo} = \frac{a}{\omega L^2} \qquad \text{Péclet:} \quad \mathbf{Pe} = \frac{vL}{a} \qquad \text{Reynolds:} \quad \mathbf{Re} = \frac{vL}{\nu}$$

$$\text{Prandtl:} \quad \mathbf{Pr} = \frac{\nu}{a} \qquad \text{Nusselt:} \quad \mathbf{Nu} = \frac{L\alpha}{\kappa} \qquad \text{Eckert:} \qquad \mathbf{Ec} = \frac{v^2}{c\Delta T}$$

Here, $\nu = \eta/\varrho$ is the *kinematic viscosity*, c is the speed of sound and L is a characteristic length of the system. α follows from the equation for heat transport $\kappa \partial_y T = \alpha \Delta T$ and $a = \kappa/\varrho c$ is the thermal diffusion coefficient.

These numbers can be interpreted as follows:

- Re: (stationary inertial forces)/(viscous forces)

- Sr: (non-stationary inertial forces)/(stationary inertial forces)

- Fr: (stationary inertial forces)/(gravity)

- Fo: (heat conductance)/(non-stationary change in enthalpy)

- Pe: (convective heat transport)/(heat conductance)

- Ec: (viscous dissipation)/(convective heat transport)

- Ma: (velocity)/(speed of sound): objects moving faster than approximately Ma = 0,8 produce shockwaves which propagate with an angle θ with the velocity of the object. For this angle holds Ma$= 1/\arctan(\theta)$.

- Pr and Nu are related to specific materials.

Now, the dimensionless Navier-Stokes equation becomes, with $x_d = x/L$, $\vec{v}_d = \vec{v}/V$, $\boldsymbol{\nabla}_d = L\boldsymbol{\nabla}_d$, $\boldsymbol{\nabla}_d^2 = L^2\boldsymbol{\nabla}^2$ and $t_d = t\omega$:

$$\text{Sr}\frac{\partial \vec{v}_d}{\partial t_d} + (\vec{v}_d \cdot \boldsymbol{\nabla}_d)\vec{v}_d = -\boldsymbol{\nabla}_d p_d + \frac{\vec{g}}{\text{Fr}} + \frac{\boldsymbol{\nabla}_d^2 \vec{v}_d}{\text{Re}} \tag{4.29}$$

4.7   Dynamic similarity: Application in engineering*

Engineering models are used to study complex fluid dynamics problems where calculations and computer simulations are not reliable. Models are usually smaller than the final design, but not always. Scale models allow testing of a design prior to building, and in many cases are a critical step in the development process. Construction of a scale model, however, must be accompanied by an analysis to determine what conditions it is tested under. While the geometry may be simply scaled, other parameters, such as pressure, temperature or the velocity and type of fluid may need to be altered. Similitude is achieved when testing conditions are created such that the test results are applicable to the real design. The following criteria are required:

1. Geometric similarity: The model is the same shape as the application, usually scaled.
2. Kinematic similarity: Fluid flow of both the model and real application must undergo similar time rates of change motions. (fluid streamlines are similar)
3. Dynamic similarity: Ratios of all forces acting on corresponding fluid particles and boundary surfaces in the two systems are constant.

Dimensional analysis is used to express the system with as few independent variables and as many dimensionless parameters as possible. The values of the dimensionless parameters are held to be the same for both the scale model and application. The design of marine vessels remains more of an art than a science in large part because dynamic similitude is especially difficult to attain for a vessel that is partially submerged: a ship is affected by wind forces in the air above it, by hydrodynamic forces within the water under it, and especially by wave motions at the interface

Variable	Application	Scaled model	Units
L (diameter of submarine)	1	1/40	(m)
V (speed)	5	calculate	(m/s)
ρ (density)	1028	988	$(kg/m3)$
μ (dynamic viscosity)	$1.88 \cdot 10^{-3}$	$1.0 \cdot 10^{-3}$	$Pa \cdot s(Ns/m^2)$
F (force)	calculate	to be measured	N (kgm/s^2)

Table 4.2: Table shows the typical scales for the submarine model.

between the water and the air. The scaling requirements for each of these phenomena differ, so models cannot replicate what happens to a full sized vessel nearly so well as can be done for an aircraft or submarin–each of which operates entirely within one medium.

As an example, consider a submarine modeled at 1/40th scale. The application operates in sea water at $0.5°C$, moving at $5m/s$. The model will be tested in fresh water at $20°C$. Find the power required for the submarine to operate at the stated speed. A free body diagram is constructed and the relevant relationships of force and velocity are formulated. The variables which describe the system are listed in Table 4.2. This example has five independent variables and three fundamental units. The fundamental units are: metre, kilogram, second. Invoking the Buckingham π theorem shows that the system can be described with two dimensionless numbers and one independent variable. Dimensional analysis is used to re-arrange the units to form the Reynolds number (Re) and so-called pressure coefficient (pc). The pressure coefficient is a parameter for studying the flow of incompressible fluids such as water, and also the low-speed flow of compressible fluids such as air. The relationship between the dimensionless coefficient and the dimensional numbers is

$$pc = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{p - p_\infty}{p_0 - p_\infty} \qquad (4.30)$$

where:

p is the static pressure at the point at which pressure coefficient is being evaluated

p_∞ is the static pressure in the freestream (i.e. remote from any disturbance)

p_0 is the stagnation pressure in the freestream (i.e. remote from any disturbance)

ρ_∞ is the freestream fluid density

V_∞ is the freestream velocity of the fluid, or the velocity of the body through the fluid.

Scaling laws:

$$Re = \left(\frac{\rho V L}{\mu}\right) \quad\quad\quad \longrightarrow V_{\text{model}} = V_{\text{application}} \times \left(\frac{\rho_a}{\rho_m}\right) \times \left(\frac{L_a}{L_m}\right) \times \left(\frac{\mu_m}{\mu_a}\right)$$

$$\tag{4.31}$$

$$pc = \left(\frac{2\Delta p}{\rho V^2}\right), F = \Delta p L^2 \quad \longrightarrow F_{\text{application}} = F_{\text{model}} \times \left(\frac{\rho_a}{\rho_m}\right) \times \left(\frac{V_a}{V_m}\right)^2 \times \left(\frac{L_a}{L_m}\right)^2.$$

$$\tag{4.32}$$

The pressure (p) is not one of the five variables, but the force (F) is. The pressure difference has thus been replaced with (F/L^2) in the pressure coefficient. This gives a required test velocity of:

$$V_{\text{model}} = V_{\text{application}} \times 21.9.$$

A model test is then conducted at that velocity and the force that is measured in the model (F_{model}) is then scaled to find the force that can be expected for the real application $(F_{application})$:

$$F_{\text{application}} = F_{\text{model}} \times 3.44$$

The power P in Watt required by the submarine is then:

$$P[\text{W}] = F_{\text{application}} \times V_{\text{application}} = F_{\text{model}}[\text{N}] \times 17.2 \text{ m/s}$$

Note that even though the model is scaled smaller, the water velocity needs to be increased for testing. This remarkable result shows how similitude in nature is often counterintuitive.

Similitude has been well documented for a large number of engineering problems and is the basis of many textbook formulas and dimensionless quantities. These formulas and quantities are easy to use without having to repeat the laborious task of dimensional analysis and formula

derivation. Similitude can be used to predict the performance of a new design based on data from an existing, similar design. In this case, the model is the existing design. Another use of similitude and models is in validation of computer simulations with the ultimate goal of eliminating the need for physical models altogether. Another application of similitude is to replace the operating fluid with a different test fluid. Wind tunnels, for example, have trouble with air liquefying in certain conditions so helium is sometimes used. Other applications may operate in dangerous or expensive fluids so the testing is carried out in a more convenient substitute.

Chapter 5

Fluid-dynamical Examples

5.1 Potential flow

In fluid dynamics, potential flow describes the velocity field as the gradient of a scalar function: the velocity potential. As a result, a potential flow is characterized by an irrotational velocity field, which is a valid approximation for several applications. The irrotationality of a potential flow is due to the curl of a gradient always being equal to zero. In the case of an incompressible flow the velocity potential satisfies Laplace's equation. However, potential flows also have been used to describe compressible flows. The potential flow approach occurs in the modeling of both stationary as well as nonstationary flows.

A potential flow is described by means of a velocity potential, being a function of space and time. The flow velocity v is a vector field equal to the gradient of the velocity potential ϕ

$$\mathbf{v} = \boldsymbol{\nabla}\phi. \tag{5.1}$$

From vector calculus it is known, that the curl of a gradient is equal to zero:

$$\boldsymbol{\nabla} \times \boldsymbol{\nabla}\phi = \mathbf{0}, \tag{5.2}$$

and consequently the vorticity, the curl of the velocity field v, is zero:

$$\nabla \times \mathbf{v} = \mathbf{0}.$$

(5.3)

This implies that a potential flow is an irrotational flow. This has direct consequences for the applicability of potential flow. In flow regions where vorticity is known to be important, such as wakes and boundary layers, potential flow theory is not able to provide reasonable predictions of the flow. Fortunately, there are often large regions of a flow where the assumption of irrotationality is valid, which is why potential flow is used for various applications.[1]

In case of an incompressible flow[2] the velocity v has zero divergence:

$$\nabla \cdot \mathbf{v} = \mathbf{0},$$

(5.4)

with the dot denoting the inner product. As a result, the velocity potential satisfies Laplace's equation

$$\nabla^2 \phi = \mathbf{0} \quad .$$

(5.5)

In this case the flow can be determined completely from its kinematics: the assumptions of irrotationality and zero divergence of the flow. Dynamics only have to be applied afterwards, if one is interested in computing pressures: for instance for flow around airfoils through the use of Bernoulli's principle. In two dimensions, potential flow reduces to a very simple system that is analyzed using complex analysis (section 5.1.6).

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[1] For instance in: flow around aircraft, groundwater flow, acoustics and water waves.
[2] for instance of a liquid, or a gas at low Mach numbers; but not for sound waves

5.1.1   Kelvin's circulation theorem*

In fluid mechanics, Kelvin's circulation theorem states In a barotropic ideal fluid with conservative body forces, the circulation around a closed curve (which encloses the same fluid elements) moving with the fluid remains constant with time.

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = 0 \tag{5.6}$$

where the circulation Γ is the circulation around a material contour

$$\Gamma = \oint (\vec{v} \cdot \vec{e}_{\mathrm{t}}) ds \tag{5.7}$$

The circulation is the line integral of the tangential component of velocity taken about a closed curve in the flow field. The integral is taken in a counterclockwise direction about the contour C and ds is a differential length along the contour. No singularities can lie directly on the contour. The origin (center) of the potential vortex is considered as a singularity point in the flow since the velocity goes to infinity at this point. If the contour encircles the potential vortex origin, the circulation will be non-zero. If the contour does not encircle any singularities, however, the circulation will be zero. Stated more simply this theorem says that if one observes a closed contour at one instant, and follows the contour over time (by following the motion of all of its fluid elements), the circulation over the two locations of this contour are equal. This theorem does not hold in cases with viscous stresses, nonconservative body forces (for example Coriolis force) or non-barotropic pressure-density relations.

In the case of a potential flow, the vorticity is zero (5.3), Kelvin's theorem can be derived using

$$\Gamma = \iint (\nabla \times \vec{v}) \cdot \vec{n} \, d^2 A = 0 \tag{5.8}$$

Exercise 31   **–**   **Circulation theorem**

Show (5.6) using that the governing equation for an inviscid fluid with a conservative body force is

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\frac{1}{\rho}\nabla p + \nabla \Phi$$

where Φ is the potential for the body force.

Hint: Potential flow, Kelvin's theorem

5.1.2   Streamlines

For a 2-dimensional flow a flow function $\psi(x, y)$ can be defined: $u = \partial\psi/\partial y, v = -\partial\psi/\partial x$.
With Ψ_{AB} the amount of liquid flowing through a curve s between the points A and B:

$$\Psi_{AB} \;=\; \int_A^B (\vec{v} \cdot \vec{n})ds = \int_A^B (udy - vdx) \tag{5.9}$$

$$=\; \int_A^B d\psi = \psi(B) - \psi(A) \quad . \tag{5.10}$$

The lines of constant $\phi = 0$ are called potential lines of the flow.

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy = udx + vdy \tag{5.11}$$

Since $d\phi = 0$ along a potential line, we have

$$\frac{dy}{dx} = -\frac{u}{v} \tag{5.12}$$

Recall that streamlines are lines everywhere tangent to the velocity,

$$\frac{dy}{dx} = \frac{u}{v} \tag{5.13}$$

so potential lines are perpendicular to the streamlines. For inviscid and irrotational flow is indeed quite pleasant to use potential function to represent the velocity field. As a point to note here, many texts use stream function instead of potential function as it is slightly more intuitive to consider a line that is everywhere tangent to the velocity.

5.1.3   Bernoulli's equations*

Starting with the momentum equation one can find for a non-viscous medium for stationary flows, with

$$(\vec{v} \cdot \nabla)\vec{v} = \tfrac{1}{2}\nabla(v^2) + (\mathbf{rot}\vec{v}) \times \vec{v}$$

and the potential equation $\vec{g} = -\nabla(gh)$ that:

$$\tfrac{1}{2}v^2 + gh + \int \frac{dp}{\varrho} = \text{constant along a streamline}$$

For compressible flows holds: $\tfrac{1}{2}v^2 + gh + p/\varrho =$ constant along a line of flow. If also holds $\mathbf{rot}\vec{v} = 0$ and the entropy is equal on each streamline holds $\tfrac{1}{2}v^2 + gh + \int dp/\varrho =$ constant everywhere. For incompressible flows this becomes:

$$\tfrac{1}{2}v^2 + gh + p/\varrho = \text{constant everywhere.} \tag{5.14}$$

For ideal gases with constant C_p and C_V holds, with $\gamma = C_p/C_V$:

$$\tfrac{1}{2}v^2 + \frac{\gamma}{\gamma - 1}\frac{p}{\varrho} = \tfrac{1}{2}v^2 + \frac{c^2}{\gamma - 1} = \text{constant}$$

With a velocity potential defined by $\vec{v} = \mathbf{grad}\phi$ holds for instationary flows:

$$\frac{\partial \phi}{\partial t} + \tfrac{1}{2}v^2 + gh + \int \frac{dp}{\varrho} = \text{constant everywhere}$$
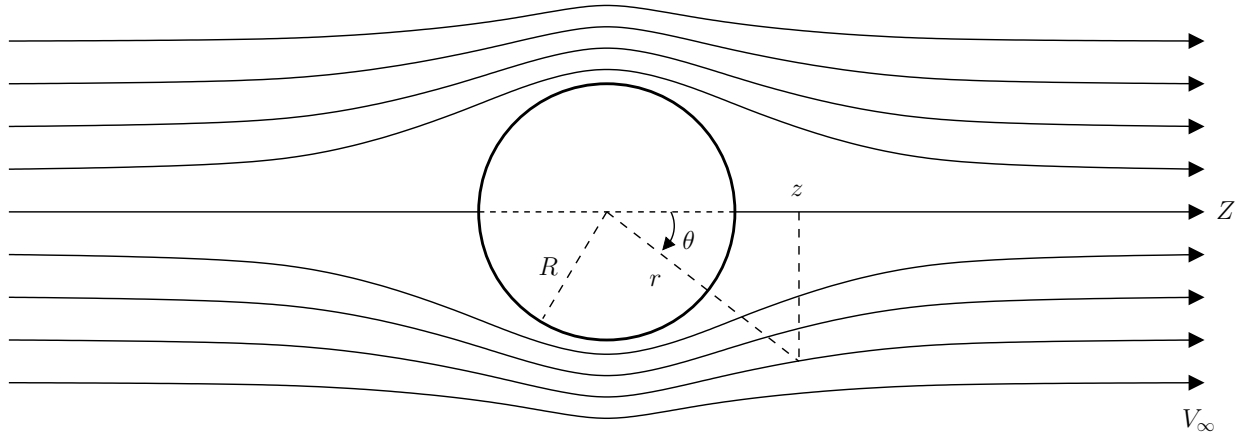
Figure 5.1: Streamlines for the incompressible potential flow around a circular cylinder in a uniform onflow.

5.1.4   Bernoulli flow*

The solution for ϕ is obtained in polar coordinates r and θ, related to conventional Cartesian coordinates by $x = r\cos\theta$ and $y = r\sin\theta$. In polar coordinates, Laplace's equation is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0 \tag{5.15}$$

The solution that satisfies the boundary conditions is

$$\phi(r,\theta) = U\left(r + \frac{R^2}{r}\right)\cos\theta. \tag{5.16}$$

The velocity components in polar coordinates are obtained from the components of $\nabla\phi$ in polar coordinates:

$$V_r = \frac{\partial\phi}{\partial r} = U\left(1 - \frac{R^2}{r^2}\right)\cos\theta \tag{5.17}$$

and

$$V_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -U\left(1 + \frac{R^2}{r^2}\right)\sin\theta. \tag{5.18}$$

Being invisicid and irrotational, Bernoulli's equation (5.14) allows the solution for pressure field to be obtained directly from the velocity field:

$$p = \frac{1}{2}\rho \left(U^2 - V^2\right) + p_\infty, \tag{5.19}$$

where the constants U and p_∞ appear so that $p \to p_\infty$ far from the cylinder, where $V = U$. Using

$$V^2 = V_r^2 + V_\theta^2, \tag{5.20}$$

$$p = \frac{1}{2}\rho U^2 \left(2\frac{R^2}{r^2}\cos(2\theta) - \frac{R^4}{r^4}\right) + p_\infty. \tag{5.21}$$

In Fig. 5.2, the colorized field referred to as "pressure" is a plot of

$$2\frac{p - p_\infty}{\rho U^2} = 2\frac{R^2}{r^2}\cos(2\theta) - \frac{R^4}{r^4}. \tag{5.22}$$

On the surface of the cylinder, or $r = R$, pressure varies from a maximum of 1 (red color) at the stagnation points at $\theta = 0$ and $\theta = \pi$ to a minimum of -3 (purple) on the sides of the cylinder, at $\theta = \frac{1}{2}\pi$ and $\theta = \frac{3}{2}\pi$. Likewise, V varies from $V = 0$ at the stagnation points to $V = 2U$ on the sides, in the low pressure.

The flow being incompressible, a stream function can be found such that $\vec{V} = \boldsymbol{\nabla}\psi \times \hat{k}$. It follows from this definition, using vector identities, $\vec{V} \cdot \boldsymbol{\nabla}\psi = 0$. Therefore a contour of a constant value of ψ will also be a stream line, a line tangent to $\vec{V}$. For the flow past a cylinder, we find:

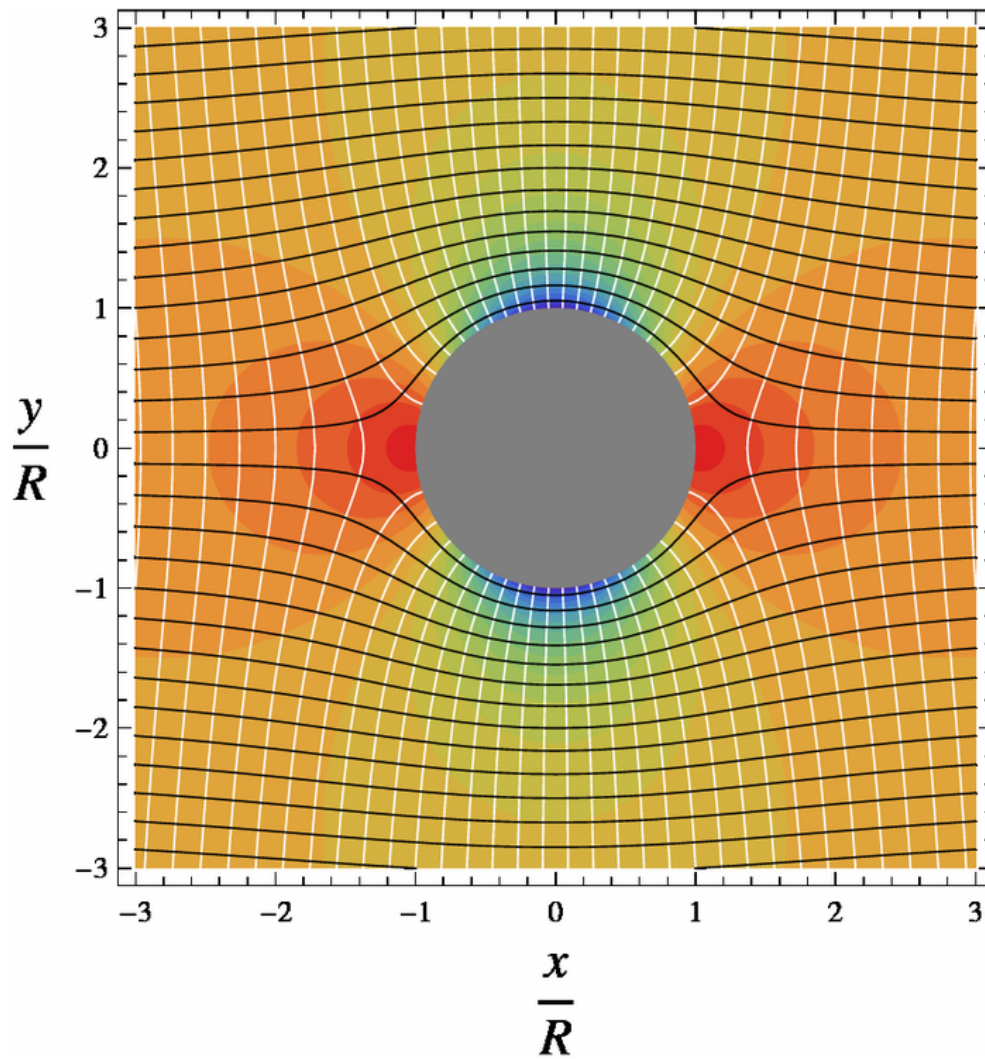$$\psi = U \left(r - \frac{R^2}{r}\right)\sin\theta. \tag{5.23}$$

Figure 5.2: Pressure field (colors), stream function (black) with contour interval of $0.2Ur$ from bottom to top, velocity potential (white) with contour interval $0.2Ur$ from left to right.

Physical interpretation

Laplace's equation is linear, and is one of the most elementary partial differential equations. The dynamic pressure at the upstream stagnation point has value of $\rho U^2/2$, a value needed to decelerate the free stream flow of speed U. This same value appears at the downstream stagnation point, this high pressure is again need to decelerate the flow to zero speed. This symmetry arises only because the flow is completely frictionless. The low pressure on sides on the cylinder is needed to provide the centripetal acceleration of the flow:

$$\frac{\partial p}{\partial r} = \frac{\rho V^2}{L},\tag{5.24}$$

where L is the radius of curvature of the flow. But $L \approx R$, and $V \approx U$. The integral of the equation for centripetal acceleration, which will over a distance $\Delta r \approx R$ will thus yield

$$p - p_\infty \approx -\rho U^2.\tag{5.25}$$

The exact solution has, for the lowest pressure,

$$p - p_\infty = -\frac{3}{2}\rho U^2.\tag{5.26}$$

The low pressure, which must be present to provide the centripetal acceleration, will also increase the flow speed as the fluid travels from higher to lower values of pressure. Thus we find the maximum speed in the flow, $V = 2U$, in the low pressure on the sides of the cylinder. A value of $V > U$ is consistent with conservation of the volume of fluid. With the cylinder blocking some of the flow, V must be greater than U somewhere in the plane through the center of the cylinder and transverse to the flow.

5.1.5 Comparison with flow of a real fluid past a cylinder*

This symmetry of this ideal solution has the peculiar property of having zero net drag on the cylinder, a property known as d'Alembert's paradox. Unlike an ideal inviscid fluid, a viscous flow past a cylinder, no matter how small the viscosity, will acquire vorticity in a thin boundary layer adjacent to the cylinder. Boundary layer separation can occur, and a trailing wake will occur behind the cylinder. The pressure will be lower on the wake side of the cylinder, than on the upstream side, resulting in a drag force in the downstream direction. A particular aspect are the Von Karman Vortices.

Fig. 5.3 features a ubiquitous occurrence in the motion of fluids–a vortex street, which is a linear chain of spiral eddies called von Karman vortices. Von Karman vortices are named after Theodore von Karman, who first described the phenomenon in the atmosphere. von Karman vortices form nearly everywhere that fluid flow is disturbed by an object and form at all scales of fluid motion. The "object" that is disturbing the fluid flow is an island or group of islands. As a prevailing wind encounters the island, the disturbance in the flow propagates downstream of the island in the form of a double row of vortices which alternate their direction of rotation.

As a fluid particle flows toward the leading edge of a cylinder, the pressure on the particle rises from the free stream pressure to the stagnation pressure. The high fluid pressure near the leading edge impels flow about the cylinder as boundary layers develop about both sides. The high pressure is not sufficient to force the flow about the back of the cylinder at high Reynolds numbers. Near the widest section of the cylinder, the boundary layers separate from each side of the cylinder surface and form two shear layers that trail aft in the flow and bound the wake. Since the innermost portion of the shear layers, which is in contact with the cylinder, moves much more slowly than the outermost portion of the shear layers, which is in contact with the free flow, the shear layers roll into the near wake, where they fold on each other and coalesce into discrete swirling vortices. A regular pattern of vortices, called a vortex street, trails aft in the wake.
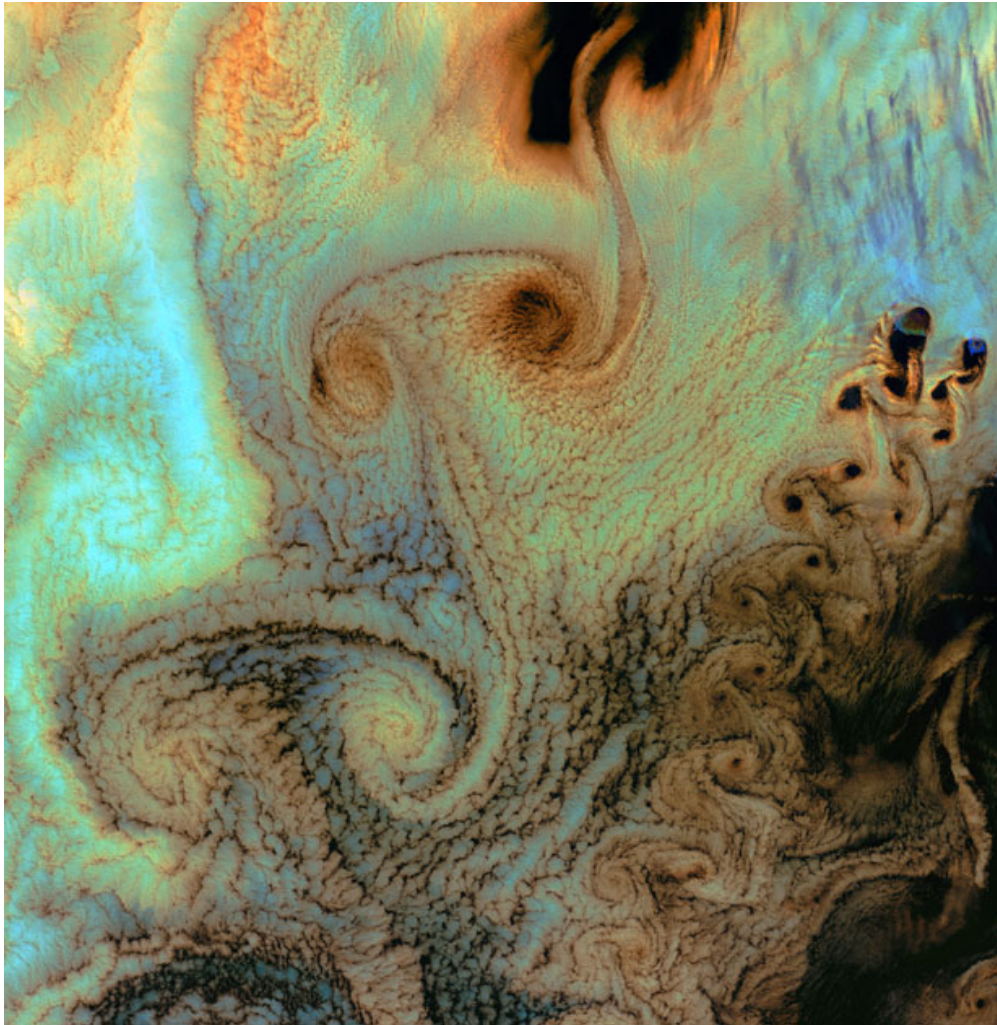
Figure 5.3:   Von Karman Vortices - As air flows over and around objects in its path, spiraling eddies, known as Von Karman vortices, may form. The vortices in this image were created when prevailing winds sweeping east across the northern Pacific Ocean encountered Alaska's Aleutian Islands. The image is from the Landsat 7 satellite.

5.1.6 Analysis for two-dimensional flow using conformal mapping*

Potential flow does not include all the characteristics of flows that are encountered in the real world. Potential flow theory cannot be applied for viscous internal flows. Richard Feynman considered potential flow to be so unphysical that the only fluid to obey the assumptions was "dry water" (quoting John von Neumann). More precisely, potential flow cannot account for the behaviour of flows that include a boundary layer. Nevertheless, understanding potential flow is important in many branches of fluid mechanics. In particular, simple potential flows (called elementary flows) such as the free vortex and the point source possess ready analytical solutions. These solutions can be superposed to create more complex flows satisfying a variety of boundary conditions. These flows correspond closely to real-life flows over the whole of fluid mechanics; in addition, many valuable insights arise when considering the deviation (often slight) between an observed flow and the corresponding potential flow. Potential flow finds many applications in fields such as aircraft design. For instance, in computational fluid dynamics, one technique is to couple a potential flow solution outside the boundary layer to a solution of the boundary layer equations inside the boundary layer.

Potential flow in two dimensions is simple to analyze using conformal mapping, by the use of transformations of the complex plane. The basic idea is to use a holomorphic (also called analytic) or meromorphic function f, which maps the physical domain (x,y) to the transformed domain (ϕ, ψ). While x, y, ϕ, ψ are all real valued, it is convenient to define the complex quantities $z = x + iy$ and $w = \phi + i\psi$. Now, if we write the mapping f as $f(x + iy) = \phi + i\psi$ or $f(z) = w$. Then, because f is a holomorphic function, it has to satisfy the Cauchy-Riemann equations

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{5.27}$$

The velocity components (u,v), in the (x,y) directions respectively, can be obtained directly from f

by differentiating with respect to z. That is

$$\frac{df}{dz} = u - iv \tag{5.28}$$

So the velocity field (u,v) is specified by

$$u = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \qquad v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}. \tag{5.29}$$

Both φ and ψ then satisfy Laplace's equation:

$$\Delta\varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad \text{and} \qquad \Delta\psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0. \tag{5.30}$$

So φ can be identified as the velocity potential and ψ is called the stream function. Lines of constant ψ are known as streamlines and lines of constant φ are known as equipotential lines. Streamlines and equipotential lines are orthogonal to each other, since

$$\nabla\phi \cdot \nabla\psi = \frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y}\frac{\partial \psi}{\partial y} = \frac{\partial \psi}{\partial y}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \psi}{\partial y} = 0. \tag{5.31}$$

Thus the flow occurs along the lines of constant ψ and at right angles to the lines of constant φ. It is interesting to note that $\Delta\psi = 0$ is also satisfied, this relation being equivalent to $\nabla \times v = 0$.

Exercise 32   –   Conformal mapping

We note that the complex velocity potential must be an analytic function respecting the boundary conditions, and once we have it, we can easily obtain the flow field. Let us see how we can use this fact to solve some basic fluid mechanics problems. In case the following power-law conformal map is applied, from $z = x + iy$ to $w = \phi + i\psi$:

$$w = Az^n, \tag{5.32}$$

then, writing z in polar coordinates as $z = x + iy = re^{i\Theta}$, we have

$$\varphi = Ar^n \cos(n\theta) \text{ and } \psi = Ar^n \sin(n\theta). \tag{5.33}$$

Study the cases $n = 1/2, 2/3, 3/2, 2, 3$ and draw the streamlines and equipotential!

Hint: web site for conformal mapping

Solution $n = 1$: uniform flow

Uniform flow: $v = V$ If w = Az, that is, a power law with n = 1, the streamlines (i.e. lines of constant ψ) are a system of straight lines parallel to the x-axis. This is easiest to see by writing in terms of real and imaginary components: $f(x + iy) = Uz = Ux + iUy$ thus giving $\phi = Ux$ and $\psi = Uy$. This flow may be interpreted as uniform flow parallel to the x-axis.

Think on the problem of flow around a corner. What is a consistent flow pattern past a corner according to the ideal fluid conditions? $f(z) = Uz^2$ Why ? One uses analytic functions to map a fluids problem (or more generally a Laplace equation problem) from a given domain to a domain on which the problem is solved.

Another problem where we know the solution from the last section: Flow around a cylinder with $f(z) = U(z + 1/z)$.

One of the more important potential flow results obtained using conformal mapping begins with the known solution for the flow past a circular cylinder (with circulation) and maps the circle into an airfoil shape using waht is called the Joukowski mapping.

5.2   More on fluid flows*

5.2.1   Tube flows*

For tube flows holds: they are laminar if Re$<$ **2300** with dimension of length the diameter of the tube, and turbulent if Re is larger. For an incompressible laminar flow through a straight, circular tube holds for the velocity profile:

$$v(r) = -\frac{1}{4\eta}\frac{dp}{dx}(R^2 - r^2)$$

For the volume flow holds: $\Phi_V = \int\limits_0^R v(r)2\pi r dr = -\frac{\pi}{8\eta}\frac{dp}{dx}R^4$

The *entrance length* L_e is given by:

1. $500 < \mathrm{Re}_D < 2300$: $L_e/2R = 0.056\mathrm{Re}_D$

2. $\mathrm{Re} > 2300$: $L_e/2R \approx 50$

For gas transport at low pressures (Knudsen-gas) holds: $\Phi_V = \dfrac{4R^3\alpha\sqrt{\pi}}{3}\dfrac{dp}{dx}$

For flows at a small Re holds: $\nabla p = \eta\nabla^2\vec{v}$ and div$\vec{v} = 0$. For the total force on a sphere with radius R in a flow then holds: $F = 6\pi\eta Rv$. For large Re holds for the force on a surface A: $F = \frac{1}{2}C_W A\varrho v^2$.

5.2.2   Boundary layers*

If for the thickness of the boundary layer holds: $\delta \ll L$ holds: $\delta \approx L/\sqrt{\mathrm{Re}}$. With v_∞ the velocity of the main flow it follows for the velocity $v_y \perp$ the surface: $v_y L \approx \delta v_\infty$. Blasius' equation for the boundary layer is, with $v_y/v_\infty = f(y/\delta)$: $2f''' + ff'' = 0$ with boundary conditions $f(0) = f'(0) = 0$, $f'(\infty) = 1$. From this follows: $C_W = 0.664\,\mathrm{Re}_x^{-1/2}$.

The momentum theorem of Von Karman for the boundary layer is: $\dfrac{d}{dx}(\vartheta v^2) + \delta^* v\dfrac{dv}{dx} = \dfrac{\tau_0}{\varrho}$

where the displacement thickness $\delta^* v$ and the momentum thickness ϑv^2 are given by:

$$\vartheta v^2 = \int_0^\infty (v - v_x) v_x dy \;, \quad \delta^* v = \int_0^\infty (v - v_x) dy \; \text{ and } \; \tau_0 = -\eta \left. \frac{\partial v_x}{\partial y} \right|_{y=0}$$

The boundary layer is released from the surface if $\left(\dfrac{\partial v_x}{\partial y} \right)_{y=0} = 0$. This is equivalent with $\dfrac{dp}{dx} = \dfrac{12\eta v_\infty}{\delta^2}$.

If the thickness of the temperature boundary layer $\delta_T \ll L$ holds: 1. If $\mathrm{Pr} \leq 1$: $\delta/\delta_T \approx \sqrt{\mathrm{Pr}}$.

2. If $\mathrm{Pr} \gg 1$: $\delta/\delta_T \approx \sqrt[3]{\mathrm{Pr}}$.

5.2.3 Heat conductance*

For non-stationairy heat conductance in one dimension without flow holds:

$$\frac{\partial T}{\partial t} = \frac{\kappa}{\varrho c} \frac{\partial^2 T}{\partial x^2} + \Phi$$

where Φ is a source term. If $\Phi = 0$ the solutions for harmonic oscillations at $x = 0$ are:

$$\frac{T - T_\infty}{T_{\max} - T_\infty} = \exp\left(-\frac{x}{D} \right) \cos\left(\omega t - \frac{x}{D} \right)$$

with $D = \sqrt{2\kappa/\omega\varrho c}$. At $x = \pi D$ the temperature variation is in anti-phase with the surface. The one-dimensional solution at $\Phi = 0$ is

$$T(x,t) = \frac{1}{2\sqrt{\pi a t}} \exp\left(-\frac{x^2}{4at} \right)$$

This is mathematical equivalent to the diffusion problem:

$$\frac{\partial n}{\partial t} = D \nabla^2 n + P - A$$

where P is the production of and A the discharge of particles. The flow density $J = -D\nabla n$.

5.2.4   Turbulence*

The time scale of turbulent velocity variations τ_t is of the order of: $\tau_t = \tau\sqrt{\mathrm{Re}}/\mathrm{Ma}^2$ with τ the molecular time scale. For the velocity of the particles holds: $v(t) = \langle v \rangle + v'(t)$ with $\langle v'(t) \rangle = 0$. The Navier-Stokes equation now becomes:

$$\frac{\partial \langle \vec{v} \rangle}{\partial t} + (\langle \vec{v} \rangle \cdot \nabla) \langle \vec{v} \rangle = -\frac{\nabla \langle p \rangle}{\varrho} + \nu\nabla^2 \langle \vec{v} \rangle + \frac{\mathrm{div}\mathsf{S}_R}{\varrho}$$

where $\mathsf{S}_{Rij} = -\varrho \langle v_i v_j \rangle$ is the turbulent stress tensor. Boussinesq's assumption is: $\tau_{ij} = -\varrho \left\langle v_i' v_j' \right\rangle$. It is stated that, analogous to Newtonian media: $\mathsf{S}_R = 2\varrho\nu_t \langle \mathsf{D} \rangle$. Near a boundary holds: $\nu_t = 0$, far away of a boundary holds: $\nu_t \approx \nu\mathrm{Re}$.

For a (semi) two-dimensional flow holds:

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + J(\omega, \psi) = \nu\nabla^2\omega \tag{5.34}$$

With $J(\omega, \psi)$ the Jacobian. So if $\nu = 0$, ω is conserved. Further, the kinetic energy$/mA$ and the enstrofy V are conserved: with $\vec{v} = \nabla \times (\vec{k}\psi)$

$$E \sim (\nabla\psi)^2 \sim \int\limits_0^\infty \mathcal{E}(k, t)dk = \text{constant} \;,\;\; V \sim (\nabla^2\psi)^2 \sim \int\limits_0^\infty k^2\mathcal{E}(k, t)dk = \text{constant}$$

From this follows that in a two-dimensional flow the energy flux goes towards large values of k: larger structures become larger at the expanse of smaller ones. In three-dimensional flows the situation is just the opposite.

5.2.5 Couette flow*

Couette flow refers to the laminar flow of a viscous fluid in the space between two parallel plates, one of which is moving relative to the other. The flow is driven by virtue of viscous drag force acting on the fluid and the applied pressure gradient parallel to the plates. This type of flow is named in honor of Maurice Marie Alfred Couette, a Professor of Physics at the French university of Angers in the late 19th century. Couette flow is frequently used in undergraduate physics and engineering courses to illustrate shear-driven fluid motion. The simplest conceptual configuration finds two infinite, parallel plates separated by a distance h. One plate, say the top one, translates with a constant velocity u0 in its own plane. Neglecting pressure gradients, the Navier-Stokes equations simplify to

$$\frac{d^2 u}{dy^2} = 0, \tag{5.35}$$

where y is a spatial coordinate normal to the plates and u (y) is the velocity distribution. This equation reflects the assumption that the flow is uni-directional. That is, only one of the three velocity components (u,v,w) is non-trivial. If y originates at the lower plate, the boundary conditions are $u(0) = 0$ and $u(h) = u_0$. The exact solution

$$u(y) = u_0 \frac{y}{h} \tag{5.36}$$

can be found by integrating twice and solving for the constants using the boundary conditions.

A notable aspect of this model is that shear stress is constant throughout the flow domain. In particular, the first derivative of the velocity, u_0/h, is constant. (This is implied by the straight-line profile in the figure.) According to Newton's Law of Viscosity (Newtonian fluid), the shear stress is the product of this expression and the (constant) fluid viscosity.

A more general Couette flow situation arises when a pressure gradient is imposed in a direction
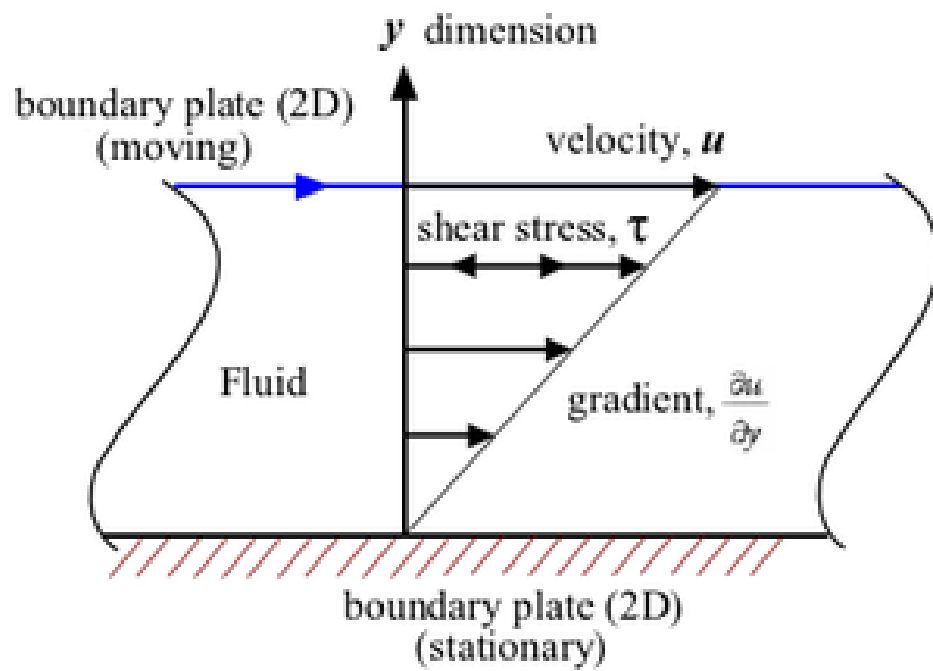
Figure 5.4: Simple Couette configuration using two infinite flat plates.

parallel to the plates. The Navier-Stokes equations, in this case, simplify to

$$\frac{d^2u}{dy^2} = \frac{1}{\mu}\frac{dp}{dx}, \tag{5.37}$$

where dp/dx is the pressure gradient parallel to the plates and μ is fluid viscosity. Integrating the above equation twice and applying the boundary conditions (same as in the case of Couette flow without pressure gradient) to yield the following exact solution

$$u(y) = u_0\frac{y}{h} + \frac{1}{2\mu}\left(\frac{dp}{dx}\right)(y^2 - hy). \tag{5.38}$$

The shape of the above velocity profile depends on the dimensionless parameter

$$P = -\frac{h^2}{2\mu u_0}\left(\frac{dp}{dx}\right). \tag{5.39}$$

The pressure gradient can be positive (adverse pressure gradient) or negative (favorable pressure gradient). It may be noted that in the limiting case of stationary plates, the flow is referred to as plane Poiseuille flow with a symmetric (with reference to the horizontal mid-plane) parabolic velocity profile.

In fluid dynamics, the Taylor-Couette flow consists of a viscous fluid confined in the gap between two rotating cylinders. For low angular velocities, measured by the Reynolds number Re, the flow is steady and purely azimuthal. This basic state is known as circular Couette flow, after Maurice Marie Alfred Couette who used this experimental device as a means to measure viscosity. Sir Geoffrey Ingram Taylor investigated the stability of the Couette flow in a ground-breaking paper which has been a cornerstone in the development of hydrodynamic stability theory.

5.3   Convection in the Rayleigh-Bénard system

5.3.1   Geometry and set up

Here, we shall introduce a system of three ordinary differential equations whose solutions afford the simplest example of deterministic flow that we are aware of. The system is a simplification of the one derived by Saltzman [1962], to study finite-amplitude convection. Although our present interest is in the non-periodic nature of its solutions rather than in its contributions to the convection problem, we shall describe its physical background briefly.
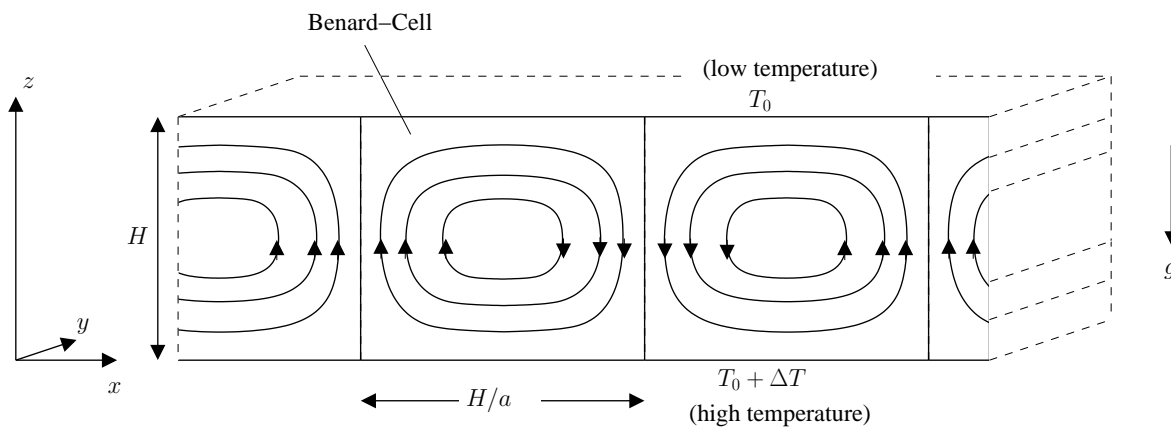
Figure 5.5: Geometry of the Rayleigh-Bénard system (see text for details).

Rayleigh [1916] studied the flow occurring in a layer of fluid of uniform depth H, when the temperature difference between the upper- and lower-surfaces is maintained at a constant value ΔT.

$$T(x, y, z = H) = T_0$$
$$T(x, y, z = 0) = T_0 + \Delta T \tag{5.40}$$

The Boussinesq approximation is used, which results in a buoyancy force term which couples the thermal and fluid velocity fields. Therefore

$$\rho \approx \rho_0 = const. \tag{5.41}$$

except in the buoyancy term, where:

$$\varrho = \varrho_0(1 - \alpha(T - T_0)) \text{ with } \alpha > 0 \quad . \tag{5.42}$$

ρ_0 is the fluid density in the reference state. This assumption reflects a common feature of geophysical flows, where the density fluctuations caused by temperature variations are small, yet they are the ones driving the overall flow. We have the following relations. Furthermore, we assume that the density depends linearly on temperature T.

For some experiments go to the trailer 1, trailer 2, trailer in German, KIT trailer: Rayleigh Benard Thermal Convection 3D Simulation.

This system possesses a steady-state solution in which there is no motion, and the temperature varies linearly with depth:

$$u = w = 0$$
$$T_{eq} = T_0 + \left(1 - \frac{z}{H}\right)\Delta T \tag{5.43}$$

When this solution becomes unstable, convection should develop.

5.3.2   The usual approach: Elimination of pressure and vorticity dynamics

In the case where all motions are parallel to the $x - z$-plane, and no variations in the direction of the y-axis occur, the governing equations may be written (see Saltzman [1962]) as:

$$D_t u = -\frac{1}{\rho_0}\partial_x p + \nu\nabla^2 u \tag{5.44}$$

$$D_t w = -\frac{1}{\rho_0}\partial_z p + \nu\nabla^2 w + g(1 - \alpha(T - T_0)) \tag{5.45}$$

$$D_t T = \kappa\nabla^2 T \tag{5.46}$$

$$\partial_x u + \partial_z w = 0 \tag{5.47}$$

where w and u are the vertical and horizontal components of the velocity, $D_t u = \partial_t u + u\partial_x u + w\partial_z u$, $D_t w = \partial_t w + u\partial_x w + w\partial_z w$, and $D_t T = \partial_t T + u\partial_x T + w\partial_z T$ respectively. Furthermore, $\nu = \eta/\rho_0$, $\kappa = \lambda/(\rho_0 C_v)$ the momentum diffusivity (kinematic viscosity) and thermal diffusivity, respectively.

Now, compare the procedure with the elimination of the pressure term in section 4.4 where we derive the vorticity equation $D_t\left(\nabla^2\psi\right) = \nu\nabla^4\psi$. Here, it is useful to define the stream function Ψ for the two-dimensional motion, i.e.

$$\frac{\partial\Psi}{\partial x} = w \tag{5.48}$$

$$\frac{\partial\Psi}{\partial z} = -u \quad . \tag{5.49}$$

We take the

$$\frac{\partial}{\partial x}(5.45) - \frac{\partial}{\partial z}(5.44) = \frac{\partial}{\partial x}D_t w - \frac{\partial}{\partial z}D_t u = D_t\frac{\partial w}{\partial x} - D_t\frac{\partial u}{\partial z} \tag{5.50}$$

$$= = D_t\frac{\partial^2\Psi}{\partial x^2} - D_t\frac{\partial^2\Psi}{\partial z^2} = D_t\nabla^2\Psi \quad . \tag{5.51}$$

Note that $D_t \nabla^2 \Psi$ does not contain u, w anymore:

$$D_t \left(\nabla^2 \Psi \right) = \partial_t \nabla^2 \Psi - \frac{\partial \Psi}{\partial z} \frac{\partial \nabla^2 \Psi}{\partial x} + \frac{\partial \Psi}{\partial x} \frac{\partial \nabla^2 \Psi}{\partial z} \quad .$$

Furthermore, we introduce the function Θ as the departure of temperature from that occurring in the state of no convection (5.43):

$$T = T_{eq} + \Theta \tag{5.52}$$

In the temperature term in $\frac{\partial}{\partial x}$(5.45) on the right hand side:

$$\frac{\partial}{\partial x} g(1 - \alpha(T_{eq} + \Theta - T_0)) = -g\alpha \frac{\partial}{\partial x} \Theta$$

The left hand side of (5.46) reads

$$D_t T = D_t T_{eq} + D_t \Theta = w \cdot \frac{-\Delta T}{H} + D_t \Theta = -\frac{\Delta T}{H} \frac{\partial \Psi}{\partial x} + D_t \Theta$$

Then, the dynamics can be formulated as

$$D_t \left(\nabla^2 \Psi \right) = \nu \nabla^4 \Psi - g\alpha \frac{\partial \Theta}{\partial x} \tag{5.53}$$

$$D_t \Theta \quad = \frac{\Delta T}{H} \frac{\partial \Psi}{\partial x} + \kappa \nabla^2 \Theta \quad . \tag{5.54}$$

Non-dimensionalization of the problem yields equations including the dimensionless Prandtl number[3] σ and the Rayleigh number R_a which are the control parameters of the problem. One can

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[3]The Prandtl number is a dimensionless number; the ratio of momentum diffusivity (kinematic viscosity) to thermal diffusivity. It is named after the German physicist Ludwig Prandtl. Note that whereas the Reynolds number and Grashof number are subscripted with a length scale variable, the Prandtl number contains no such length scale in its definition and is dependent only on the fluid and the fluid state. As such, the Prandtl number is often found in property tables alongside other properties such as viscosity and thermal conductivity. Typical values for are:
1) Low - thermal diffusivity dominant: 13.4 and 7.2 for seawater at $0°$ and $20°$ Celsius respectively.
2) High - momentum diffusivity dominant: For mercury, heat conduction is very effective compared to convection: thermal diffusivity is dominant. For engine oil, convection is very effective in transferring energy from an area,

take the layer thickness H as the length of unit, the time $T = H^2/\kappa$ of vertical diffusion of heat as the unit of time, and the temperature difference ΔT as the unit of temperature. See exercise 33 for the non-dimensionalization procedure.

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compared to pure conduction: momentum diffusivity is dominant.

In heat transfer problems, the Prandtl number controls the relative thickness of the momentum and thermal boundary layers. When σ is small, it means that the heat diffuses very quickly compared to the velocity (momentum). This means that for liquid metals the thickness of the thermal boundary layer is much bigger than the velocity boundary layer. The Rayleigh number is named after Lord Rayleigh and is defined as the product of the Grashof number, which describes the relationship between buoyancy and viscosity within a fluid, and the Prandtl number, which describes the relationship between momentum diffusivity and thermal diffusivity. Hence the Rayleigh number itself may also be viewed as the ratio of buoyancy and viscosity forces times the ratio of momentum and thermal diffusivities.

Exercise 33   –   **Non-dimensional Rayleigh-Bénard system**

Write down the non-dimensional version of the Rayleigh-Bénard. Non-dimensionalization yields equations including the dimensionless Prandtl number σ and the Rayleigh number R_a which is also the control parameter. One can take the layer thickness H as the length of unit, the time $T = H^2/\kappa$ of vertical diffusion of heat as the unit of time, $U = H/T = \kappa/H$ the unit of velocity, and the temperature difference ΔT as the unit of temperature. Rayleigh and Prandtl numbers are

$$R_a = \frac{g\alpha H^3 \Delta T}{\nu\kappa} \quad , \tag{5.55}$$

$$\sigma = \frac{\nu}{\kappa} \quad . \tag{5.56}$$

The Prandtl number is a dimensionless number describing the ratio of momentum diffusivity (kinematic viscosity) to thermal diffusivity.

Hint: One shall first write down the equations for u, w, Θ. Then, introduce a background pressure $p_0(z)$ related to the temperature equilibrium solution T_{eq} :

$$\frac{d}{dz}p_0(z) = g\varrho_0(1 - \alpha(T_{eq}(z) - T_0)) \tag{5.57}$$

and $p(x, z) = p_0(z) + p'(x, z)$. Since $p_0(z)$ has no x-dependence, $\frac{\partial}{\partial x}p = \frac{\partial}{\partial x}p'$.

Ansatz: The solution in non-dimensional form shall be something like

$$\frac{1}{\sigma}D_{\text{t,d}}u_d = \cdots + \nabla_d^2 u_d \tag{5.58}$$

$$\frac{1}{\sigma}D_{\text{t,d}}w_d = \cdots - R_a\Theta_d + \nabla_d^2 w_d \tag{5.59}$$

$$D_{\text{t,d}}\Theta_d = w_d + \nabla_d^2\Theta_d \tag{5.60}$$

$$\frac{\partial}{\partial x_d}u_d + \frac{\partial}{\partial z_d}w_d = 0 \quad . \tag{5.61}$$

where $D_t = T^{-1}D_{\text{t,d}}$ and $\nabla = H^{-1}\nabla_d$.

Solution 1 of exercise 33

- **Write the characteristic scales of the RB system**

$$\text{length: } H \qquad\qquad\qquad \rightarrow \quad x = H x_{\mathrm{d}} \qquad\qquad (5.62\text{a})$$

$$\text{time: } T = \frac{H^2}{\kappa} \qquad\qquad \rightarrow \quad t = \frac{H^2}{\kappa} t_{\mathrm{d}} \qquad\qquad (5.62\text{b})$$

$$\text{velocity: } U = \frac{H}{T} = \frac{\kappa}{H} \qquad \rightarrow \quad u = \frac{\kappa}{H} u_{\mathrm{d}} \qquad\qquad (5.62\text{c})$$

$$\text{pressure: } P = \rho_0 \frac{H^2}{T^2} = \rho_0 \frac{\kappa^2}{H^2} \qquad \rightarrow \quad p = \rho_0 \frac{\kappa^2}{H^2} p_{\mathrm{d}} \qquad\qquad (5.62\text{d})$$

$$\text{temperature: } \Delta T \qquad\qquad \rightarrow \quad T = \Delta T \; T_{\mathrm{d}} \qquad\qquad (5.62\text{e})$$

Furthermore the following definitions for the Rayleigh (R_a) and Prandtl (σ) numbers are given:

$$R_a = \frac{g \alpha H^3 \Delta T}{\nu \kappa} \qquad\qquad (5.63)$$

and

$$\sigma = \frac{\nu}{\kappa}. \qquad\qquad (5.64)$$

- **Modifications of temperature and pressure terms**

We introduce the temperature difference from the equilibrium solution:

$$\theta = T - T_{\mathrm{eq}} \quad \rightarrow \quad T = \theta + T_{\mathrm{eq}}, \qquad\qquad (5.65)$$

where T_{eq} is given as:

$$T_{\mathrm{eq}} = T_0 + (1 - \frac{z}{H}) \Delta T. \qquad\qquad (5.66)$$

Later we will need the following terms:

$$\partial_z T_{\mathrm{eq}} = -\frac{\Delta T}{H} \qquad\qquad (5.67)$$

and

$$\nabla^2 T_{\mathrm{eq}} = 0. \tag{5.68}$$

In the pressure term we introduce the background pressure $p_0(z)$:

$$p(x, z) = p_0(z) + p'(x, z). \tag{5.69}$$

Later we need the x-derivative of p_0, which is zero, and the z-derivative of p_0:

$$\partial_z p_0 = g\rho_0 \left(1 - \alpha(T_{\mathrm{eq}} - T_0)\right) \tag{5.70}$$

- **Continuity equation**

$$\nabla \cdot \mathbf{u} = 0$$

$$\rightarrow \quad \frac{\kappa}{H^2} \nabla_{\mathrm{d}} \cdot \mathbf{u}_{\mathrm{d}} = 0$$

$$\boxed{\nabla_{\mathrm{d}} \cdot \mathbf{u}_{\mathrm{d}} = 0}$$

- **Horizontal component of momentum equation**

$$D_t u = -\frac{1}{\rho_0} \partial_x p + \nu \nabla^2 u$$

Using Eq. (5.69) and using the fact that $\partial_x p_0 = 0$ we get:

$$\frac{\kappa^2}{H^3} D_{t,\mathrm{d}} u_{\mathrm{d}} = -\frac{\kappa^2}{H^3} \partial_{x,\mathrm{d}} p_{\mathrm{d}}' + \frac{\kappa\nu}{H^3} \nabla_{\mathrm{d}}^2 u_{\mathrm{d}}$$

$$\frac{\kappa}{\nu} D_{t,\mathrm{d}} u_{\mathrm{d}} = -\frac{\kappa}{\nu} \partial_{x,\mathrm{d}} p_{\mathrm{d}}' + \nabla_{\mathrm{d}}^2 u_{\mathrm{d}}$$

$$\boxed{\frac{1}{\sigma} D_{t,\mathrm{d}} u_{\mathrm{d}} = -\frac{1}{\sigma} \partial_{x,\mathrm{d}} p_{\mathrm{d}}' + \nabla_{\mathrm{d}}^2 u_{\mathrm{d}}}$$

- **Vertical component of momentum equation**

$$D_t w = -\frac{1}{\rho_0}\partial_z p + \nu\nabla^2 w + g\left((1 - \alpha(T - T_0))\right)$$

Using again Eq. (5.69) and $\partial_z p_0$ from Eq.*(5.70) we get:

$$D_t w = -\frac{1}{\rho_0}\partial_z p' + \nu\nabla^2 w - g\alpha(T - T_{\text{eq}})$$
$$= -\frac{1}{\rho_0}\partial_x p' + \nu\nabla^2 w - g\alpha\theta$$

Now we rewrite the equation in a dimensionless form:

$$\frac{\kappa^2}{H^3}D_{t,d}w_d = -\frac{\kappa^2}{H^3}\partial_{z,d}p'_d + \frac{\kappa\nu}{H^3}\nabla_d^2 w_d - g\alpha\Delta T\theta_d$$
$$\frac{\kappa}{\nu}D_{t,d}w_d = -\frac{\kappa}{\nu}\partial_{z,d}p'_d + \nabla_d^2 w_d - \frac{g\alpha\Delta T H^3}{\kappa\nu}\theta_d$$

$$\boxed{\frac{1}{\sigma}D_{t,d}w_d = -\frac{1}{\sigma}\partial_{z,d}p'_d + \nabla_d^2 w_d - R_a\theta_d}$$

- **Thermal energy equation**

$$D_t T = D_t\theta + D_t T_{\text{eq}} = D_t\theta + \partial_t T_{\text{eq}} + u\partial_x T_{\text{eq}} + w\partial_z T_{\text{eq}}$$
$$= \kappa\nabla^2 T = \kappa(\nabla^2\theta + \nabla^2 T_{\text{eq}})$$

Since T_{eq} has no time and no x dependence $\partial_t T_{\text{eq}}$ and $\partial_x T_{\text{eq}}$ are zero and also the term $\nabla^2 T_{\text{eq}}$ vanishes (see eq. (5.68)). Using Eq. (5.67) we can replace $\partial_z T_{\text{eq}}$:

$$D_t T = D_t\theta - w\frac{\Delta T}{H} = \kappa\nabla^2\theta$$
$$D_t\theta = \kappa\nabla^2\theta + w\frac{\Delta T}{H}$$

By introducing the dimensionless variables (Eq. (5.62)) we get:

$$\frac{\kappa \Delta T}{H^2} D_{\mathrm{t,d}}\theta_{\mathrm{d}} = \frac{\kappa \Delta T}{H^2}\nabla_{\mathrm{d}}^2\theta_{\mathrm{d}} + \frac{\kappa \Delta T}{H^2}w_{\mathrm{d}}$$

$$\boxed{D_{\mathrm{t,d}}\theta_{\mathrm{d}} = \nabla_{\mathrm{d}}^2\theta_{\mathrm{d}} + w_{\mathrm{d}}}$$

Solution 2 of exercise 33

A shorter solution is to use the (5.53, 5.54) system.

$$\frac{1}{T}\frac{1}{L^2}\frac{L^2}{T}D_{\mathrm{t,d}}\left(\nabla_d^2\Psi_d\right) = \nu\frac{1}{L^4}\frac{L^2}{T}\nabla_d^4\Psi_d - g\alpha\frac{\Delta T}{L}\frac{\partial \Theta_d}{\partial x_d} \tag{5.71}$$

$$\frac{\Delta T}{T}D_{\mathrm{t,d}}\Theta_d = \frac{\Delta T}{H}\frac{L^2}{TL}\frac{\partial \Psi_d}{\partial x_d} + \kappa\frac{\Delta T}{L^2}\nabla_d^2\Theta_d \quad . \tag{5.72}$$

This yields (remember $L = H$)

$$D_{\mathrm{t,d}}\left(\nabla_d^2\Psi_d\right) = \nu\frac{T}{H^2}\nabla_d^4\Psi_d - g\alpha\frac{T^2\Delta T}{H}\frac{\partial \Theta_d}{\partial x_d} \tag{5.73}$$

$$D_{\mathrm{t,d}}\Theta_d = \frac{\partial \Psi_d}{\partial x_d} + \kappa\frac{T}{H^2}\nabla_d^2\Theta_d \quad . \tag{5.74}$$

Inserting $T = H^2/\kappa$, gives

$$D_{\mathrm{t,d}}\left(\nabla_d^2\Psi_d\right) = \frac{\nu}{\kappa}\nabla_d^4\Psi_d - g\alpha\frac{H^3\Delta T}{\kappa^2}\frac{\partial \Theta_d}{\partial x_d} \tag{5.75}$$

$$D_{\mathrm{t,d}}\Theta_d = \frac{\partial \Psi_d}{\partial x_d} + \nabla_d^2\Theta_d \quad . \tag{5.76}$$

Finally, inserting the Rayleigh $R_a = \frac{g\alpha H^3\Delta T}{\nu\kappa}$ and Prandtl $\sigma = \frac{\nu}{\kappa}$ numbers:

$$\boxed{D_{\mathrm{t,d}}\left(\nabla_d^2\Psi_d\right) = \sigma\nabla_d^4\Psi_d - R_a\sigma\frac{\partial \Theta_d}{\partial x_d}}$$

5.3.3   Boundary conditions

We shall now discuss the boundary conditions: $\Theta = 0$ at $z = 0, H$. As further boundary condition, the normal component could be set to zero straightaway and we have $\mathbf{v}_{normal} = w = 0$ at $z = 0, H$.

In many applications, one may assume **no-slip boundary condition** as the appropriate conditions for velocity components at the wall. In general, while the tangential component is set to the velocity of the wall:

$$\mathbf{v}_{tangential} = \mathbf{v}_{wall} \quad . \tag{5.77}$$

The fluid velocity at all fluid-solid boundaries is equal to that of the solid boundary. Conceptually, one can think of the outermost molecules of fluid as stuck to the surfaces past which it flows. Because the solution is prescribed at given locations, this is an example of a Dirichlet boundary condition. Particles close to a surface do not move along with a flow when adhesion is stronger than cohesion. At the fluid-solid interface, the force of attraction between the fluid particles and solid particles (adhesive forces) is greater than that between the fluid particles (cohesive forces). This force imbalance brings down the fluid velocity to zero. In our case: since the wall is not moving $u = 0$ at $z = 0, H$.

Another boundary condition is called **free boundary condition.** All the normal velocities normal to the wall should be zero, and furthermore the gradient of velocity parallel to wall should be zero:

$$\frac{\partial}{\partial z}\mathbf{v}_{tangential} = 0 \tag{5.78}$$

Here, we assume a free surface both the upper- and the lower-boundaries because then the problem is most analytically tractable.[4] In our case this means no tangential stress is for $\frac{\partial u}{\partial z} = \frac{\partial^2 \psi}{\partial z^2} = 0$ One can show that in which case Ψ and $\nabla^2\Psi$ vanish at both boundaries. The basis functions can

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[4]In practical applications, the boundaries are not free and there is friction.

be chosen as sinus and cosinus as orthogonal set of base functions. In chapter 3.4, the dynamics is solved numerically using the Lattice Boltzmann approach. Other techniques and experimental approaches are summarized in Tritton [1988].

5.3.4 Galerkin approximation: Obtaining the Lorenz system

Saltzman [1962] derived a set of ordinary differential equations by expanding Ψ and Θ in double Fourier series in x and z, with functions of t alone for coefficients, and substituting these series into (5.53) and (5.54) A complete Galerkin approximation

$$\Psi(x, z, t) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \Psi_{k,l}(t) \; \sin\left(\frac{k\pi a}{H}x\right) \times \sin\left(\frac{l\pi}{H}z\right) \tag{5.79}$$

$$\Theta(x, z, t) = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \Theta_{k,l}(t) \cos\left(\frac{k\pi a}{H}x\right) \times \sin\left(\frac{l\pi}{H}z\right) \tag{5.80}$$

yields an infinite set of ordinary differential equations for the time coefficients. He arranged the right-hand sides of the resulting equations in double Fourier-series form, by replacing products of trigonometric functions of x (or z) by sums of trigonometric functions, and then equated coefficients of similar functions of x and z. He then reduced the resulting infinite system to a finite system by omitting reference to all but a specified finite set of functions of t. He then obtained time-dependent solutions by numerical integration. In certain cases all, except three of the dependent variables, eventually tended to zero, and these three variables underwent irregular, apparently non-periodic fluctuations. These same solutions would have been obtained if the series had been at the start truncated to include a total of three terms. Accordingly, in this study we shall let

$$\frac{a}{1+a^2}\kappa \;\; \Psi = X\sqrt{2}\sin\left(\frac{\pi a}{H}x\right)\sin\left(\frac{\pi}{H}z\right) \tag{5.81}$$

$$\pi\frac{R_a}{R_c}\frac{1}{\Delta T} \;\; \Theta = Y\sqrt{2}\cos\left(\frac{\pi a}{H}x\right)\sin\left(\frac{\pi}{H}z\right) - Z\sin\left(2\frac{\pi}{H}z\right) \tag{5.82}$$

where $X(t)$, $Y(t)$, and $Z(t)$ are functions of time alone.

It is found that fields of motion of this form would develop if the Rayleigh number

$$R_a = \frac{g\alpha H^3 \Delta T}{\nu\kappa} \quad , \tag{5.83}$$

exceeds a critical value

$$R_c = \pi^4 a^{-2}(1 + a^2)^3 \quad . \tag{5.84}$$

The minimum value of R_c, namely $27\pi^4/4 = 657.51$, occurs when $a^2 = 1/2$. In fluid mechanics, the Rayleigh number for a fluid is a dimensionless number associated with the relation of buoyancy and viscosity in a flow. When the Rayleigh number is below the critical value for that fluid, heat transfer is primarily in the form of conduction; when it exceeds the critical value, heat transfer is primarily in the form of convection.

When the above truncation (5.81,5.82) is substituted into the dynamics, we obtain the equations

$$\dot{X} = -\sigma X + \sigma Y \tag{5.85}$$

$$\dot{Y} = rX - Y - XZ \tag{5.86}$$

$$\dot{Z} = -bZ + XY \tag{5.87}$$

Here a dot denotes a derivative with respect to the dimensionless time $t_d = \pi^2 H^{-2}(1 + a^2)\kappa t$, while $\sigma = \nu\kappa^{-1}$ is the Prandtl number, $r = R_a/R_c$, and $b = 4(1 + a^2)^{-1}$.

Except for multiplicative constants, our variables X, Y, Z are the same as Saltzman's variables A, D, and G. These equations are the convection equations whose solutions we shall study. In these equations X is proportional to the intensity of the convective motion, while Y is proportional to the temperature difference between the ascending and descending currents, identical signs of X and Y denoting that warm fluid is rising and cold fluid is descending. The variable Z is proportional to the distortion of the vertical temperature-profile from linearity, a positive value indicating that the strongest gradients occur near the boundaries.

Chapter 6

Atmosphere and Ocean Dynamics

Here, we shortly introduce the basic concepts of oceanography: the fundamental equations and approximations. Basically, we deal with the dynamical equations on a rotating sphere. Several approximations can be done since the scales of the components in the dynamical equations differ in the orders of magnitude. A more detailled description can be found in text books [Von Arx, 1962; Gill, 1982; Holton, 2004; Pickard and Emery, 1990; Stewart, 2005; Dijkstra, 2000; Olbers et al., 2012].

6.1   Pseudo forces and the Coriolis effect

A pseudo force on an object arises when the frame of reference used to describe the object's motion is accelerating compared to a non-accelerating frame. It acts on all masses whose motion is described using a non-inertial frame of reference, such as a rotating reference frame. The inertial frame is the Sun and not the Earth.[1] Assuming Newton's second law in the form F = ma, pseudo

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[1]Galilean invariance or Galilean relativity states that the laws of motion are the same in all inertial frames. Galileo Galilei first described this principle in 1632 in his Dialogue Concerning the Two Chief World Systems using the example of a ship travelling at constant velocity, without rocking, on a smooth sea; any observer doing experiments below the deck would not be able to tell whether the ship was moving or stationary. Galilean relativity can be shown as follows. Consider two inertial frames S and S' . A physical event in S will have position coordinates $r = (x, y, z)$ and time t; similarly for S' . By the second axiom above, one can synchronize the clock in the two frames and assume $t = t'$. Suppose S' is in relative uniform motion to S with velocity v. Consider a point object whose position is

forces are always proportional to the mass m. The surface of the Earth is a rotating reference frame. To solve classical mechanics problems exactly in an Earth-bound reference frame, three pseudo forces must be introduced, the Coriolis force, the centrifugal force (described below) and the Euler force. The Euler force is typically ignored because the variations in the angular velocity of the rotating Earth surface are usually insignificant. Both of the other pseudo forces are weak compared to most typical forces in everyday life, but they can be detected under careful conditions. For example, Foucault was able to show the Coriolis force that results from the Earth's rotation using the Foucault pendulum (see Exercise 3). If the Earth were to rotate a thousand times faster (making each day only $\approx$ 86 seconds long), people could easily get the impression that such fictitious forces are pulling on them, as on a spinning carousel.

In the rotating framework, we have the Coriolis and centrifugal forces which stem from the rotating framework. We derive from the simple relation for the time derivative in the inertial system (i) to the Earth system (e)

$$(d_t A)_i = (d_t A)_e + \Omega \times A \tag{6.1}$$

where the $\times$ symbol represents the cross product operator. For the case $A = r$, it follows for the velocity

$$v_i = v_e + \Omega \times r \tag{6.2}$$

and the relation for the acceleration (case $A = v_i$)

$$
\begin{aligned}
a_i &= \quad (d_t v_i)_e \quad + \quad \Omega \times v_i \\
&= \quad d_t v_e + \Omega \times v_e + \Omega \times (v_e + \Omega \times r) = a_e + 2\Omega \times v_e + \Omega \times \Omega \times r
\end{aligned}
\tag{6.3}
$$

given by $r'(t) = r(t)$ in S. We see that $r'(t) = r(t) - vt$. and acceleration is identical in the two frames $a'(t) = \frac{d^2}{dt^2} r'(t) = \frac{d^2}{dt^2} r(t) = a(t)$. A side remark: All approximations of the dynamical equations shall be Galilean invariant. In numerical examples, the lack of invariance for unresolved solutions is because the truncation error is not Galilean invariant. While advanced methods reduce the truncation error, none of them eliminate it entirely, and therefore formally solutions will still violate Galilean invariance at the level of the truncation error.

At a given rate of rotation of the observer, the magnitude of the Coriolis acceleration of the object is proportional to the velocity of the object and also to the sine of the angle between the direction of movement of the object and the axis of rotation. In the following the subscript e is dropped, since we are only interested in the dynamics in the rotating Earth system. The forces in the rotating system are thus the forces in the inertial system plus the Coriolis and centrifugal forces:

$$\boldsymbol{F} = \boldsymbol{F}_i + \boldsymbol{F}_C + \boldsymbol{F}_{cf} \tag{6.4}$$

where

$$\boldsymbol{F}_C = -2\,m\,\boldsymbol{\Omega} \times \boldsymbol{v}. \tag{6.5}$$

$\boldsymbol{\Omega}$ is the angular velocity vector which has magnitude equal to the rotation rate ω and is directed along the axis of rotation of the rotating reference frame. The formula implies that the Coriolis acceleration is perpendicular both to the direction of the velocity of the moving mass and to the frame's rotation axis.

The centrifugal term is equal to

$$\boldsymbol{F}_{cf} = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = -\omega^2 \boldsymbol{R}, \tag{6.6}$$

where $\boldsymbol{r}$ is the space vector and $\boldsymbol{R}$ the component of $\boldsymbol{r}$ perpenticular to the axis of rotation. This term can be absorbed into the gravitation is then called gravity. One can introduce the gravitational potential

$$\phi = gz - \frac{\omega^2 R^2}{2} = gz - \frac{\omega^2 (a+z)^2 \cos^2(\varphi)}{2} \simeq gz - \frac{\omega^2 a^2 \cos^2(\varphi)}{2} \quad . \tag{6.7}$$

where a is the Earth radius and φ the latitude. The combined vector $\nabla\phi$ shows only minor modification with respect to the vertical coordinate defined by the gravitation. In practice, the gravity is used for the vertical coordinate.

6.2   Scaling of the dynamical equations

As we will see now, the Coriolis effect is one of the dominating forces for the large-scale dynamics of the oceans and the atmosphere. It is convenient to work in the rotating frame of reference of the Earth. The equation can be scaled by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. Starting from (4.9), we can estimate the relative contributions in units of m/s^2 in the horizontal momentum equations:

$$\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{U/T \sim 10^{-8}} + \underbrace{\mathbf{v} \cdot \nabla \mathbf{v}}_{U^2/L \sim 10^{-8}} = \underbrace{-\frac{1}{\rho} \nabla p}_{\delta P/(\rho L) \sim 10^{-5}} + \underbrace{2\Omega \times \mathbf{v}}_{f_0 U \sim 10^{-5}} + \underbrace{fric}_{\nu U/H^2 \sim 10^{-13}} \tag{6.8}$$

where fric denotes the contributions of friction due to eddy stress divergence (usually $\sim \nu \nabla^2 \mathbf{v}$). Typical values are given in Table 6.2. The values have been taken for the **ocean.** You may repeat the estimate for the atmosphere using Table 6.2.

It is useful to think about the orders of magnitude: Because of the continuity equation $U/L \sim W/H$ and since the horizonatal scales are orders of magnitude larger than the vertical ones, the vertical velocity is very small relative to the horizontal. For small scale motion (like small-scale ocean convection or cumuls clouds) the horizontal length scale is of the same order as the vertical one and therefore the vertical motion is in the same order of magnitude as the horizontal motion. The timescales are related to $T \sim L/U \sim H/W$.

It is already useful to think about the relative importance of the different terms in the momentum balance (6.8). The Rossby number Ro is the ratio of inertial (the left hand side) to Coriolis (second term on the right hand side) terms

$$Ro = \frac{(U^2/L)}{(fU)} = \frac{U}{fL} \quad . \tag{6.9}$$

It is used in the oceans and atmosphere, where it characterizes the importance of Coriolis accelerations arising from planetary rotation. It is also known as the Kibel number. Ro is small when the flow is in a so-called geostrophic balance. This will be the subject in the next paragraphs.

	Quantity	Atmosphere	Ocean
horizontal velocity	U	$10 \ ms^{-1}$	$10^{-1} \, ms^{-1}$
vertical velocity	W	$10^{-1} \, ms^{-1}$	$10^{-4} \, ms^{-1}$
horizontal length	L	$10^6 \, m$	$10^6 \, m$
vertical length	H	$10^4 \, m$	$10^3 \, m$
horizonal Pressure changes	δP (horizontal)	$10^3 \, Pa$	$10^4 \, Pa$
mean pressure	P_0	$10^5 \, Pa$	$10^7 \, Pa$
time scale	T	$10^5 \, s$	$10^7 \, s$
gravity (gravitation+centrifugal)	g	$10 \, ms^{-2}$	$10 \, ms^{-2}$
Earth radius	a	$10^7 \, m$	$10^7 \, m$
Coriolis parameter at $45°$N	$f_0 = 2\Omega \sin \varphi_0$	$10^{-4} \, s^{-1}$	$10^{-4} \, s^{-1}$
2nd Coriolis parameter at $45°$N	$f_1 = 2\Omega \cos \varphi_0$	$10^{-4} \, s^{-1}$	$10^{-4} \, s^{-1}$
density	ρ	$1 \ kgm^{-3}$	$10^3 \, kgm^{-3}$
viscosity (turbulent)	ν	$10^{-5} \, kgm^{-3}$	$10^{-6} \, kgm^{-3}$

Table 6.1: Table shows the typical scales in the atmosphere and ocean system. Using these orders of magnitude, one can derive estimates of the different terms in (6.8).

Exercise 34   –   Non-dimensional system

a) Write down the non-dimensional version of (6.8) ! What are the characteristic numbers?

b) Use Table 6.2 to estimate the order of magnitude of the characteristic numbers !

c) Compare the procedure to exercise 33.

6.3   The coordinate system

The equations have to be solved on a proper coordinate system. Consider a location with latitude φ on a sphere that is rotating around the north-south axis. A local coordinate system is set up with the x axis horizontally due east, the y axis horizontally due north and the z axis vertically upwards. The axis of rotation is then expressed by a y-component $\sim \cos \varphi$ and a z-component $\sim \sin \varphi$. The rotation vector expressed in this local coordinate system is

$$\Omega = \Omega \begin{pmatrix} 0 \\ \cos \varphi \\ \sin \varphi \end{pmatrix} . \tag{6.10}$$

Likewise, the components of the velocity vector are listed in the order East (u), North (v) and Upward (w):

$$v = \begin{pmatrix} u \\ v \\ w \end{pmatrix} , \tag{6.11}$$

and Coriolis acceleration is therefore in this coordinate system

$$a_C = -2\Omega \times v = 2\,\Omega \begin{pmatrix} v \sin \varphi - w \cos \varphi \\ -u \sin \varphi \\ u \cos \varphi \end{pmatrix} . \tag{6.12}$$

In the following, $f = 2\Omega \sin \varphi$ is called the Coriolis parameter, $f^{(2)} = 2\Omega \cos \varphi$ is called the second Coriolis parameter.

When considering atmospheric or oceanic dynamics, the vertical velocity is small and therefore the vertical component of the Coriolis acceleration is small compared to gravity (see table 6.2 and the following paragraph). For such cases, only the horizontal (East and North) components matter.
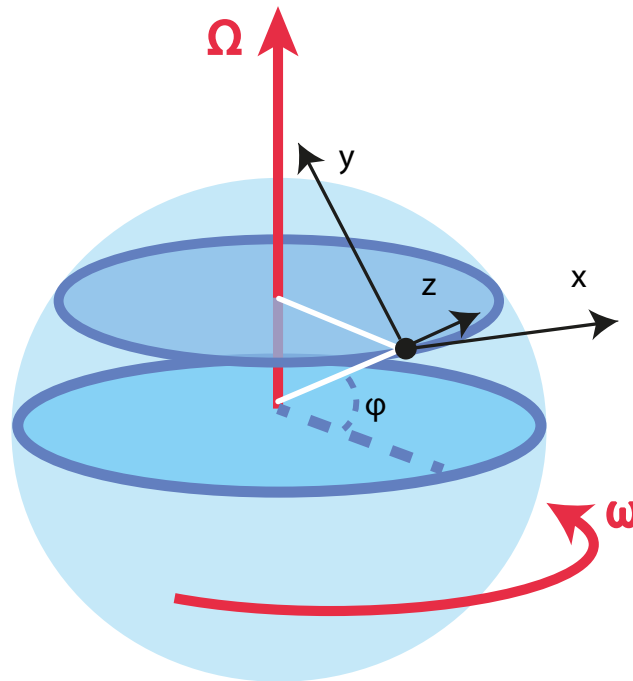
Figure 6.1: Coordinate system at a local latitude φ with x-axis east, y-axis north and z-axis upward (that is, radially outward from center of sphere). $(x, y, z) = (a\lambda \cos\varphi, a\varphi, z)$ where (λ, φ, z) denote longitude, latitude, hight. a is the Earth radius. Ω is the the Earth rotation and equal to $2\pi/(24h)$. Note that the axis of rotation has a y- and z-component in this coordinate system (see text for details).

If we further assume $v = 0$, it can be seen immediately that (for positive φ) a movement to the east results in an acceleration to south. Similarly, for $u = 0$, it is seen that a movement due north results in an acceleration due east. In general, observed horizontally, looking along the direction of the movement causing the acceleration, the acceleration always is turned $90°$ to the right on the Northern Hemisphere (left on the Southern Hemisphere) and of the same size regardless of the horizontal orientation.

Exercise 35   –   Double Vector Product

Examine the double vector product $\Omega \times (\Omega \times r)$ with vectors $\Omega = (0, 0, \omega)$, $r = (x, y, z)$.

Solution

$$
\Omega \times (\Omega \times r) = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \left(\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right)
$$

$$
= \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} -\omega y \\ \omega x \\ 0 \end{pmatrix}
$$

$$
= \begin{pmatrix} -\omega^2 y \\ -\omega^2 x \\ 0 \end{pmatrix} = -\|\Omega\|^2 \mathbf{R}
$$

with $\mathbf{R} = (x, y, 0)^T$ and $\|\Omega\|^2 = \omega^2$.

Exercise 36   –   Some Questions Dynamics

1. Consider the heat diffusion-advection equation

$$
\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + u \frac{\partial T}{\partial x}
$$

and determine the time evolution with initial conditions

a) $T(x, 0) = \exp(-x^2/a)$ with a = constant.

b) $T(x, 0) = T_0$ for $x \geq 0$ and $T(x, 0) = 0$ elsewhere.

Discuss the special cases $k = 0$ (no diffusion) and $u = 0$ (no advection).

2. A tornado rotates with constant angular velocity ω. Show that the surface pressure at the center of the tornado is given by:

$$p = p_0 \exp(-\omega^2 r_0{}^2/(2RT))$$

   where p_0 is the surface pressure at the distance r_0 from the center and T is the temperature (assumed constant). [Hint: What are the dominant forces? Pressure gradient and centrifugal force.]

   If temperature is 288K, pressure at 100m from the center is 10^2 kPa, and wind speed at 100m from the center is 100m/s, what is the central pressure?

3. Suppose a 1kg parcel of dry air is rising at a constant vertical velocity. If the parcel is being heated by radiation at a rate of $10^{-1} W/kg$, what must the speed of rise be in order to maintain the parcel at a constant temperature. [Hint: Energy equation.]

4. Show that for an atmosphere with an adiabatic lapse rate (i.e. constant potential temperature), the geopotential $Z(z) := \Phi(z)/g_0$ is given by

$$Z = H_\Theta[1 - (p/p_0)^a]$$

   where p_0 is the pressure at $Z = 0$ and $H_\Theta = c_p\Theta/g_0$ is the total geopotential in the atmosphere. $a = R/c_p$.

Exercise 37   –   Some questions

1. Please write down the equation of state for the ocean and atmosphere!

2. What is the hydrostatic approximation in the momentum equations?

3. Please clarify: On the Northern Hemisphere, particles tend to go to the right or left relative to the direction of motion due to the Coriolis force?

6.4   Geostrophy

The momentum equations (6.8) can be also written in the coordinate system (Fig. 6.1) above as

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u - \frac{uv \tan \varphi}{a} - \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - f^{(2)}w + \nu \nabla^2 u \qquad (6.13)$$

$$\frac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla v - \frac{u^2 \tan \varphi}{a} - \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \qquad\qquad + \nu \nabla^2 v \qquad (6.14)$$

complemented by the dynamics for the vertical component w :

$$\underbrace{\frac{\partial w}{\partial t}}_{W/T \sim 10^{-11}} + \underbrace{\mathbf{v} \cdot \nabla w}_{UW/L \sim 10^{-11}} - \underbrace{\frac{u^2 + v^2}{a}}_{U^2/a \sim 10^{-9}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{P_0/(\rho H) \sim 10} + \underbrace{g}_{\sim 10} + \underbrace{f^{(2)}u}_{\sim 10^{-5}} + \underbrace{\nu \partial_z^2 w}_{\nu W/H^2 \sim 10^{-16}} \qquad (6.15)$$

As boundary conditions, equations (6.13, 6.14) are complemented by the horizontal wind stresses $\partial_z \tau_{xz}$ and $\partial_z \tau_{yz}$ at the ocean surface, respectively.

$$\frac{\partial u}{\partial t} + \mathbf{v} \cdot \nabla u + \ldots = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - f^{(2)}w + \nu \nabla^2 u + \frac{1}{\rho}\partial_z \tau_{xz} \qquad (6.16)$$

$$\frac{\partial v}{\partial t} + \mathbf{v} \cdot \nabla v + \ldots = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \qquad\qquad + \nu \nabla^2 v + \frac{1}{\rho}\partial_z \tau_{yz} \qquad (6.17)$$

It should be noted that due to sperical coordinates (see Fig. 6.1), one has metric terms, e.g. on the left hand sides of (6.13,6.14,6.15): $-\frac{uv \tan \varphi}{a} - \frac{uw}{a}$, $\frac{u^2 \tan \varphi}{a} - \frac{vw}{a}$, and $\frac{u^2+v^2}{a}$, respectively. In the geostropic approximation, one can drop these terms.[2]

A small Rossby number signifies a system which is strongly affected by Coriolis forces, and a large Rossby number signifies a system in which inertial forces dominate. For example, in tornadoes, the Rossby number is large ($\approx 10^3$), in atmospheric low-pressure systems it is low ($\approx 0.1 - 1$), but depending on the phenomena can range over several orders of magnitude ($\approx 10^{-2} - 10^2$).[3] Using the values in table 6.2, Ro in oceanic systems is of the order of 10^{-3}.

---

[2]Task: Calculate the order of magnitude of the metric terms in (6.13,6.14) by using table 6.2.

[3]As a result, in tornadoes the Coriolis force is negligible, and balance is between pressure and centrifugal forces (called cyclostrophic balance). This balance also occurs at the outer eyewall of a tropical cyclone.

When the Rossby number is large (either because f is small, such as in the tropics and at lower latitudes; or because L is small, that is, for small-scale motions such as flow in a bathtub; or for large speeds), the effects of planetary rotation are unimportant and can be neglected. Repeating: When the Rossby number is small, then the effects of planetary rotation are large and the net acceleration is comparably small allowing the use of the so-called geostrophic approximation: The force balance is largely between the pressure gradient force acting towards the low-pressure area and the Coriolis force acting away from the center of the low pressure in equation (6.8). By scaling arguments, one can derive the geostrophic horizontal flow components (u_g, v_g) as:

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y} \tag{6.18}$$

$$v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x} \tag{6.19}$$

The validity of this approximation depends on the local Rossby number. It is invalid at the equator, because $f = 2\Omega \sin\varphi$ is equal to zero there, and therefore generally not used in the tropics.

Equations (6.18,6.19) show that large-scale motions in the atmosphere and ocean tend to occur perpendicular to the pressure gradient, instead of flowing down the gradient. This circulation is called geostrophic flow. On a non-rotating planet, fluid would flow along the straightest possible line, quickly eliminating pressure gradients.[4]

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[4]Task: Think how the geostrophy can be derived in the inertial system with a fixed reference frame, e.g. the Sun. The final result shall be independent on the reference system used!
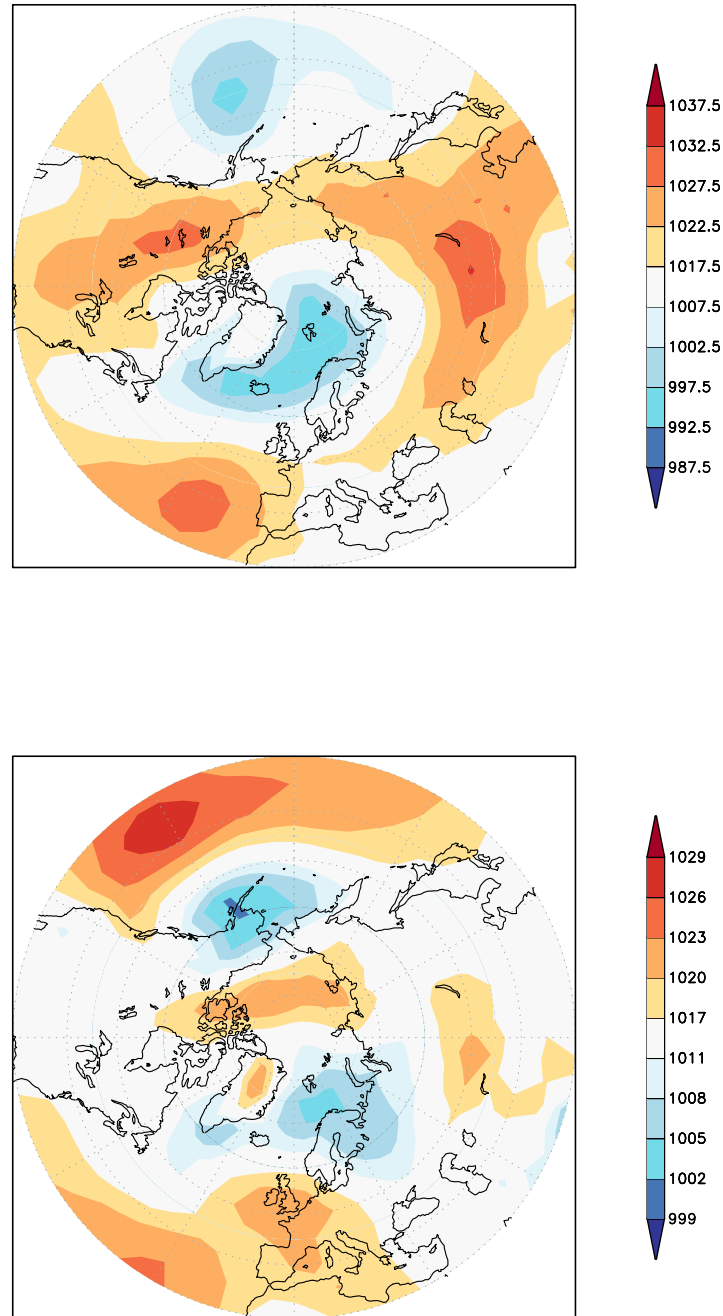
Figure 6.2: Sea level pressure (hPa) field for February (upper) and April (lower) 2015. In February, the circulation is characterized by a low pressure over the Greenland-Iceland-Norwegian Sea, and a surrounded high pressure. In April, the circulation was dominated by a high pressure over northern France and the subtropical Atlantic and Pacific Oceans, a low pressure over Scandianavia and the Aleutian Islands. Task: Draw the direction of large-scale motions in the atmosphere using the geostrophic balance (6.18,6.19). Data are from Trenberth and Paolino (1980).

Stream Lines and the Stream Function

At each instant in time, we can represent a flow field by a vector velocity at each point in space. The instantaneous curves that are everywhere tangent to the direction of the vectors are called the *stream lines* of the flow. If the flow is unsteady, the pattern of stream lines change with time. The trajectory of a fluid particle, the path followed by a Lagrangian drifter, is called the path line in fluid mechanics. The path line is the same as the stream line for steady flow, and they are different for an unsteady flow. We can simplify the description of two-dimensional, incompressible flows by using the *stream function* ψ defined by:

$$u \equiv \frac{\partial \psi}{\partial y}, \qquad v \equiv -\frac{\partial \psi}{\partial x}, \tag{6.20}$$

The stream function is often used because it is a scalar from which the vector velocity field can be calculated. This leads to simpler equations for some flows.

The volume rate of flow between any two stream lines of a steady flow is $d\psi$, and the volume rate of flow between two stream lines $\psi 1$ and ψ_2 is equal to $\psi_1 - \psi_2$. To see this, consider an arbitrary line $dx = (dx, dy)$ between two stream lines (Fig. 6.3). The volume rate of flow between the stream lines is:

$$v \; dx + (-u) \; dy = -\frac{\partial \psi}{\partial x} \; dx - \frac{\partial \psi}{\partial y} \; dy = -d\psi \tag{6.21}$$

and the volume rate of flow between the two stream lines is numerically equal to the difference in their values of ψ.

Now, lets apply the concepts to satellite-altimeter maps of the oceanic topography. One can show that

$$u_s = -\frac{g}{f}\frac{\partial \eta}{\partial y}, \qquad v_s = -\frac{g}{f}\frac{\partial \eta}{\partial x}, \tag{6.22}$$

where g is gravity, f is the Coriolis parameter, and η is the height of the sea surface above a level
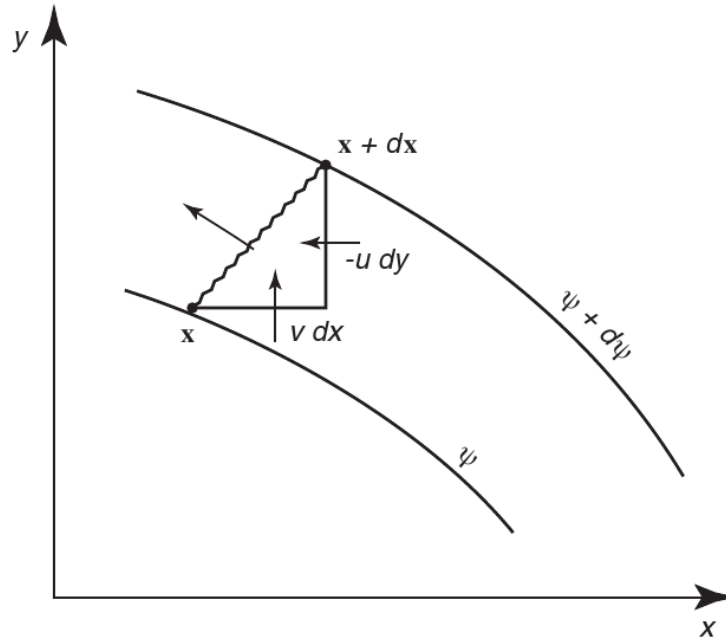
Figure 6.3: Volume transport between stream lines in a two-dimensional, steady flow. After Kundu (1990: 68).

surface. Comparing 6.22 with 6.20 it is clear that

$$\psi = -\frac{g}{f}\,\eta \tag{6.23}$$

and the sea surface is a stream function scaled by g/f. The lines of constant height are stream lines, and flow is along the lines. The surface geostrophic transport is proportional to the difference in height, independent of the distance between the stream lines. The transport is relative to transport at the **1000** decibars surface, which is roughly one kilometer deep.

In addition to the stream function, oceanographers use the mass-transport stream function Ψ defined by:

$$M_x \equiv \frac{\partial \Psi}{\partial y}, \qquad M_y \equiv -\frac{\partial \Psi}{\partial x} \tag{6.24}$$
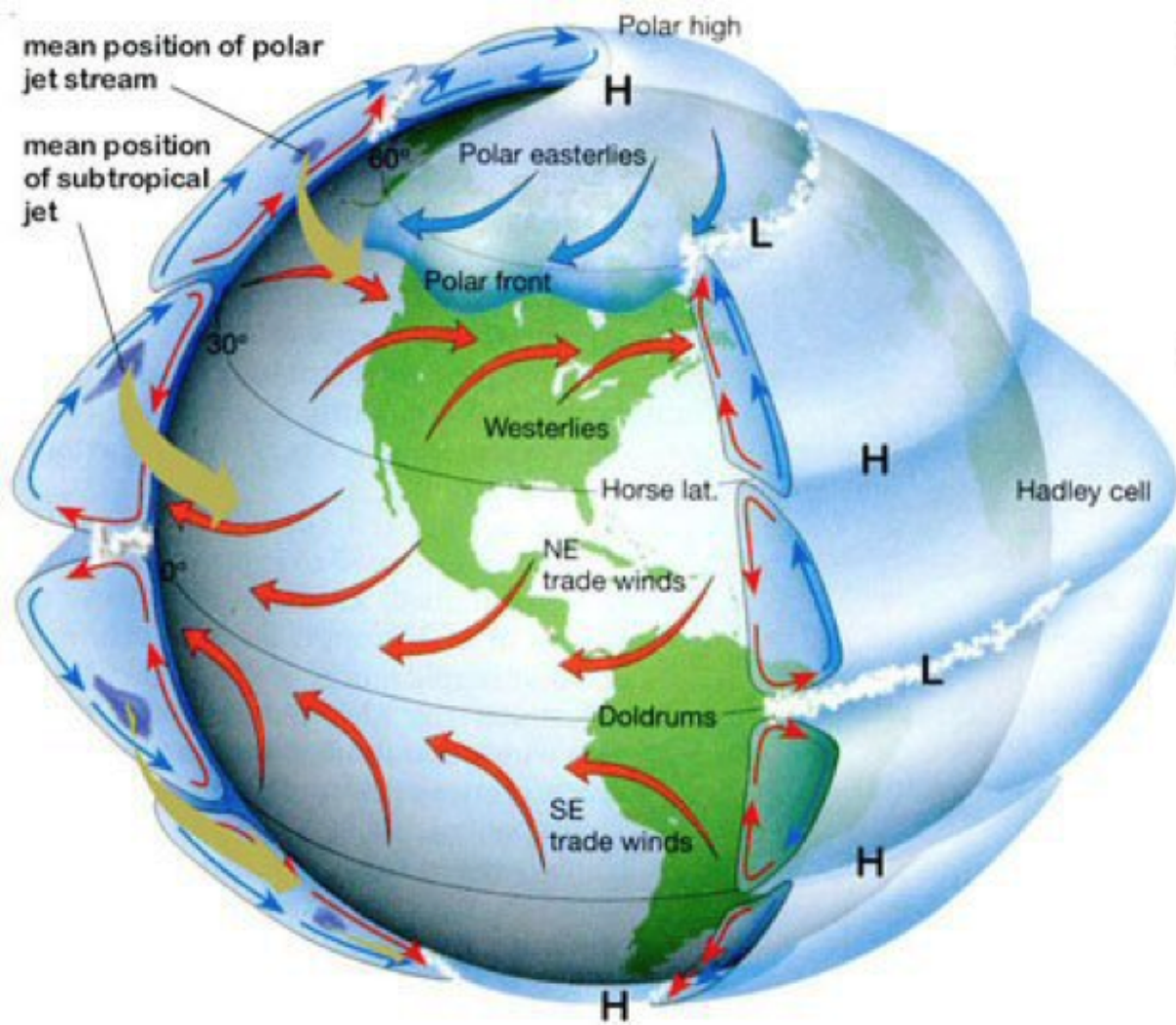
Figure 6.4: Schematic picture of the Hadley Cell and the jet streams.

Exercise 38   –   Angular momentum and Hadley Cell

Consider a zonally symmetric circulation (i.e., one with no longitudinal variations) in the atmosphere. In the inviscid upper troposphere one expects such a flow to conserve absolute angular momentum, i.e.,
$$\frac{DA}{Dt} = 0,$$

where A is the absolute angular momentum per unit mass (parallel to the Earth's rotation axis)

$$A = r\left(u + \Omega r\right) = \Omega a^2 \cos^2 \varphi + ua \cos \varphi \quad .$$

Ω is the Earth rotation rate, u the eastward wind component, $r = a \cos \varphi$ is the distance from the rotation axis, a the Earth's radius, and φ latitude.

1. Show, for inviscid zonally symmetric flow, that the relation $\frac{DA}{Dt} = 0$ is consistent with the zonal component of the equation of motion (using our standard notation, with F_x the x-component of the friction force per unit mass)

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + F_x$$

   in (x, y, z) coordinates, where $y = a\varphi$.

2. Use angular momentum conservation to describe how the existence of the Hadley circulation explains the existence of both the subtropical jet in the upper troposphere.

   Hint: In the upper troposphere, the flow leaves the rising branch of the Hadley cell at the equator (cf. Fig. 6.4) with angular momentum density $A_0 = \Omega a^2$, if we assume that the flow rises from the ground there with no relative motion. The zonal flow can then be described as $u = \Omega a \sin^2 \varphi / \cos \varphi$. Show that the zonal flow will be greatest at the edge of the cell, where φ is greatest, thus producing the subtropical jet.

   Note that $\Omega a^2 = \frac{2\pi}{86400\,s} \cdot (6.371 \cdot 10^6 m)^2 = 3 \cdot 10^9 m^2 s^{-1}$.

3. Describe the near-surface trade winds by recognizing that this low-level flow is under the

influence of surface friction and A will therefore be progressively reduced. Show that

$$u_{low} < \Omega a \frac{\sin^2 \varphi}{\cos \varphi}$$

and some φ that u_{low} becomes negative (eastward winds).

4. If the Hadley circulation is symmetric about the equator, and its edge is at $20°$ latitude, determine the strength of the subtropical jet.

   (The observed zonal winds are weaker than the value. In reality, non-axisymmetric atmospheric eddies act to reduce angular momentum in the outflow, and hence reduce the strength of the jets.)

5. Consider the tropical Hadley circulation in northern winter. The circulation rises at $10°S$, moves northward across the equator in the upper troposphere, and sinks at $20°N$. Assuming that the circulation, outside the near-surface boundary layer, is zonally symmetric (independent of x) and inviscid (and thus conserves absolute angular momentum about the Earth's rotation axis), and that it leaves the boundary layer at $10°S$ with zonal velocity u = 0, calculate the zonal wind in the upper troposphere and provide the numbers for the equator, $10°N$, and $20°N$.

Solution of Exercise 38

1. For inviscid axisymmetric flow, conservation of angular momentum implies

$$D_t(\Omega a^2 \cos^2 \varphi + ua \cos \varphi) = 0$$

Remember that $y = a\varphi, dx = a \cos \varphi \, d\lambda$. Here, we reformulate the planetary term:

$$D_t(\Omega a^2 \cos^2 \varphi) = v\partial_y(\Omega a^2 \cos^2 \varphi) \tag{6.25}$$

$$= \Omega a v \partial_\varphi(\cos^2 \varphi) \tag{6.26}$$

$$= -2\Omega a v \sin \varphi \cos \varphi \tag{6.27}$$

$$= -fv \cdot a \cos \varphi \tag{6.28}$$

Similar

$$D_t(ua \cos \varphi) = a \cos \varphi D_t u + u \cdot v \, \partial_\varphi \cos \varphi \tag{6.29}$$

where in the coordinate system

$$D_t u = (\partial_t + u\partial_x + v\partial_y)u + \frac{uv}{a} \tan \varphi \tag{6.30}$$

(the last term is a metric term). Therefore and under the assumption $\partial_x p = 0$:

$$D_t u - fv = 0 \tag{6.31}$$

2. Use angular momentum conservation to describe how the existence of the Hadley circulation explains the existence of both the subtropical jet in the upper troposphere. In the upper troposphere, the flow leaves the rising branch of the Hadley cell at the equator (cf. Fig. 6.4) with angular momentum density $A_0 = \Omega a^2$, if we assume that the flow rises from the ground there with no relative motion. We have

$$A = \Omega a^2 \cos^2 \varphi + ua \cos \varphi = A_0 = \Omega a^2$$

and therefore the zonal flow can then be described as

$$u = \Omega a \sin^2 \varphi / \cos \varphi \, .$$

The zonal flow will be greatest at the edge of the cell, where φ is greatest, thus producing the subtropical jet.

3. If the return flow, in the lower troposphere, were inviscid and thus also conserved angular momentum with $A_{low} = A_0$, then at a given latitude the low level flow would be the same as that aloft, since in u is a function of φ only. However, in reality this low-level flow is under the influence of surface friction and A will therefore be progressively reduced. Thus,

$$A_{low} = \Omega a^2 \cos^2 \varphi + u_{low} a \cos \varphi < A_0 = \Omega a^2$$

$$\text{Thus} \quad u_{low} < \Omega a \frac{\sin^2 \varphi}{\cos \varphi}$$

and some φ_0 north of the equator that u_{low} becomes negative (eastward winds), and so the low level flow will be equatorward and eastward there. (Note that $\Omega a^2 = \frac{2\pi}{86400\,s} \cdot (6.371 \cdot 10^6 m)^2 = 3 \cdot 10^9 m^2 s^{-1}$.)

4. Assume the Hadley circulation is symmetric about the equator, and its edge is at $20°$ latitude, determine the strength of the subtropical jet by

$$u(20°) = \Omega a \sin^2(20°)/\cos(20°) = 57.6\,m\,s^{-1}$$

The observed zonal winds are weaker than the value. In reality, non-axisymmetric atmospheric eddies act to reduce angular momentum in the outflow, and hence reduce the strength of the jets.

5. Consider the angular momentum $A = \Omega a^2 \cos^2 \varphi + ua \cos \varphi = A_0 = \Omega a^2$ with $A_0 = \Omega a^2 \cos^2(10°S) = 2.952 \cdot 10^9 m^2 s^{-1}$.

$$\text{Therefore} \quad u = \frac{A_0 - \Omega a^2 \cos^2 \varphi}{a \cos \varphi}$$

At $\varphi = 0°, u = -13.9\,ms^{-1}$, at $\varphi = 10°, u = 0\,ms^{-1}$, at $\varphi = 20°, u = 42.8\,ms^{-1}$.

6.5   Conservation of vorticity

In simple words, vorticity is the rotation of the fluid. The rate of rotation can be defined in various ways. Consider a bowl of water sitting on a table in a laboratory. The water may be spinning in the bowl. In addition to the spinning of the water, the bowl and the laboratory are rotating because they are on a rotating earth. The two processes are separate and lead to two types of vorticity.

Everything on earth, including the ocean, the atmosphere, and bowls of water, rotates with the earth. This rotation is the *planetary vorticity* f. It is twice the local rate of rotation of earth:

$$f \equiv 2\,\Omega \sin\varphi \quad \left(\frac{1}{s}\right) = 2\,\sin\varphi \quad \left(\frac{cycles}{day}\right) \tag{6.32}$$

Planetary vorticity is also called the Coriolis parameter. It is greatest at the poles where it is twice the rotation rate of earth. Note that the vorticity vanishes at the equator and that the vorticity in the Southern Hemisphere is negative because φ is negative.

The ocean and atmosphere do not rotate at exactly the same rate as the Earth. They have some rotation relative to Earth due to currents and winds. *Relative vorticity* ζ is the vorticity due to currents in the ocean.[5] Mathematically it is:

$$\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{6.33}$$

where we have assumed that the flow is two-dimensional.

For a rigid body rotating at rate Ω, $\zeta = 2\,\Omega$. Of course, the flow does not need to rotate as a rigid body to have relative vorticity. Vorticity can also result from shear. For example, at a north/south western boundary in the ocean, $u = 0$, $v = v(x)$ and $\zeta = \partial v(x)/\partial x$.

ζ is usually much smaller than f. To make an estimate for ζ : It is greatest at the edge of fast currents such as the Gulf Stream. To obtain some understanding of the size of ζ, consider the edge

---

[5]ζ is the vertical component of the threedimensional vorticity vector ω, and it is sometimes written ω_z. ζ is positive for counter-clockwise rotation viewed from above. This is the same sense as Earth's rotation in the Northern Hemisphere. One could use ω_z for relative vorticity, but ω is also commonly used to mean frequency in radians per second.

of the Gulf Stream off Cape Hatteras where the velocity decreases by $1\,ms^{-1}$ in 100km at the boundary. The curl of the current is approximately

$$\zeta = \frac{\partial v}{\partial x} = \frac{1\,ms^{-1}}{100\,km} = 0.14\;\;\frac{cycles}{day} = 1\;\;\frac{cycle}{week} = 1.62 \cdot 10^{-6}\,\frac{1}{s}. \quad (6.34)$$

Hence even this large relative vorticity is still almost seven times smaller than f (compare 6.32). A more typical value of relative vorticity, such as the vorticity of eddies, is a cycle per month. The sum of the planetary and relative vorticity is called absolute vorticity:

$$\text{Absolute Vorticity} \equiv (\zeta + f) \quad (6.35)$$

We can obtain an equation for absolute vorticity in the ocean by manipulating the equations of motion for frictionless flow. We begin with:

$$\frac{Du}{Dt} - f\,v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad (6.36)$$

$$\frac{Dv}{Dt} + f\,u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad (6.37)$$

If we expand the substantial derivative, and if we subtract $\partial/\partial y$ of (6.36) from $\partial/\partial x$ of (6.37) to eliminate the pressure terms, we obtain

$$\frac{D}{Dt}\left(\partial_x v - \partial_y u\right) \;+\; \left(\partial_x u \partial_x v + \partial_x v \partial_y v\right) - \left(\partial_y u \partial_x u + \partial_y v \partial_y u\right)$$
$$+\; f\left(\partial_x u + \partial_y v\right) + v\,\partial_y f \;=\; 0 \quad (6.38)$$

Using $\frac{D}{Dt} f = v\,\partial_y f$:

$$\frac{D}{Dt}\zeta \;+\; \partial_x v\left(\partial_x u + \partial_y v\right) - \partial_y u\left(\partial_x u + \partial_y v\right)$$
$$+\; f\left(\partial_x u + \partial_y v\right) + \frac{D}{Dt} f \;=\; 0 \quad (6.39)$$

this yields

$$\frac{D}{Dt}(\zeta + f) + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad . \tag{6.40}$$

Exercise 39   –   **Non-dimensional system of the vorticity dynamics**

a) For cnstant depth, derive the the non-dimensional version of the vorticity equation

$$\frac{D}{Dt}(\zeta + f) = \nu \nabla^2 \zeta \quad .$$

Hint: Repeat exercise 30. b) What are the characteristic numbers?

c) Estimate the order of magnitude of the characteristic numbers for the atmosphere and ocean !

You can use Table 6.2 and other references.

6.5.1   Potential vorticity equation $(\zeta + f)/h$

The rotation rate of a column of fluid changes as the column is expanded or contracted. This changes the vorticity through changes in ζ. To see how this happens, consider barotropic, geostrophic flow in an ocean with depth $h(x, y, t)$, where h is the distance from the sea surface to the bottom. That is, we allow the surface to have topography (Fig. 6.5). Integrating the continuity equation from the bottom to the top of the ocean gives:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\int_b^{b+h} dz + w\big|_b^{b+h} = 0 \tag{6.41}$$

where b is the topography of the bottom, and h is the depth of the water. Notice that $\partial u/\partial x$ and $\partial v/\partial y$ are independent of z because they are barotropic, and the terms can be taken outside the integral. The boundary conditions require that flow at the surface and the bottom be along the
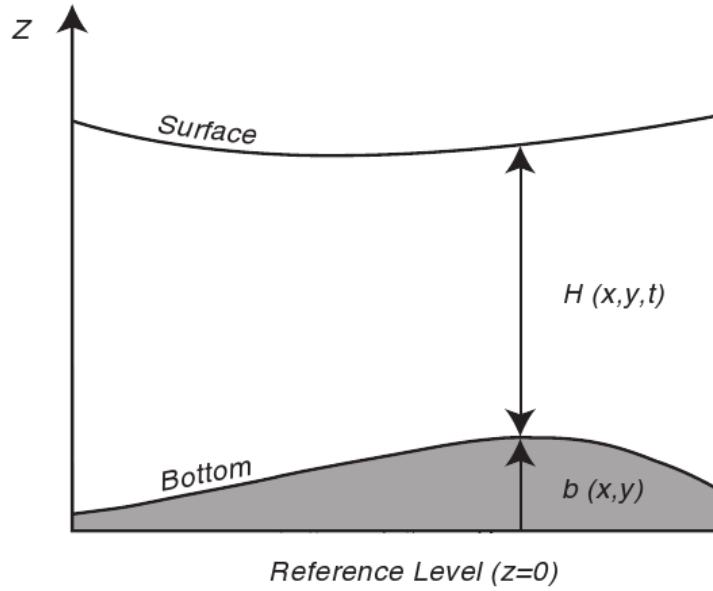
Figure 6.5: Sketch of fluid flow used for deriving conservation of potential vorticity. Here $H = h$. After Cushman-Roisin (1994: 55).

surface and the bottom. Thus the vertical velocities at the top and the bottom are:

$$w|_{b+h} = D_t(b+h) = \frac{\partial (b+h)}{\partial t} + u\frac{\partial (b+h)}{\partial x} + v\frac{\partial (b+h)}{\partial y} \tag{6.42}$$

$$w|_b = D_b = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} \tag{6.43}$$

where we used $\partial b/\partial t = 0$ because the bottom does not move, and $\partial h/\partial z = 0$. Substituting (6.42) and (6.43) into (6.41) we obtain

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{h}\frac{Dh}{Dt} = 0 \tag{6.44}$$

Substituting this into (6.40) gives:

$$\frac{D}{Dt}(\zeta + f) - \frac{(\zeta + f)}{h}\frac{Dh}{Dt} = 0 \tag{6.45}$$
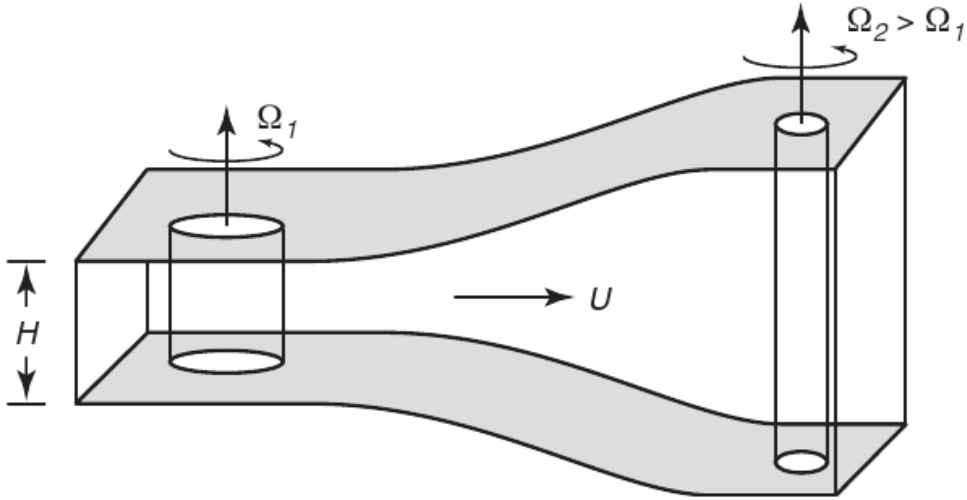
Figure 6.6: Sketch of the production of relative vorticity by change in the height of a fluid column. As the vertical fluid column moves from left to right, vertical stretching reduces the moment of inertia of the column, causing it to spin faster.

which can be rewritten as

$$\frac{1}{h}\frac{D}{Dt}\left(\zeta + f\right) - \left(\zeta + f\right)\frac{D_t h}{h^2} \;=\; 0 \tag{6.46}$$

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) \;=\; 0 \;\;. \tag{6.47}$$

The quantity within the parentheses must be constant. It is called *potential vorticity* Π. Potential vorticity is conserved along a fluid trajectory:

$$\text{Potential Vorticity} = \Pi \equiv \frac{\zeta + f}{h} \tag{6.48}$$

The angular momentum of any isolated spinning body is conserved. The spinning body can be an eddy in the ocean or the earth in space. If the spinning body is not isolated, that is, if it is linked to another body, then angular momentum can be transferred between the bodies. The conservation of potential vorticity couples changes in depth, relative vorticity, and changes in latitude. All three interact:
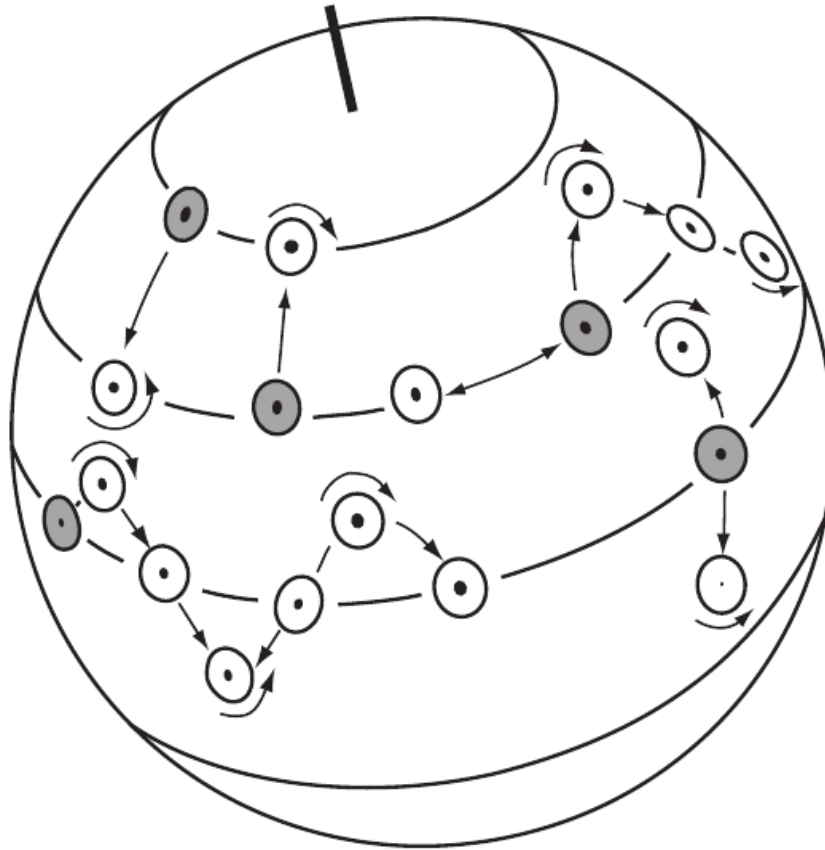
Figure 6.7: Angular momentum tends to be conserved as columns of water change latitude. This changes the relative vorticity of the columns. After von Arx (1962).

- Changes in the depth h of the flow results in change of the relative vorticity. The concept is analogous with the way figure skaters decrease their spin by extending their arms and legs. The action increases their moment of inertia and decreases their rate of spin (Fig. 6.6).

- Changes in latitude require a corresponding change in ζ. As a column of water moves equatorward, f decreases, and ζ must increase (Fig. 6.7). If this seems somewhat mysterious, von Arx (1962) suggests we consider a barrel of water at rest at the north pole. If the barrel is moved southward, the water in it retains the rotation it had at the pole, and it will appear to rotate counterclockwise at the new latitude where f is smaller.

Potential vorticity conservation $(\zeta + f)/h$: Implications

The concept of conservation of potential vorticity has far reaching consequences, and its application to fluid flow in the ocean gives a deeper understanding of ocean currents.

Flow Tends to be Zonal

In the ocean f tends to be much larger than ζ and thus $f/h =$ constant. This requires that the flow in an ocean of constant depth be zonal. Of course, depth is not constant, but in general, **currents tend to be east-west rather than north-south**. Wind makes small changes in ζ, leading to a small meridional component of the flow (see Fig. 6.7).

Topographic Steering

Barotropic flows are diverted by sea floor features. Consider what happens when a flow that extends from the surface to the bottom encounters a sub-sea ridge (Fig. 6.8). As the depth decreases, $\zeta + f$ must also decrease, which requires that f decrease, and the flow is turned toward the equator. This is called topographic steering. If the change in depth is sufficiently large, no change in latitude will be sufficient to conserve potential vorticity, and the flow will be unable to cross the ridge. This is called topographic blocking.

Streamfunction f/h

In the ocean, f tends to be much larger than ζ and

$$\frac{D}{Dt}\left(\frac{f}{h}\right) = 0 \tag{6.49}$$
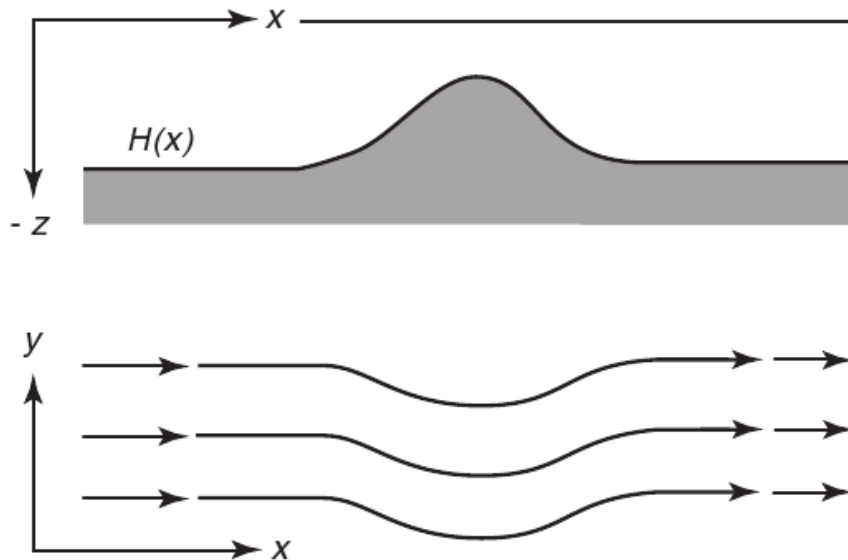
Figure 6.8: Barotropic flow over a sub-sea ridge is turned equatorward to conserve potential vorticity. After Dietrich et al. (1980: 333).

implies $f/h = $ constant along the flow. In this case, we have a streamfunction Ψ and pressure p that are functions of f/h:

$$\Psi = \Psi(f/h) \quad ; \qquad p = p(f/h). \tag{6.50}$$

This requires that the flow in an ocean of constant depth be zonal. Of course, depth is not constant, but in general, currents tend to be east-west rather than north-south. Wind makes small changes in ζ, leading to a small meridional component of the flow (see figure 6.7). The geostrophic contours f/h turn out to be an interesting combination of latitude circles and bottom topographic contours. Over small horizontal distances[6] and at high latitude topography, h tends to dominate (as in the example in Fig.6.10), but over longer distances or in the tropics, the latitude-variation of f dominates.

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[6]Then $\frac{D}{Dt}\left(\frac{f}{h}\right) = 0$ can be transformed into $\frac{D}{Dt}h = 0$.

Exercise 40   –   Differential operators

Deriving the vorticity equation

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0 \quad ,$$

we need to evaluate the terms $\partial_y \frac{D}{Dt} u$ and $\partial_x \frac{D}{Dt} v$. Write down the explicit terms!

Exercise 41   –   Potential vorticity in the atmosphere

An air column at $53°$N with $\zeta = 0$ initially streches from the surface to a fixed tropopause at 10 km height. If the air column moves until it is over a mountain barrier 2.5 km hight at $30°$N, what is its absolute vorticity and relative vorticity as it passes the mountain top?
Assume: $\sin 53° = 0.8$; $\sin 30° = 0.5$. The angular velocity of the Earth $\Omega = 2\pi/(1 day)$.

Exercise 42   –   f/h contours

Geostrophic contours using available topography data. Barotropic flows are diverted by sea floor features. Consider what happens when a flow that extends from the surface to the bottom encounters a sub-sea ridge.

1. Show the f/h contours for the North Atlantic Ocean! See Fig. 6.9.
2. Show it for low latitudes regions: region around $20°S$ to $20°N$ in the Atlantic and Pacific Ocean. One problem is that the geostrophic contours bump into continents, so that ocean currents running along them have a serious difficulty there. Actually all such f/h contours head toward the Equator as they run up into shallow water (as $h \to 0 \quad f \to 0$ also, hence $\varphi \to 0$). This shows that we need more terms in the vorticity dynamics to describe the ocean circulation.
3. The examination of tidal rhythmites and theories about the Earth-Moon dynamics suggest that the length of day 900 million years ago was 18 h instead of 24h. How are the results of the vorticity dynamics are affected?
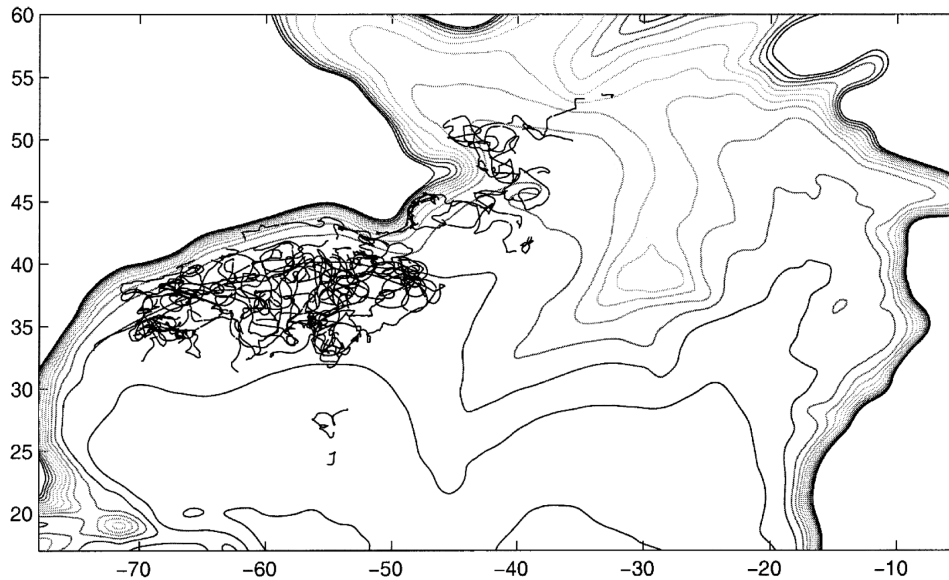
Figure 6.9: Floats in the northwestern NorthAtlantic below 1000m. The trajectories, superimposed on the smoothed f/h contours (LaCasce, 2000).
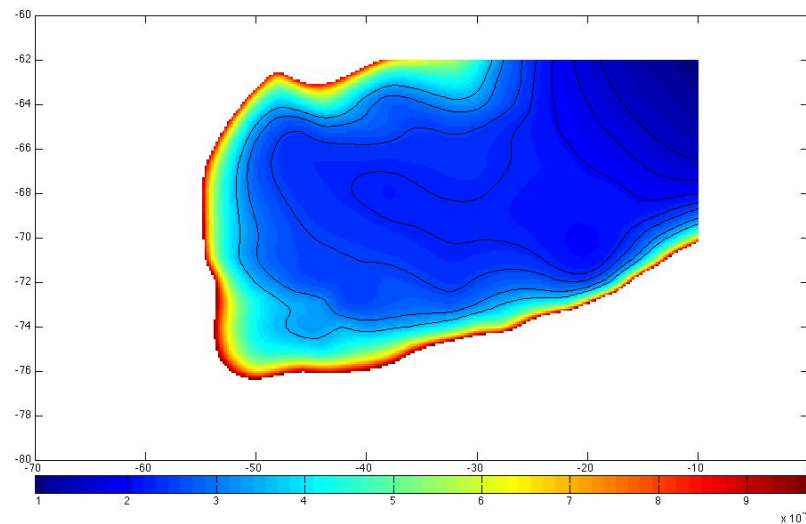
Figure 6.10: f/h countour in the Weddell Sea for 34 Ma $(34 \cdot 10^6$ years before present).

4. For the Miocene (about 34 Million years ago), the topography data were provided in the course. Calculate the f/h-contours! The length of the day was nearly as today. See Fig. 13.3.

Baroclinic flow in a continuously stratified fluid

For baroclinic flow in a continuously stratified fluid, the potential vorticity can be written (Pedlosky, 1987)

$$\Pi = \frac{\zeta + f}{\rho} \cdot \nabla \lambda \tag{6.51}$$

where λ is any conserved quantity for each fluid element. In particular, if $\lambda = \rho$ then:

$$\Pi = \frac{\zeta + f}{\rho} \frac{\partial \rho}{\partial z} \tag{6.52}$$

assuming the horizontal gradients of density are small compared with the vertical gradients, a good assumption in the thermocline. In most of the interior of the ocean, $f \gg \zeta$ and (6.52) is written (Pedlosky, 1996)

$$\Pi = \frac{f}{\rho} \frac{\partial \rho}{\partial z} \tag{6.53}$$

This allows the potential vorticity of various layers of the ocean to be determined directly from hydrographic data without knowledge of the velocity field.

6.5.2 Taylor-Proudman Theorem

The influence of vorticity due to Earth's rotation is most striking for geostrophic flow of a fluid with constant density ρ_0 on a plane with constant rotation $f = f_0$. The components of the geostrophic and hydrostatic pressure equations are:

$$-f\,v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{6.54}$$

$$f\,u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{6.55}$$

$$g = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \tag{6.56}$$

and the continuity equation is:

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{6.57}$$

Taking the z derivative of (6.54) and using (6.56) gives:

$$-f_0\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{\rho_0}\frac{\partial p}{\partial z}\right) = \frac{\partial g}{\partial x} = 0 \tag{6.58}$$

Therefore for $f_0 \neq 0$

$$\frac{\partial v}{\partial z} = 0$$

Similarly, for the u-component of velocity (6.55). Thus, the vertical derivative of the horizontal velocity field must be zero.

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \tag{6.59}$$

The flow is two-dimensional and does not vary in the vertical direction. This is the *Taylor-Proudman Theorem*, which applies to slowly varying flows in a homogeneous, rotating, inviscid fluid. The theorem places strong constraints on the flow[7]. The physical origin of this strangely

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[7]Taylor (1921): If therefore any small motion be communicated to a rotating fluid the resulting motion of the fluid must be one in which any two particles originally in a line parallel to the axis of rotation must remain so, except for

constrained flow is in the stiffness endowed to the fluid by rapid rotation of the Earth, which has a peculiarly strong sense along the axis of rotation. Taylor's laboratory experiments showed how homogeneous fluid tends to move in vertical columns. Dye in the water forms curtains, and viewing the dye from above shows fine twists and whirls that are vertically coherent.

Hence, rotation greatly stiffens the flow! Geostrophic flow cannot go over a seamount, it must go around it. Taylor [1917] explicitly derived (6.59) and (6.61) below. Proudman [1916] independently derived the same theorem but not as explicitly.

Laboratory experiments showing the formation of a Taylor column, go to 2:50, other material: vorticity and circulation, boundary layers, good introduction, Taylor column

Vertical velocity in the the Taylor-Proudman theorem

Further consequences of the theorem can be obtained by eliminating the pressure terms from (6.54, 6.55) to obtain:

$$
\begin{aligned}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= -\frac{\partial}{\partial x}\left(\frac{1}{f_0\rho_0}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f_0\rho_0}\frac{\partial p}{\partial x}\right) \\
&= \frac{1}{f_0\rho_0}\left(-\frac{\partial^2 p}{\partial x \partial y} + \frac{\partial^2 p}{\partial x \partial y}\right) = 0
\end{aligned}
\tag{6.60}
$$

Because the fluid is incompressible, the continuity equation (6.57) requires

$$
\frac{\partial w}{\partial z} = 0
\tag{6.61}
$$

Furthermore, because $w = 0$ at the sea surface and at the sea floor, if the bottom is level, there can be no vertical velocity on an f-plane.

---

possible small oscillations about that position.

Geostrophic flow: Vertical velocity leads to north-south currents

If the Taylor-Proudman theorem in (6.61) is true, the flow cannot expand or contract in the vertical direction, and it is indeed as rigid as a steel bar. Since we observe gradients of vertical movements, one of the constraints used in deriving (6.61) must be violated, i.e. our assumption that $f = f_0$ can not be a good approximation.

Going back to (6.40):

$$\frac{D}{Dt}(\zeta + f) + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad . \tag{6.62}$$

we obtain

$$\beta\, v \quad + \quad f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad . \tag{6.63}$$

Using the continuity equation, we obtain

$$f\frac{\partial w_g}{\partial z} = \beta\, v \tag{6.64}$$

where we have used the subscript $_g$ to emphasize that (6.64) applies to the ocean's interior, geostrophic flow. Thus the variation of Coriolis force with latitude allows vertical velocity gradients in the geostrophic interior of the ocean, and the vertical velocity leads to north-south currents.

6.6 Wind-driven ocean circulation

What drives the ocean currents? At first, we might answer, the winds drive the circulation. But if we think more carefully about the question, we might not be so sure. We might notice, for example, that strong currents, such as the North Equatorial Countercurrents in the Atlantic and Pacific Ocean go upwind. Spanish navigators in the 16th century noticed strong northward currents along the Florida coast that seemed to be unrelated to the wind. How can this happen? And, why are strong currents found offshore of east coasts but not offshore of west coasts?[8]

Friction is essential for the transfer of momentum in a fluid. Friction transfers momentum from the atmosphere to the ocean through the thin, frictional, Ekman layer at the sea surface. Friction transfers momentum from the ocean to the solid earth through the Ekman layer at the sea floor. Friction along the sides of subsea mountains leads to pressure differences on either side of the mountain which causes another kind of drag called *form drag*. This is the same drag that causes wind force on cars moving at high speed. In the vast interior of the ocean, however, the flow is frictionless, and vorticity is conserved. Such a flow is said to be *conservative*. Here, we apply the vorticity dynamics for the ocean and include the wind stress term in (6.16, 6.17):

$$D_t u - f v \; = \; -\frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{1}{\rho}\partial_z \tau_{xz} \tag{6.65}$$

$$D_t v + f u \; = \; -\frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\partial_z \tau_{yz} \tag{6.66}$$

in order to get the modified vorticity balance (6.45):

$$\frac{D}{Dt}(\zeta + f) - \frac{(\zeta + f)}{h}\frac{D}{Dt}h \; = \; \frac{1}{\rho}\,\mathrm{curl}_z\,\partial_z\tau \; = \; \frac{1}{\rho}\left(\frac{\partial}{\partial x}\,\partial_z\tau_{yz} - \frac{\partial}{\partial y}\,\partial_z\tau_{xz}\right) \tag{6.67}$$

---

[8]Answers to the questions can be found in a series of three remarkable papers published from 1947 to 1951. In the first, Harald Sverdrup (1947) showed that the circulation in the upper kilometer or so of the ocean is directly related to the curl of the wind stress. Henry Stommel (1948) showed that the circulation in oceanic gyres is asymmetric because the Coriolis force varies with latitude. Finally, Walter Munk (1950) added eddy viscosity and calculated the circulation of the upper layers of the Pacific. Together the three oceanographers laid the foundations for a modern theory of ocean circulation.

	Quantity	Ocean
horizontal velocity	U	$1.6 \cdot 10^{-2}\, ms^{-1}$
horizontal length	L	$10^6\, m$
vertical length	D	$10^3\, m$
wind stress	τ_0	$1.5 \cdot 10^{-1}\, Pa$
Coriolis parameter at 45°N	$f_0 = 2\Omega \sin \varphi_0$	$10^{-4}\, s^{-1}$
density	ρ_0	$10^3\, kgm^{-3}$
viscosity (turbulent)	A_H	$10^2 - 10^4\, m^2 s^{-1}$
Reynolds number	Re	$1.6 - 160$
wind stress strength number	α	$1 \cdot 10^3$

Table 6.2: Table shows the typical scales in the ocean system for exercise 43.

The formulation 'wind stress curl' stands for the z-component of

$$
\nabla \times \begin{pmatrix} \tau_x \\ \tau_y \\ 0 \end{pmatrix} = curl \begin{pmatrix} \tau_x \\ \tau_y \\ 0 \end{pmatrix} \quad .
$$

Exercise 43   –   Non-dimensional vorticity dynamics including wind stress

a) Derive the the non-dimensional version of the vorticity equation (6.67) assuming that h is not varying and include friction! We can vertically integrate (6.67) over depth $\int_{-D}^{0} dz$:

$$
\frac{D}{Dt}(\zeta + f) = A_H \nabla^2 \zeta + \frac{1}{\rho D}\left(\frac{\partial}{\partial x}\tau_y - \frac{\partial}{\partial y}\tau_x \right) \quad . \tag{6.68}
$$

Include the Reynolds number $Re = UL/A_H$, Rossby number $Ro = U/(f_0 L)$, and the wind stress strength number $\alpha = \tau_0 L/(\rho_0 D U^2)$. *Compare to exercises 30, 39.*

b) Estimate the order of magnitude of the characteristic numbers for the ocean ! You can use Table 6.6.

Solution of Non-dimensional vorticity dynamics including wind stress

Starting from (6.68),

$$\frac{1}{T}\frac{D}{Dt_d}\left(\frac{1}{T}\zeta_d + f_0 f_d\right) = A_H \frac{1}{L^2 T}\nabla_d^2 \zeta_d + \frac{\tau_0}{\rho_0 DL}\left(\frac{\partial}{\partial x_d}\tau_{y,d} - \frac{\partial}{\partial y_d}\tau_{x,d}\right) \quad (6.69)$$

$$\frac{D}{Dt_d}\left(\frac{1}{T}\zeta_d + f_0 f_d\right) = A_H \frac{1}{L^2}\nabla_d^2 \zeta_d + \frac{\tau_0 T}{\rho_0 DL}\left(\frac{\partial}{\partial x_d}\tau_{y,d} - \frac{\partial}{\partial y_d}\tau_{x,d}\right) \quad (6.70)$$

Multiplying with T and using $T = L/U$, we obtain

$$\frac{D}{Dt_d}\left(\zeta_d + \frac{f_0 L}{U} f_d\right) = \frac{A_H}{UL}\nabla_d^2 \zeta_d + \frac{\tau_0 L}{\rho_0 DU^2}\left(\frac{\partial}{\partial x_d}\tau_{y,d} - \frac{\partial}{\partial y_d}\tau_{x,d}\right) \quad (6.71)$$

and finally

$$\frac{D}{Dt_d}\left(\zeta_d + \frac{1}{Ro} f_d\right) = \frac{1}{Re}\nabla_d^2 \zeta_d + \alpha\left(\frac{\partial}{\partial x_d}\tau_{y,d} - \frac{\partial}{\partial y_d}\tau_{x,d}\right) \quad (6.72)$$

6.6.1   Sverdrup relation

Suppose for simplicity that h is constant, so the only 'topography' is that of the spherical shape of the ocean. This produces a simple version of the vorticity balance in the ocean circulation, the Sverdrup relation. For most regions, the right hand side in (6.67) or the left hand side in (6.72) is dominated by the term $v\partial_y f = v\beta$.

$$v\beta = \frac{1}{\rho}\left(\frac{\partial}{\partial x}\partial_z \tau_{yz} - \frac{\partial}{\partial y}\partial_z \tau_{xz}\right) \quad . \quad (6.73)$$

Integrating over z, we receive

$$\beta\int_h^0 dz\, v = \beta V = \frac{1}{\rho}\mathrm{curl}_z\, \tau(z=0) = \frac{1}{\rho}\left(\frac{\partial\tau_{yz}}{\partial x} - \frac{\partial\tau_{xz}}{\partial y}\right) \quad . \quad (6.74)$$

Mass transport and Stream Lines of Sverdrup's Theory*

While Sverdrup was analyzing observations of equatorial currents, he derived the relation by the the wind stress to mass transport within the upper ocean. It is assumed that the flow is stationary, that lateral friction and molecular viscosity are small, that non-linear terms such as $u \partial u / \partial x$ are small. With these assumptions, the horizontal components of the momentum equation are:

$$\frac{\partial p}{\partial x} = f \rho v + \frac{\partial \tau_{\text{xz}}}{\partial z} \tag{6.75}$$

$$\frac{\partial p}{\partial y} = -f \rho u + \frac{\partial \tau_{\text{yz}}}{\partial z} \tag{6.76}$$

Sverdrup integrated these equations from the surface to a depth $-D$ equal to or greater than the depth at which the horizontal pressure gradient becomes zero. We can define

$$\frac{\partial P}{\partial x} = \int_{-D}^{0} \frac{\partial p}{\partial x} \, dz, \qquad \frac{\partial P}{\partial y} = \int_{-D}^{0} \frac{\partial p}{\partial y} \, dz, \tag{6.77}$$

M_x, M_y are the mass transports in the wind-driven layer extending down to an assumed depth of no motion:

$$M_x \equiv \int_{-D}^{0} \rho \, u \, (z) \, dz, \qquad M_y \equiv \int_{-D}^{0} \rho \, v \, (z) \, dz \quad . \tag{6.78}$$

The horizontal boundary condition at the sea surface is the wind stress, and the boundary at depth $-D$ is zero stress because the currents go to zero:

$$\tau_{xz} \, (0) = \tau_x \qquad \tau_{xz} \, (-D) = 0 \tag{6.79}$$

$$\tau_{yz} \, (0) = \tau_y \qquad \tau_{yz} \, (-D) = 0 \quad . \tag{6.80}$$

Using these definitions and boundary conditions, (6.75, 6.76) becomes:

$$\frac{\partial P}{\partial x} = f \, M_y + \tau_x \tag{6.81}$$

$$\frac{\partial P}{\partial y} = -f \, M_x + \tau_y \quad . \tag{6.82}$$

In a similar way, Sverdrup integrated the continuity equation over the same vertical depth, assuming the vertical velocity at the surface and at depth $-D$ are zero, to obtain:

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_y}{\partial y} = 0 \tag{6.83}$$

Differentiating (6.81) with respect to y and (6.82) with respect to x, subtracting, and using (6.83) gives:

$$\beta \, M_y \quad = \quad \frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y} = \mathrm{curl}_z \, (\tau) \tag{6.84}$$

$$\text{where} \quad \beta \equiv \frac{\partial f}{\partial y} = \frac{2\Omega \cos \varphi}{R} \tag{6.85}$$

with R as earth's radius and φ as latitude. Over much of the open ocean, the wind is zonal and $\partial \tau_y / \partial x$ is sufficiently small that

$$M_y \approx -\frac{1}{\beta} \frac{\partial \tau_x}{\partial y} \tag{6.86}$$

Substituting (6.86) into (6.83), and (6.85) we obtain

$$\frac{\partial M_x}{\partial x} = -\frac{1}{2 \, \Omega \, \cos \varphi} \left(\frac{\partial \tau_x}{\partial y} \tan \varphi + \frac{\partial^2 \tau_x}{\partial y^2} R \right) \tag{6.87}$$

Sverdrup integrated this equation from a north-south eastern boundary at $x = 0$, assuming no flow into the boundary. This requires $M_x = 0$ at $x = 0$. Then

$$M_x \quad = \quad -\frac{1}{2 \, \Omega \, \cos \varphi} \left(\left[\int_0^x \frac{\partial \tau_x}{\partial y} dx' \right] \tan \varphi + \left[\int_0^x \frac{\partial^2 \tau_x}{\partial y^2} dx' \right] R \right) \tag{6.88}$$

$$= \quad -\frac{1}{2 \, \Omega \, \cos \varphi} \left(\tan \varphi \, \frac{\partial}{\partial y} \left[\int_0^x \tau_x dx' \right] + R \frac{\partial^2}{\partial y^2} \left[\int_0^x \tau_x dx' \right] \right) \quad . \tag{6.89}$$

If τ_x can be approximated by its zonal mean, then

$$M_x = -\frac{\Delta x}{2 \, \Omega \, \cos \varphi} \left[\tan \varphi \, \frac{\partial \langle \tau_x \rangle}{\partial y} + R \frac{\partial^2 \langle \tau_x \rangle}{\partial y^2} \right] \tag{6.90}$$

where Δx is the distance from the eastern boundary of the ocean basin, and brackets indicate zonal averages of the wind stress.
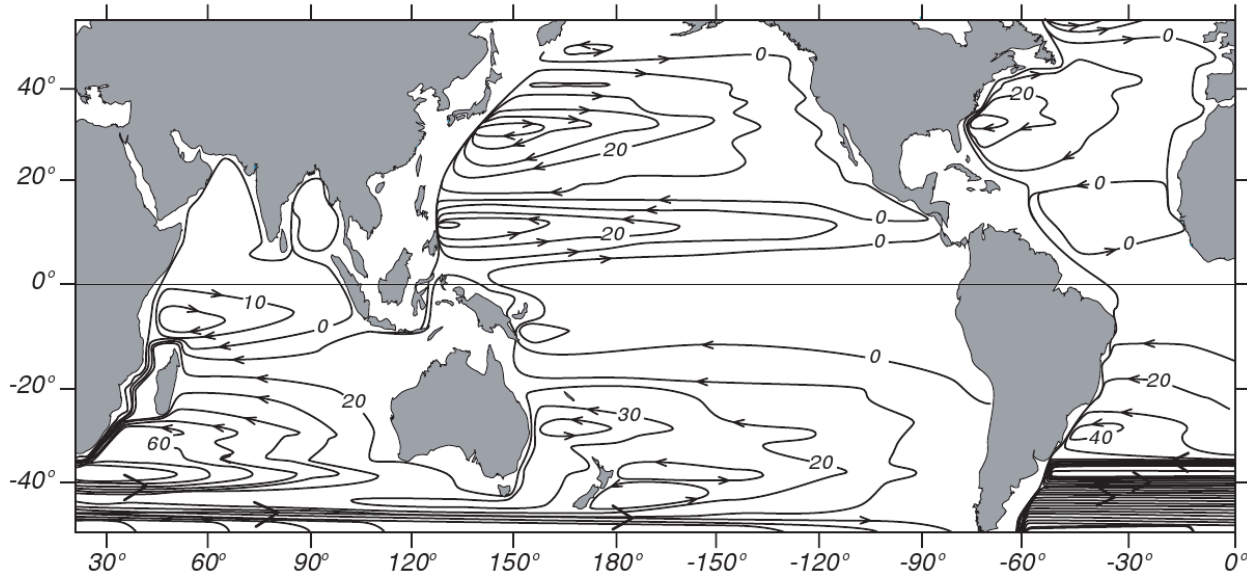
Figure 6.11: Depth-integrated Sverdrup transport applied globally using the wind stress from Hellerman and Rosenstein (1983). Contour interval is **10** Sverdrups (Tomczak and Godfrey, 1994).

6.6.2 Ekman Pumping

Ekman Pumping in a thin Ekman layer

Let us come now to the wind-driven forcing at the surface, the Ekman Pumping. The pressure terms are small because the Ekamn layer is thin. The Ekman layer near the surface of the ocean extends only about 10-20 meters deep[9]. The Ekman transports V_E, U_E describe the dynamics in the upper mixed layer:

$$fV_E = -\tau_x/\rho \tag{6.91}$$

$$fU_E = \tau_y/\rho \tag{6.92}$$

---

[9]The instrumentation sensitive enough to observe a velocity profile in such a shallow depth has only been available since around 1980. Also, wind waves modify the flow near the surface, and make observations close to the surface rather difficult.

where $U_E = \int_{-E}^{0} u\,dz$ and $V_E = \int_{-E}^{0} v\,dz$ are the depth-integrated velocities in the thin friction-dominated Ekman layer at the sea surface. The vertical velocity at the surface is zero and denote w_E as the Ekman vertical velocity the bottom of the Ekman layer.

$$-\int_{-E}^{0} \frac{\partial w}{\partial z}dz = w_E = \frac{\partial}{\partial x}U_E + \frac{\partial}{\partial y}V_E \tag{6.93}$$

The curl of the wind stress τ produces a divergence of the Ekman transports leading to a vertical velocity w_E at the bottom of the Ekman layer.

$$w_E = \text{curl}\left(\frac{\tau}{\rho\,f}\right) = \frac{\partial}{\partial x}\left(\frac{\tau_y}{\rho\,f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_x}{\rho\,f}\right) \quad . \tag{6.94}$$

The order of magnitude of the Ekman vertical velocity w_E can be estimated as from a typical wind stress variation of $0.2 N m^{-2}$ per 2000 km in y-direction:

$$w_E \simeq -\frac{\Delta\tau_x}{\rho\,f_0\Delta y} \simeq \frac{1}{10^3 kg m^{-3}}\frac{0.2 N m^{-2}}{10^{-4}s^{-1}\;2\cdot 10^6 m} \simeq 32\,\frac{m}{yr} \tag{6.95}$$

See for a scetch see Fig. 6.12. The center of a subtropical gyre is a high pressure zone. Circulation around the high pressure is clockwise in the northern hemisphere and counterclockwise in the southern hemisphere, due to the Coriolis effect. The high pressure in the center is due to the westerly winds on the northern side of the gyre and easterly trade winds on the southern side. These cause frictional surface currents towards the latitude at the center of the gyre. This build-up of water in the center creates flow towards the equator in the upper 2 km of the ocean. This flow is returned towards the pole in an intensified western boundary current. The boundary current of the North Atlantic Gyre is the Gulf Stream, of the North Pacific Gyre the Kuroshio Current, etc..
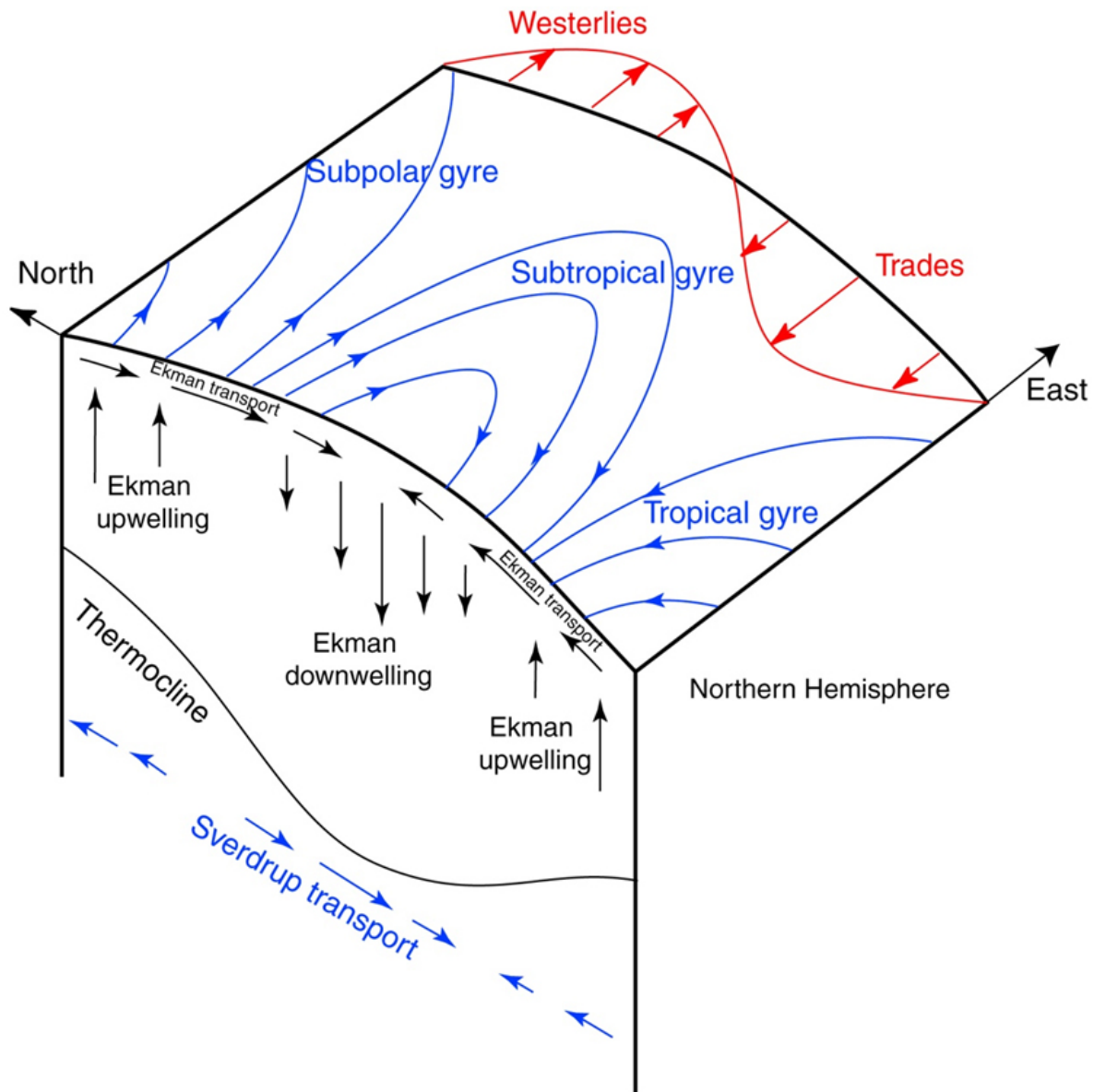
Figure 6.12: Ekman pumping that produces a downward velocity at the base of the Ekman layer forces the fluid in the interior of the ocean to move southward. Winds at the sea surface drive Ekman transports to the right of the wind in this northern hemisphere example. The converging Ekman transports driven by the trades and westerlies drive a downward geostrophic flow just below the Ekman layer.

Ekman vertical velocity and vertical geostrophic velocity

The Ekman vertical velocity must be balanced by a vertical geostrophic velocity $w_g\,(z = -E)$.

$$w_E = w_g \tag{6.96}$$

Ekman pumping drives a vertical geostrophic current in the ocean's interior. Figure 6.12 is a sketch of the cross section of the region between $10°N$ and $60°N$, and it shows the pool of warm water in the upper kilometer centered on $30°N$. Conversely, divergent transports lead to low sea level. The mean north-south pressure gradients associated with the highs and lows are balanced by the Coriolis force of east-west geostrophic currents in the upper ocean. Westerlies in the north drive a southward transport, the trades in the south drive a northward transport. The converging Ekman transports must be balanced by downward geostrophic velocity (6.96).

Figure 6.13 shows the mean zonal winds in the Pacific, together with the north-south Ekman transports driven by the zonal winds. Notice that convergence of transport leads to downwelling, which produces a thick layer of warm water in the upper kilometer of the water column, and high sea level.

Ekman Transports   Sea Surface Height and
                     Geostrophic Currents

80°

Convergence   **H**
$curl_\tau < 0$

Easterlies

AK

60°

Divergence   **L**
$curl_\tau > 0$

Westerlies

40°

Convergence   **H**
$curl_\tau < 0$

20°

NEC

Trades

Divergence   **L**
$curl_\tau > 0$ NECC

-4   0   4   Convergence   **H**
                         $curl_\tau < 0$

Mean Wind Speed (m/s)
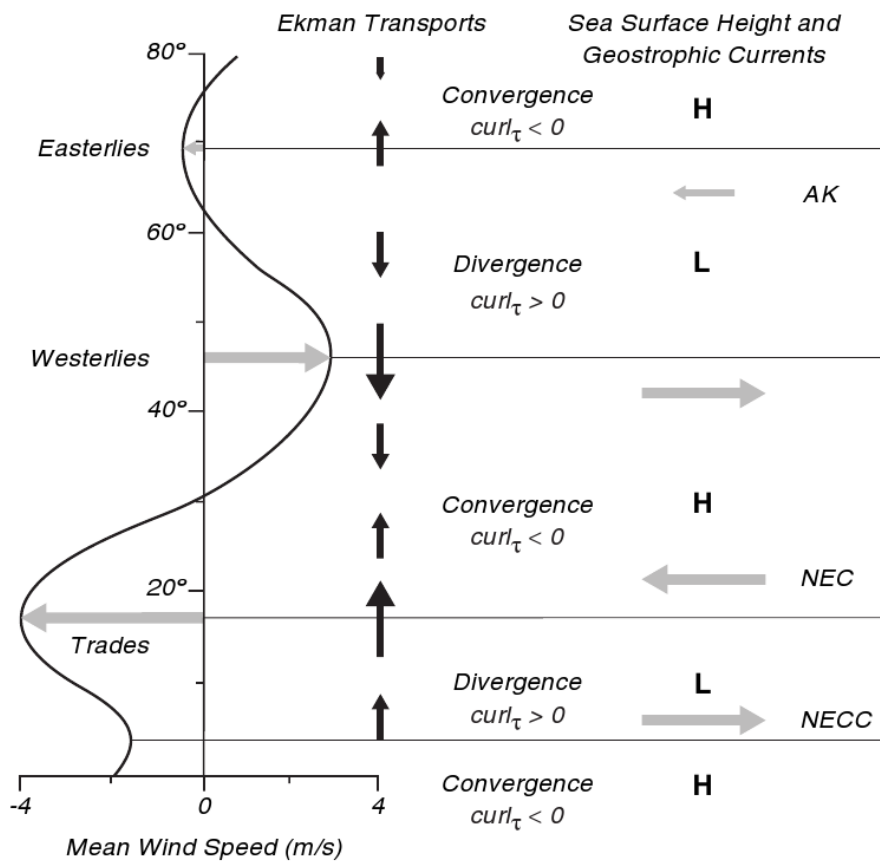
Figure 6.13: Ekman transports due to winds in the north Pacific (left) lead to Ekman pumping (center), which sets up north-south pressure gradients in the upper ocean. The pressure gradients are balanced by the Coriolis force due to east-west geostrophic currents (right). Horizontal lines indicate regions where the curl of the zonal wind stress changes sign. AK: Alaskan Current, NEC: North Equatorial Current, NECC: North Equatorial Counter Current.

Exercise 44   –   Ekman transports and pumping

The Ekman transports V_E, U_E describe the dynamics in the upper mixed layer:

$$fV_E = -\tau_x/\rho_0 \tag{6.97}$$

$$fU_E = \tau_y/\rho_0 \tag{6.98}$$

Derive the Ekman pumping w_E velocity at the bottom of the mixed layer

$$w_E = \mathrm{curl}\left(\frac{\tau}{f\rho_0}\right) = \frac{\partial}{\partial x}\left(\frac{\tau_y}{\rho_0 f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_x}{\rho_0 f}\right) \quad . \tag{6.99}$$

Exercise 45   –   Sverdrup relation, Ekman transports and pumping

The windstress vector τ is taken zonal. Assume $\tau_x = -\tau_0 \cos \pi y/B$ for an ocean basin $0 < x < L, 0 < y < B$. The wind driven meridional ocean velocity is given by the Sverdrup relation

$$\beta V = \mathrm{curl}(\tau/\rho_0) = -\frac{\partial}{\partial y}\tau_x/\rho_0 \quad . \tag{6.100}$$

1. at what latitudes y are $|V|$ and $|V_E|$ maximum? Calculate their magnitudes. Take constant $f = 10^{-4}$ s^{-1} and $\beta = 1.8 \cdot 10^{-11}$ m^{-1}s^{-1} and $B = 5000$ km, $\tau_0/\rho_0 = 10^{-4}$ m^{2}s^{-2}.

2. Calculate the maximum of w_E for constant f (value see above). Is this measurable?
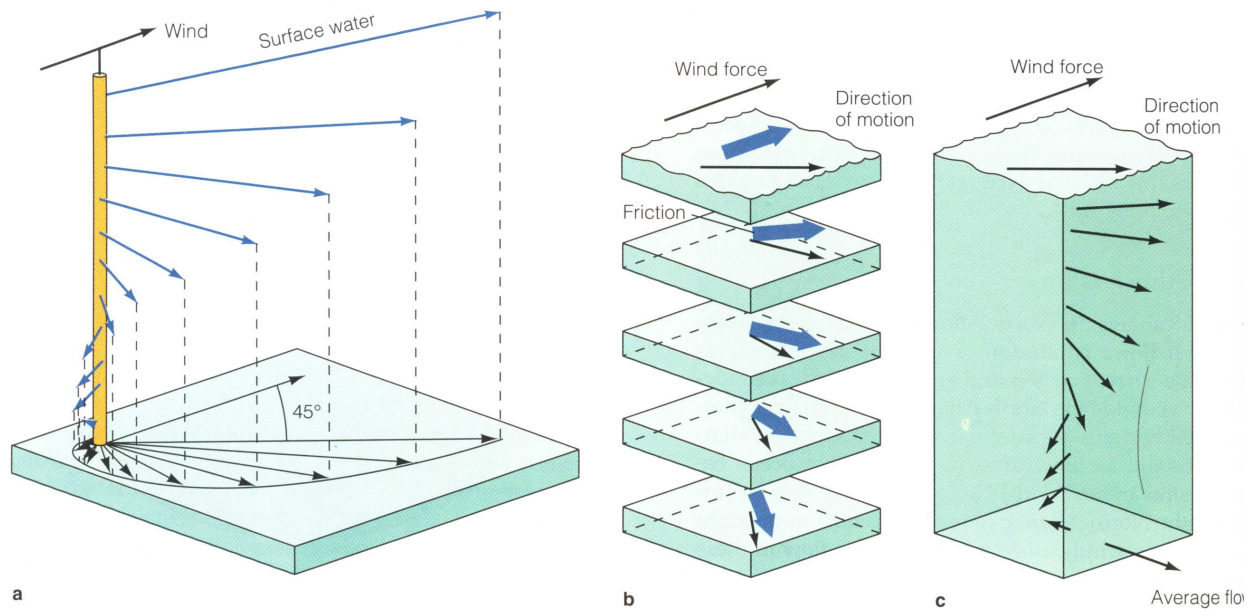
Figure 6.14: The Ekman spiral and the mechanism by which it operates. a) The Ekman spiral model. b) A body of water can be thought as a set of layers. The top layer is driven forward by the wind, and each layer below is moved by friction. Each succeeding layer moves at a slower speed, and at an angle to the layer immediately above it (to the right in the Northern Hemisphere, to the left in the Southern Hemisphere) until friction becomes negligible. (c) Though the direction of movement is different for each layer in the stack, the theoretical average direction of flow of water in the Northern Hemisphere is $90°$ to the right of the prevailing surface wind (Garrison, 1993).

6.6.3 Ekman spiral*

The Ekman spiral is a consequence of the Coriolis effect. When surface water molecules move by the force of the wind, they, in turn, drag deeper layers of water molecules below them. Each layer of water molecules is moved by friction from the shallower layer, and each deeper layer moves more slowly than the layer above it, until the movement ceases at a depth of about 100 meters. Like the surface water, however, the deeper water is deflected by the Coriolis effect-to the right in the Northern Hemisphere and to the left in the Southern Hemisphere. As a result, each successively deeper layer of water moves more slowly to the right or left, creating a spiral effect (Fig. 6.14). Because the deeper layers of water move more slowly than the shallower layers, they tend to "twist around" and flow not in the direction of the surface current.

Ekman developed the theory of the Ekman layer after Fridtjof Nansen observed that ice drifts at an angle of $20 - 40°$ to the right of the prevailing wind direction while on an Arctic expedition aboard the Fram. Nansen asked his colleague, Vilhelm Bjerknes to set one of his students upon study of the problem. Bjerknes tapped Ekman, who presented his results in 1902 as his doctoral thesis.

The mathematical formulation of the Ekman layer can be found by assuming a neutrally stratified fluid, with horizontal momentum in balance between the forces of pressure gradient, Coriolis and turbulent drag.

$$-fv = -\frac{1}{\rho_o}\frac{\partial}{\partial x}p + \nu\frac{\partial^2 u}{\partial z^2}, \tag{6.101}$$

$$fu = -\frac{1}{\rho_o}\frac{\partial}{\partial y}p + \nu\frac{\partial^2 v}{\partial z^2}, \tag{6.102}$$

where ν is the diffusive eddy viscosity, which can be derived using mixing length theory. There are many regions where an Ekman layer is theoretically plausible; they include the bottom of the atmosphere, near the surface of the earth and ocean, the bottom of the ocean, near the sea floor and at the top of the ocean, near the air-water interface. Different boundary conditions are appropriate for each of these different situations. We will consider boundary conditions of the Ekman layer in the upper ocean:

$$\text{at} \quad z = 0: \quad \nu\frac{\partial u}{\partial z} = \tau^x \quad \text{and} \quad \nu\frac{\partial v}{\partial z} = \tau^y, \tag{6.103}$$

where τ^x and τ^y are the components of the surface stress, τ, of the wind field or ice layer at the top of the ocean and u_g and v_g are the geostrophic flows as $z \rightarrow \infty : u \rightarrow u_g, v \rightarrow v_g$. In the other situations, other boundary conditions, such as the no-slip condition, may be appropriate instead.

The dynamics (6.101, 6.102) can be reformulated as

$$-fv = -fv_g + \nu \frac{\partial^2 u}{\partial z^2}, \tag{6.104}$$

$$fu = fu_g + \nu \frac{\partial^2 v}{\partial z^2}, \tag{6.105}$$

Now multiply (6.104) with i and subtract it from (6.105):

$$ifv - ifv_g + i\nu \frac{\partial^2 u}{\partial z^2} + fu - fu_g - \nu \frac{\partial^2 v}{\partial z^2} = 0 \tag{6.106}$$

Denoting $\xi = u + iv$, we get

$$f\xi - f\xi_g + i\nu \frac{\partial^2 \xi}{\partial z^2} = 0 \tag{6.107}$$

We rewite this as

$$\frac{\partial^2 \xi}{\partial z^2} - \left(\frac{if}{\nu}\right)\xi = -\left(\frac{if}{\nu}\right)\xi_g \tag{6.108}$$

We solve the inhomogenous equation (6.108) in two steps:

1. find a particular solution of the inhomogenous equation:

assume that ξ is independent on z

$$-\left(\frac{if}{\nu}\right)\xi = -\left(\frac{if}{\nu}\right)\xi_g \tag{6.109}$$

with the solution $\xi = \xi_g$

2. find a complementary function, a general solution of the homogenous part of (6.108):

$$\frac{\partial^2 \xi_h}{\partial z^2} - \left(\frac{if}{\nu}\right)\xi_h = 0 \tag{6.110}$$

$$\xi_h = C \exp(\lambda z) \quad \text{with} \quad \lambda^2 = \frac{if}{\nu} \tag{6.111}$$

Thus
$$\lambda_{\pm} = \pm\frac{1+i}{\sqrt{2}}\sqrt{\frac{f}{\nu}} = \pm(1+i)\sqrt{\frac{f}{2\nu}} = \pm(1+i)\,\gamma \tag{6.112}$$

Therefore

$$\xi_h = C_1\exp(\lambda_+ z) + C_2\exp(\lambda_- z) = C_1\exp(\gamma z)\exp(i\gamma z) + C_2\exp(-\gamma z)\exp(-i\gamma z)$$

As boundary condition ξ_h has to go to zero for $z \to \infty$, therefore $C_1 = 0$.

3. The complete solution is

$$\xi = \xi_g + C_2\exp(-\gamma z)\exp(-i\gamma z) \tag{6.113}$$

As boundary condition
$$\xi(z = 0) = 0 = \xi_g + C_2 \tag{6.114}$$

Therefore
$$\xi = \xi_g \cdot (1 - \exp(-\gamma z)\exp(-i\gamma z)) \tag{6.115}$$

For simplicity, we can assume that the geostrophic flow is zonal, so that $v_g = 0$. Then,

$$u = u_g \cdot (1 - \exp(-\gamma z)\cos(\gamma z)) \tag{6.116}$$

$$v = u_g \cdot (\exp(-\gamma z)\sin(\gamma z)) \tag{6.117}$$

This variation of horizontal velocity with depth (-z) is referred to as the Ekman spiral, diagrammed above (Fig. 6.14). If we make a Taylor expansion for small z, we see that

$$u = u_g \cdot \gamma z \tag{6.118}$$

$$v = u_g \cdot \gamma z \tag{6.119}$$

Thus the low is $45°$ to the left of the limiting zonal gestrophic flow.

By applying the continuity equation we can have the vertical velocity as following

$$w = \frac{1}{f\rho_o}\left[\left(\frac{\partial\tau^y}{\partial x} - \frac{\partial\tau^x}{\partial y}\right)\left(1 - e^{-\gamma z}\cos\left(\gamma z\right)\right) - \left(\frac{\partial\tau^x}{\partial x} + \frac{\partial\tau^y}{\partial y}\right)e^{-\gamma z}\sin\left(\gamma z\right)\right]$$

Note that when vertically integrated the volume transport associated with the Ekman spiral is to the right of the wind direction in the Northern Hemisphere.

There is much difficulty associated with observing the Ekman layer for two main reasons: the theory is too simplistic as it assumes a constant eddy viscosity, which Ekman himself anticipated, recognizing that is obvious that ν cannot generally be regarded as a constant when the density of water is not uniform within the region considered and because it is difficult to design instruments with great enough sensitivity to observe the velocity profile in the ocean.

Because the real ocean does not match the idealized conditions of the Ekman spiral, wind-induced water movements often differ appreciably from theoretical predictions. In shallow water, for example, the water depth is insufficient for the full spiral to develop so that the angle between the horizontal wind direction and surface-water movements can be as little as 15 degrees. As waters deepen, the angle increases and approaches 45 degrees. The stable pycnocline inhibits the transfer of kinetic energy to deeper waters, helping to contain wind-driven currents to the mixed layer; that is, the pycnocline acts as a permeable boundary for Ekman transport and surface currents.

Exercise 46   –   Ekman layer in the atmosphere

Consider a geostrophic flow (u_g, v_g)

$$-fv_g = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{6.120}$$

$$fu_g = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \quad . \tag{6.121}$$

The boundary-layer equations are then

$$-f(v - v_g) = \nu\frac{\partial^2 u}{\partial z^2} \tag{6.122}$$

$$f(u - u_g) = \nu\frac{\partial^2 v}{\partial z^2} \quad . \tag{6.123}$$

The boundary conditions are specified to be at the surface

$$\rho_0\nu\frac{\partial u}{\partial z} = \tau^x \tag{6.124}$$

$$\rho_0\nu\frac{\partial v}{\partial z} = \tau^y \tag{6.125}$$

and for $z \to -\infty : u = u_g, \quad v = v_g$.

1. Calculate the flow (u,v) as the departure from the interior flow (u_g, v_g)!

2. Calculate the net wind-driven horizontal transport through integration

$$V = \int_{-\infty}^{0} dz(v - v_g) \quad \text{and} \quad U = \int_{-\infty}^{0} dz(u - u_g) \quad . \tag{6.126}$$

What is the direction of U and V in terms of the surface wind stress τ?

3. For the case $f = f_0$ of constant Coriolis parameter, determine the divergence of the flow

$$\int_{-\infty}^{0} dz\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{6.127}$$

which is identical to the vertical velocity across the Ekman layer (since w(0)=0).

Exercise 47   –   Ekamn spiral in the ocean

Consider the solution (6.116,6.117) for the wind-driven Ekman layer at the surface of the ocean in the Northern Hemisphere. The geostrophic velocity is zero in this example. Show that (6.116,6.117) is a solution of (6.101, 6.102) !

Tea leaf paradox*

The tea leaf paradox describes a phenomenon where tea leaves in a cup of tea migrate to the center and bottom of the cup after being stirred rather than being forced to the edges of the cup, as would be expected in a spiral centrifuge.[10] Stirring the liquid makes it spin around the cup. Because of inertia, the pressure is higher along the rim than in the middle. However, near the bottom and outer edges the liquid is slowed by the friction against the cup. There the outward force cannot overcome the pressure gradient, so these pressure differences become more important for the water flow. This is called a boundary layer or more specifically an Ekman layer.

In a teacup, where the rotation is slower at the bottom, the pressure gradient takes over and creates an inward flow along the bottom. Higher up, the liquid flows outward instead. This secondary flow travels inward along the bottom bringing the leaves to the center, then up, out and down near the rim. The leaves are too heavy to lift upwards, so they stay in the middle. Combined with the primary rotational flow, the leaves will spiral inward along the bottom.

Besides tea leaves stirred in a cup, other classroom demonstrations can show the secondary circulation that arises due to the presence of friction. We discuss a simple laboratory experiment below, which demonstrates these ideas in a more controlled setting and can be readily applied to both atmospheric low and high pressure systems. All one needs is a rotating turntable, a cylindrical container (a large transparent beaker or a cylindrical insert inside a square container works fine), and some potassium permanganate crystals. When rotated at a constant rate, all the water comes

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[10]The formation of secondary flows in an annular channel was theoretically treated by Boussinesq in 1868.The migration of near-bottom particles in river-bend flows was experimentally investigated by A.Ya.Milovich in 1913. The solution first came from Albert Einstein in a 1926 paper where he used it to explain the erosion of river banks (Baer's law) [**?**].
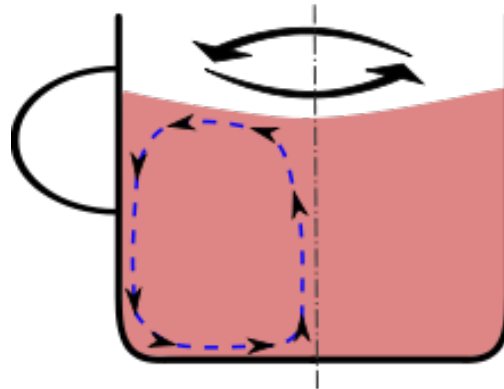
Figure 6.15:   The blue line is the secondary flow that pushes the tea leaves to the middle of the bottom.

into solid body rotation, and so there is no Coriolis or centrifugal accelerations acting. The key experimental requirement is to be able to speed up or slow down (by 10% or so) the rate of rotation of the turntable so as to induce relative motion between the water and the tank, thus creating a frictional boundary layer. The rotating platform can be used in a whole series of experiments to demonstrate atmospheric and oceanic phenomena, as presented in Marshall and Plumb (2007) and their "Weather in a Tank" website.

A tank of water is placed on the rotating platform long enough for water to reach solid body rotation, say, about 10 minutes for 10-15 rpm. Then we drop a very few potassium permanganate crystals in an equilateral triangle about the center. This shows up as three small clouds when viewed by a rotating camera. We also drop a few colored paper dots on the surface to see the flow outside the boundary layer. As the table is slowed down by a few rpm (about 10%), the permanganate on the bottom traces the near bottom circulation, which is cyclonic and inward, just like a low pressure system. The paper dots floating on the surface do not go inward. Why does this happen?

The water outside the boundary layer is still rotating with the original fast rotation rate, while the water at the bottom is rotating slower, at the new slower rotation rate. This speed differential, just like the low pressure system leads to an inward flow which is seen in the permanganate streaks at the bottom (Fig. 6.16). Similarly, the pressure gradient can be reversed by increasing the speed

Figure 6.16: To carry out the experiment we first very slightly reduce (by 10% max) the rate of rotation of the turntable. Because of the inertia of the turning fluid, it continues to spin at its original speed and so moves relative to the tank: permanganate streaks are pulled around not in circles as one might initially expect, but rather inward turning, anticlockwise spirals, as can be seen in the top panel. A beautiful symmetric pattern is remarkably easy to achieve. This is analogous to the near-surface flow in a low pressure system, as can be seen by comparing with Fig. 4.2 (see low pressure system). To visualize the flow at the upper surface, we can float a few paper dots on the surface (black dots are the most visible in this application). We observe circular, rather than spiraling, motion. To create an analogy of a high pressure system we now simply increase the speed of the turntable by 10% or so (back up to, roughly, its original speed). We observe the dye streaks on the bottom reversing and, over time, spiraling clockwise and outwards, as can be seen in the lower panel in this figure. This should be compared to the pattern of surface winds that can be seen in the high pressure system in Fig. 4.2. From Marshall and Plumb (2007).

by a few rpm to the original speed, and it leads to permanganate streaks that show an outward anti-cyclonic flow, analogous to the surface boundary layer of a High pressure system. This secondary flow in the boundary layer has important implications for movement in the vertical direction. The inward flow associated with a low pressure system leads to rising air near the center of the Low. As this air rises, it expands (pressure always decreases going upward in the atmosphere) and cools. Since the saturation of the air is very strongly dependent on the temperature, as the air cools, it may get saturated, and the water vapor may condense out to form clouds! This is why the Low pressure systems are the ones associated with stormy weather and precipitation. Conversely, high pressure systems are associated with outward motion in the boundary layer, and hence subsidence. As the air descends, it gets compressed due to the pressure increase, warms, and becomes less and less saturated. Thus the High pressure systems are fair weather systems.
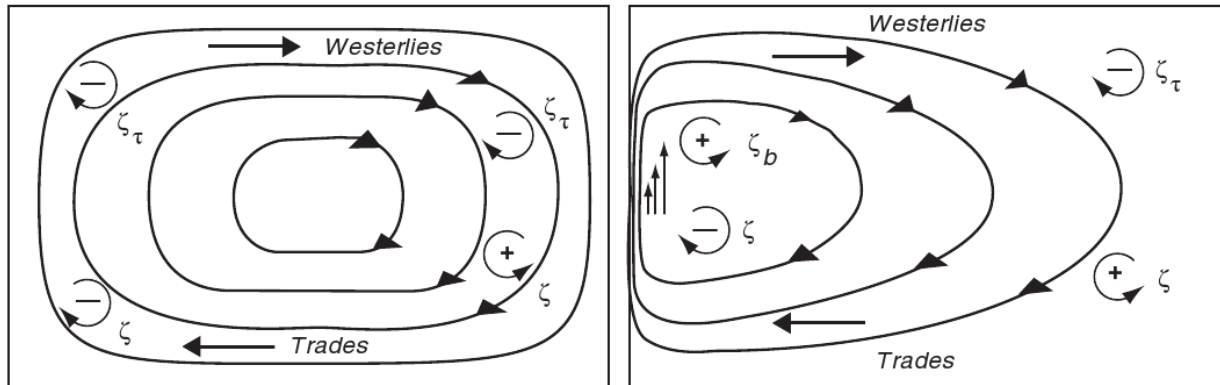
Figure 6.17: The balance of potential vorticity can clarify why western boundary currents are necessary. **Left**: Vorticity input by the wind ζ_τ balances the change in relative vorticity ζ in the east as the flow moves southward and f decreases. The two do not balance in the west where ζ must decrease as the flow moves northward and f increases. **Right**: Vorticity in the west is balanced by relative vorticity ζ_b generated by shear in the western boundary current.

6.6.4   Western Boundary Currents

The balance of vorticity provides an alternative explanation for the existence of western boundary currents. Consider the gyrescale flow in an ocean basin (Fig. 6.17), say in the North Atlantic from $10°N$ to $50°N$. The wind blowing over the Atlantic adds negative vorticity ζ_τ. As the water flows around the gyre, the vorticity of the gyre must remain nearly constant, else the flow would spin faster or slower. Overall, the negative vorticity input by the wind must be balanced by a source of positive vorticity.

Throughout most of the basin the negative vorticity input by the wind is balanced by an increase in relative vorticity. As the flow moves southward throughout the basin, f decreases and ζ must increase according to (6.48) because the depth of the wind-driven circulation does not change much.

The balance breaks down, however, in the west where the flow returns northward. In the west, f increases, ζ decreases, and a source of positive vorticity is needed. The positive vorticity ζ_b is produced by the western boundary boundary current.

Stommel's Theory of Western Boundary Currents

At the same time Sverdrup was beginning to understand circulation in the eastern Pacific, Stommel was beginning to understand why western boundary currents occur in ocean basins. To study the circulation in the north Atlantic, Stommel (1948) used essentially the same equations used by Sverdrup (6.75, 6.76, 6.77, 6.78, 6.79 and 6.80) but he added a bottom stress proportional to velocity to (6.79) and (6.80):

$$\left(A_z \frac{\partial u}{\partial z} \right)_0 = -\tau_x = -F \cos \left(\pi\, y/b \right) \qquad \left(A_z \frac{\partial u}{\partial z} \right)_D = -R\, u \qquad (6.128)$$

$$\left(A_z \frac{\partial v}{\partial z} \right)_0 = -\tau_y = 0 \qquad \left(A_z \frac{\partial v}{\partial z} \right)_D = -R\, v \qquad (6.129)$$

where F and R are constants.

Stommel calculated steady-state solutions for flow in a rectangular basin $0 \le y \le b$, $0 \le x \le \lambda$ of constant depth D filled with water of constant density. His first solution was for a non-rotating Earth. This solution had a symmetric flow pattern with no western boundary current (Fig. 6.18, left). Next, Stommel assumed a constant rotation, which again led to a symmetric solution with no western boundary current. Finally, he assumed that the Coriolis force varies with latitude. This led to a solution with western intensification (Fig. 6.18, right).

Exercise 48   –   The Stommel model of the wind-driven circulation

The wind-driven circulation in a homogeneous ocean of constant depth h is described by

$$R\nabla^2 \psi + \beta \partial_x \psi = \operatorname{curl}(\tau/\rho_0) \qquad (6.130)$$

$$= (\partial_x \tau^y - \partial_y \tau^x)/\rho_0 \qquad (6.131)$$

where R is a coefficient of bottom friction, β the derivative of the Coriolis frequency at a central latitude, and the τ the windstress vector. Finally, ψ is the streamfunction of the depth integrated
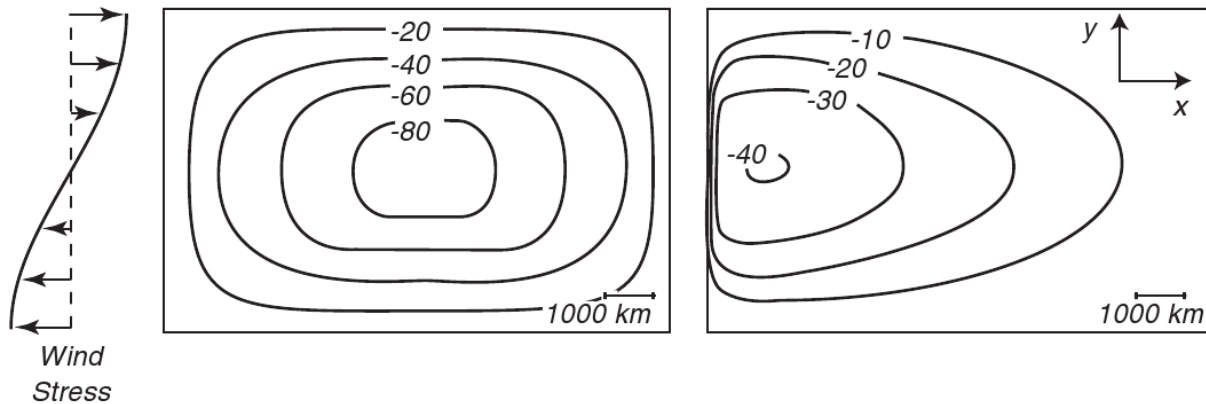
Figure 6.18: Stream function for flow in a basin as calculated by Stommel (1948). Left: Flow for non-rotating basin or flow for a basin with constant rotation. Right: Flow when rotation varies linearly with y.

velocity

$$U = (U, V) = \int_{-h}^{0} u\, dz$$

i.e.

$$U = -\partial_y \psi, V = \partial_x \psi$$

1. Derive this equation from the conservation of momentum (linearized) and mass (volume!) assuming $w = 0$ at the mean surface $z = 0$ and at the bottom $z = -h$. For simplicity take Cartesian coordinates for the horizontal, $\beta = df/dy$. Boundary condition for the flux of momentum are $\tau(z = 0) = \tau$ and $\tau(z = -h) = R(-V, U)$. Hint: integrate both equations vertically and take the curl of the integrated momentum balance.

2. in the boundary layer the terms on the left hand side of (13.86) get large. Show by scaling that the width of the layer is $W = R/\beta$.

3. how large must R be to get a width $W = 100$ km? ($\beta = 2 \times 10^{-11}$ m^{-1}s^{-1}).

Munk's Solution

Sverdrup's and Stommel's work suggested the dominant processes producing a basin-wide, wind-driven circulation. Munk (1950) built upon this foundation, adding information from Rossby (1936) on lateral eddy viscosity, to obtain a solution for the circulation within an ocean basin. Munk used Sverdrup's idea of a vertically integrated mass transport flowing over a motionless deeper layer. This simplified the mathematical problem, and it is more realistic. The ocean currents are concentrated in the upper kilometer of the ocean, they are not barotropic and independent of depth. To include friction, Munk used lateral eddy friction with constant $A_H = A_x = A_y$. Equations (6.75) (6.76) become:

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \qquad f\,v + \frac{\partial}{\partial z}\left(A_z\frac{\partial u}{\partial z}\right) + A_H\frac{\partial^2 u}{\partial x^2} + A_H\frac{\partial^2 u}{\partial y^2} \qquad (6.132)$$

$$\frac{1}{\rho}\frac{\partial p}{\partial y} = \;-\;\; f\,u + \frac{\partial}{\partial z}\left(A_z\frac{\partial v}{\partial z}\right) + A_H\frac{\partial^2 v}{\partial x^2} + A_H\frac{\partial^2 v}{\partial y^2} \qquad (6.133)$$

Munk integrated the equations from a depth $-D$ to the surface at $z = z_0$ which is similar to Sverdrup's integration except that the surface is not at $z = 0$. Munk assumed that currents at the depth $-D$ vanish, that (6.79) and (6.80) apply at the horizontal boundaries at the top and bottom of the layer, and that A_H is constant. To simplify the equations, Munk used the mass-transport stream function (6.24), and eliminated the pressure term by taking the y derivative of (6.132) and the x derivative of (6.133):

$$\underbrace{A_H\nabla^4\Psi}_{\text{Friction}} - \underbrace{\beta\frac{\partial\Psi}{\partial x}}_{\text{Sverdrup Balance}} = -\operatorname{curl}_z\tau \qquad (6.134)$$

where

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2\,\partial y^2} + \frac{\partial^4}{\partial y^4} \qquad (6.135)$$

is the biharmonic operator. Equation (6.134) is the same as (6.84) with the addition of the lateral friction term A_H. The friction term is large close to a lateral boundary where the horizontal

derivatives of the velocity field are large, and it is small in the interior of the ocean basin. Thus in the interior, the balance of forces is the same as that in Sverdrup's solution.

Equation (6.134) is a fourth-order partial differential equation, and four boundary conditions are needed. Munk assumed the flow at a boundary is parallel to a boundary and that there is no slip at the boundary:

$$\Psi_{\text{boundary}} = 0, \qquad \left(\frac{\partial \Psi}{\partial n}\right)_{\text{boundary}} = 0 \tag{6.136}$$

where n is normal to the boundary. Munk then solved (6.134) with (6.136) assuming the flow was in a rectangular basin extending from $x = 0$ to $x = r$, and from $y = -s$ to $y = +s$. He further assumed that the wind stress was zonal and in the form:

$$\tau = a \cos ny + b \sin ny + c \tag{6.137}$$

$$n = j \, \pi/s, \qquad j = 1, 2, ... \tag{6.138}$$

Munk's solution (figure 6.19) shows the dominant features of the gyre-scale circulation in an ocean basin. It has a circulation similar to Sverdrup's in the eastern parts of the ocean basin and a strong western boundary current in the west. Using $A_H = 5 \cdot 10^3 \frac{m^2}{s}$ gives a boundary current roughly $225 km$ wide with a shape similar to the flow observed in the Gulf Stream and the Kuroshio.

The transport in western boundary currents is independent of A_H, and it depends only on (6.84) integrated across the width of the ocean basin. Hence, it depends on the width of the ocean, the curl of the wind stress, and β. Using the best available estimates of the wind stress, Munk calculated that the Gulf Stream should have a transport of $36 Sv$ and that the Kuroshio should have a transport of $39 Sv$. The values are about one half of the measured values of the flow available to Munk. This is very good agreement considering the wind stress was not well known.[11]

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[11]Recent recalculations show good agreement except for the region offshore of Cape Hatteras where there is a strong recirculation. Munk's solution was based on wind stress averaged over $5°$ squares. This underestimated the curl of the stress. Leetmaa and Bunker (1978) used modern drag coefficient and $2° \times 5°$ averages of stress to obtain $32 Sv$ transport in the Gulf Stream, a value very close to that calculated by Munk.
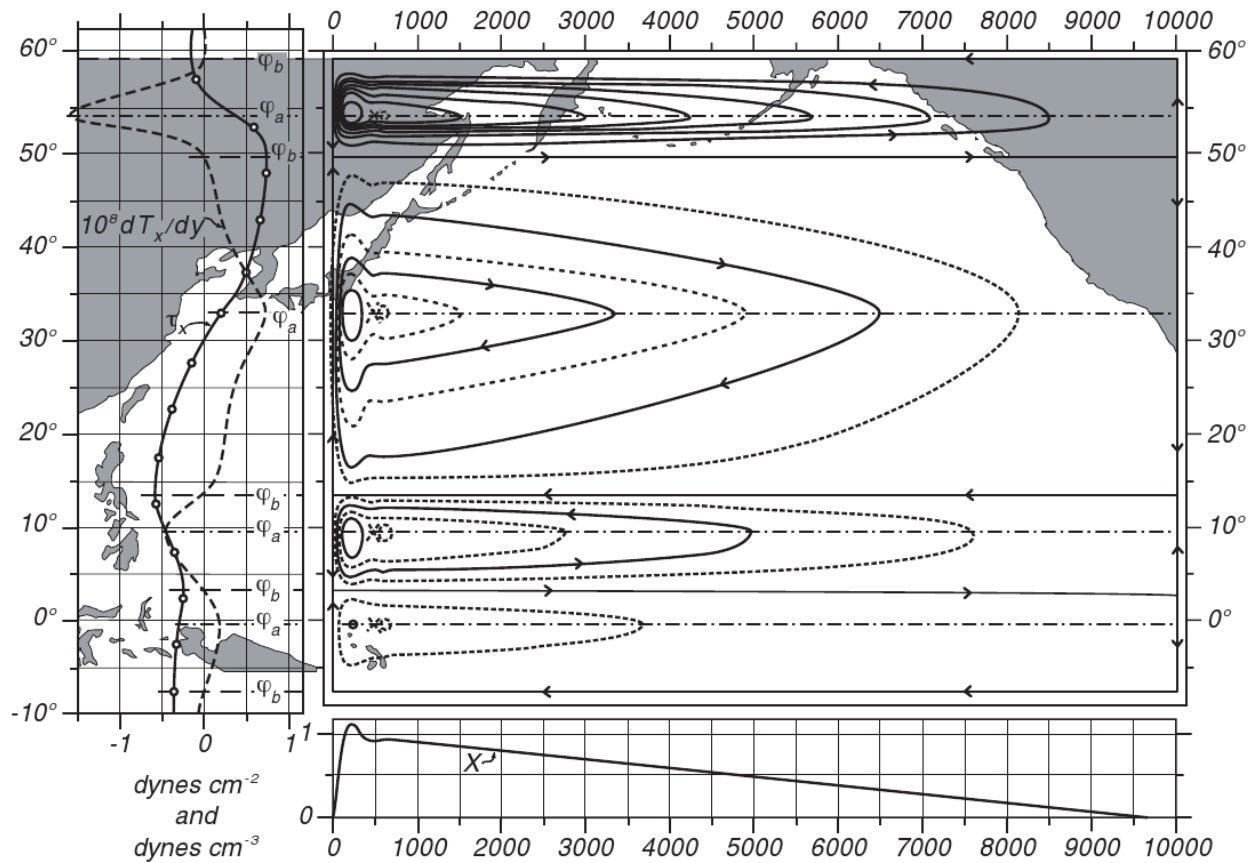
Figure 6.19: **Left**: Mean annual wind stress $\tau_x\left(y\right)$ over the Pacific and the curl of the wind stress. ϕ_b are the northern and southern boundaries of the gyres, where $M_y = 0$ and $\mathbf{curl}\,\tau = 0$. ϕ_0 is the center of the gyre. **Upper Right**: The mass transport stream function for a rectangular basin calculated by Munk (1950) using observed wind stress for the Pacific. Contour interval is **10** Sverdrups. The total transport between the coast and any point x, y is $\psi\left(x,y\right)$. The transport in the relatively narrow northern section is greatly exaggerated. **Lower Right**: North-South component of the mass transport. After Munk (1950).

Summarizing important concepts of vorticity

- Vorticity strongly constrains ocean dynamics

- Vorticity due to earth's rotation is much greater than other sources of vorticity

- Taylor and Proudman showed that vertical velocity is impossible in a uniformly rotating flow. Hence Ekman pumping requires that planetary vorticity varies with latitude.

- The curl of the wind stress adds relative vorticity to central gyres of each ocean basin. For steady state circulation in the gyre, the ocean must lose vorticity in western boundary currents.

- Positive wind stress curl leads to divergent flow in the Ekman layer. The ocean's interior geostrophic circulation adjusts through a northward mass transport.

Exercise 49   –   Cyclostrophic wind

When the flow is sufficiently near the equator so that f is small or when the Corolis force is negligible compared to the centripedal acceleration, the gradient wind equation becomes

$$\frac{v\boldsymbol{k} \times \mathbf{v}}{R} = -\frac{1}{\rho}\boldsymbol{\nabla}_z p \tag{6.139}$$

where $\boldsymbol{k}$ is the unit vector in z direction, $\mathbf{v}$ is the velocity vector, v ist the meridional velocity, R Earth radius, $\boldsymbol{\nabla}_z$ horizontal nabla operator.

1. Derive this equation!
2. What is the associated gradient wind equation including the Coriolis force?
3. What is the Rossby number?

6.7   Thermohaline ocean circulation

Water, that is dense enough to sink from the surface to the bottom, is formed when cold air blows across the ocean at high latitudes in winter in the northern North Atlantic (e.g. in the Labrador Sea and between Norway and Greenland) and near Antarctica. The wind cools and evaporates water. If the wind is cold enough, sea ice forms, further increasing the salinity of the water because sea ice is fresher than sea water and salty water remains in the water when ice is formed. Bottom water is produced only in these regions, and the deep ocean is affected by these deep water formation processes. In other regions, cold, dense water is formed, but it is not quite salty enough to sink to the bottom. At mid and low latitudes, the density, even in winter, is sufficiently low that the water cannot sink more than a few hundred meters into the ocean. The only exception are some seas, such as the Mediterranean Sea, where evaporation is so great that the salinity of the water is sufficiently great for the water to sink to intermediate depths in the seas. If these seas are can exchange water with the open ocean, the waters formed in winter in the seas spreads out to intermediate depths in the ocean. Detailed measurements of the Atlantic current structure were made by an expedition of the research vessel Meteor from 1925-1927. On the basis of these data, Wüst [1935] characterized water masses necessary to describe the Atlantic currents and tracer distribution (Fig. 6.23). Broecker proposed a circulation model based on findings of the Meteor and other expeditions. In his model, large-scale oceanic circulation is represented by the transport system of a conveyor belt (Fig. 6.21) [Broecker and Peng, 1982].

The oceans carry about one third to one half the heat out of the tropics needed to maintain earth's temperature. Heat carried by the Gulf Stream and the North Atlantic drift warms the North Atlantic, keeping it ice free in winter, and it helps warm Europe. Norway, at 60°N is far warmer than southern Greenland or northern Labrador at the same latitude. Palm trees grow on the west coast of Ireland, but not in Newfoundland which is further south. The oceanic component of the heat-transport system is also called the Global Conveyor Belt. The basic idea is that the Gulf Stream carries heat to the North Atlantic realm. There the surface water releases heat and water to the atmosphere. Some of the ocean water becomes sufficiently cold, salty, and dense that it sinks
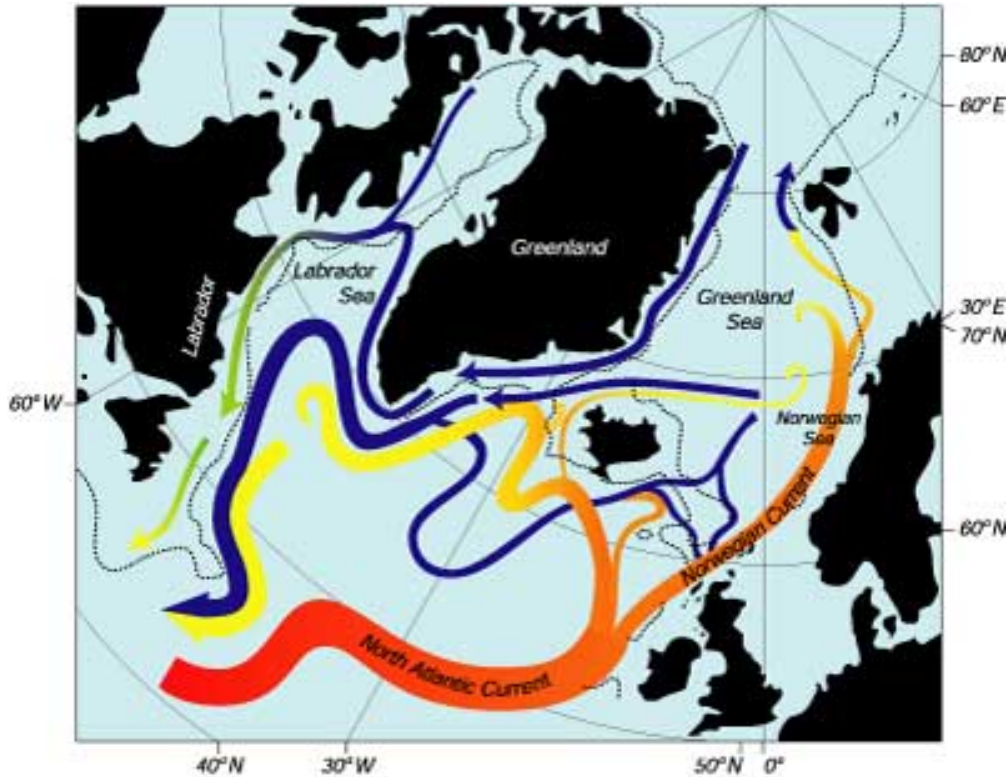
Figure 6.20: The surface (red, orange, yellow) and deep (violet, blue, green) currents in the North Atlantic. The North Atlantic Current brings warm water northward where it cools. Some sinks and returns southward as a cold, deep, western-boundary current. Some returns southward at the surface. From Woods Hole Oceanographic Institution.

to the bottom in the Norwegian and Greenland Seas. It then flows southward in very cold, bottom currents along western boundaries as a western boundary current. Some of the water remains at the surface and returns to the south in cool surface currents such as the Labrador Current and the Portugal Current (see Fig. 6.20).

The deep bottom water from the North Atlantic is mixed upward in other regions and ocean, and eventually it makes its way back to the Gulf Stream and the North Atlantic. Thus most of the water that sinks in the North Atlantic must be replaced by water from the far South Atlantic. As this surface water moves northward across the equator and eventually into the Gulf Stream, it carries heat out of the south Atlantic. So much heat is pulled northward by the formation of north-Atlantic bottom water in winter that heat transport in the Atlantic is entirely northward, even in the

southern hemisphere. Much of the solar heat absorbed by the tropical Atlantic is shipped north to warm Europe and the Northern Hemisphere. Imagine then what might happen if the supply of heat is shut off. We will get back to that topic in the next section, applying the box model.

We can make a crude estimate of the importance of the conveyor-belt circulation from a simple calculation. The Gulf Stream carries 40 Sv of $18°$C water northward. Of this, 15 Sv return southward in the deep western boundary current at a temperature of $2°$C. The flow carried by the conveyor belt must therefore lose 1 Petawatts (1 Petawatt = 10^{15} Watt = 1 PW) in the North Atlantic north of $24°$N. Although the calculation is very crude, it is remarkably close to the value of 1.2 ± 0.2 PW estimated by Rintoul and Wunsch (1991). Calculation: Exercise 55.

The production of bottom water is influenced by the salinity of surface waters in the North Atlantic. It is also influenced by the rate of upwelling due to mixing in other oceanic areas. First, let's look at the influence of salinity. Saltier surface waters form denser water in winter than less salty water. At first you may think that temperature is also important, but at high latitudes water in all ocean basins gets cold enough to freeze, so all ocean produce -$2°$C water at the surface. Of this, only the most salty will sink, and the saltiest water is in the Atlantic and under the ice on the continental shelves around Antarctica.

The the conveyor is driven by deepwater formation in the northern North Atlantic, making it the engine of conveyor belt circulation. The conveyor belt metaphor necessarily simplifies the ocean system, it is of course not a full description of the deep ocean circulation, it contains different aspects of it [Brüning and Lohmann, 1999]. Broecker's [Broecker, 1987; Broecker et al., 1991] concept provides a successful approach for global ocean circulation, although several features can be wrong like the missing Antarctic bottom water, the upwelling areas etc.. However, the global conveyor belt metaphor inspired new ideas of halting or reversing the ocean circulation and put it into a global climate context [Bryan, 1986]. This was helpful for the interpretation of Greenland ice core records (Fig. 9.9) indicating different climate states with different ocean modes of operation (like on and off states of a mechanical maschine). From the analogy, it was possible to a) identify the relevance of North Atlantic deep water production and b) realize the possibility of multiple
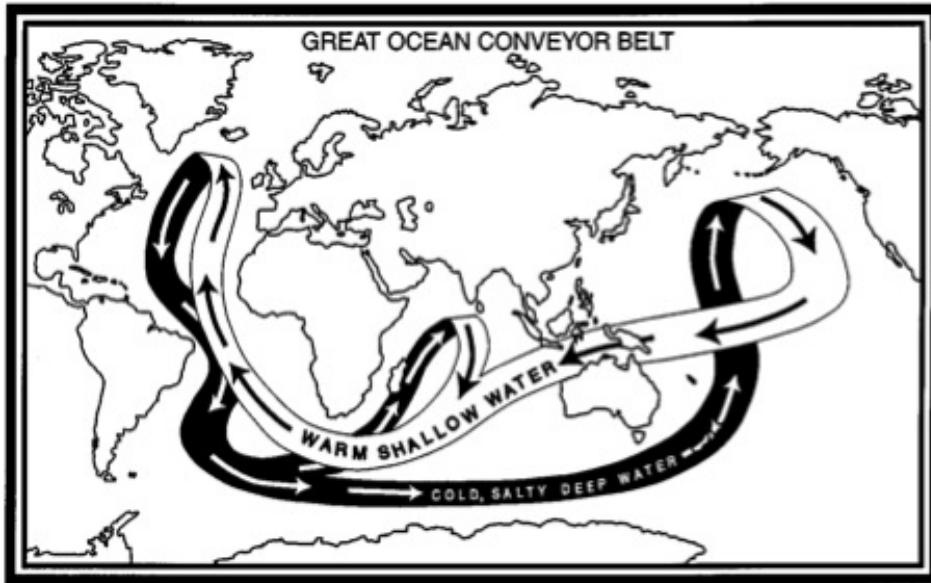
Figure 6.21: The great ocean conveyor [Broecker et al., 1991]. Warm and salty water entering the North Atlantic region is cooled. The dense water formed at the surface is convected to the deep ocean and is part of the southward return flow.

equilibria of ocean circulation states and their association with two different climatic states.

Many terms have been used to describe the deep circulation[12] and is called meridional overturning circulation. It is the zonal integral of the flow of mass plotted as a function of depth and latitude:

$$\tilde{v} = -\frac{\partial \psi}{\partial z} \tag{6.140}$$

$$\tilde{w} = \frac{\partial \psi}{\partial y} \tag{6.141}$$

with the zonally integrated velocities $\tilde{v}, \tilde{w}$, and a streamfunction $\psi(y, z)$ for the overturning circulation.

Figure 6.22 shows the meridional overturning circulation streamfunction $\psi(y, z)$ in the At-

---

[12] Abyssal circulation; Thermohaline circulation; Meridional overturning circulation; and Global conveyor. The term thermohaline circulation was once widely used, but it has disappeared almost entirely from the oceanographic literature. It is no longer used because it is clear that the flow is not only density driven.
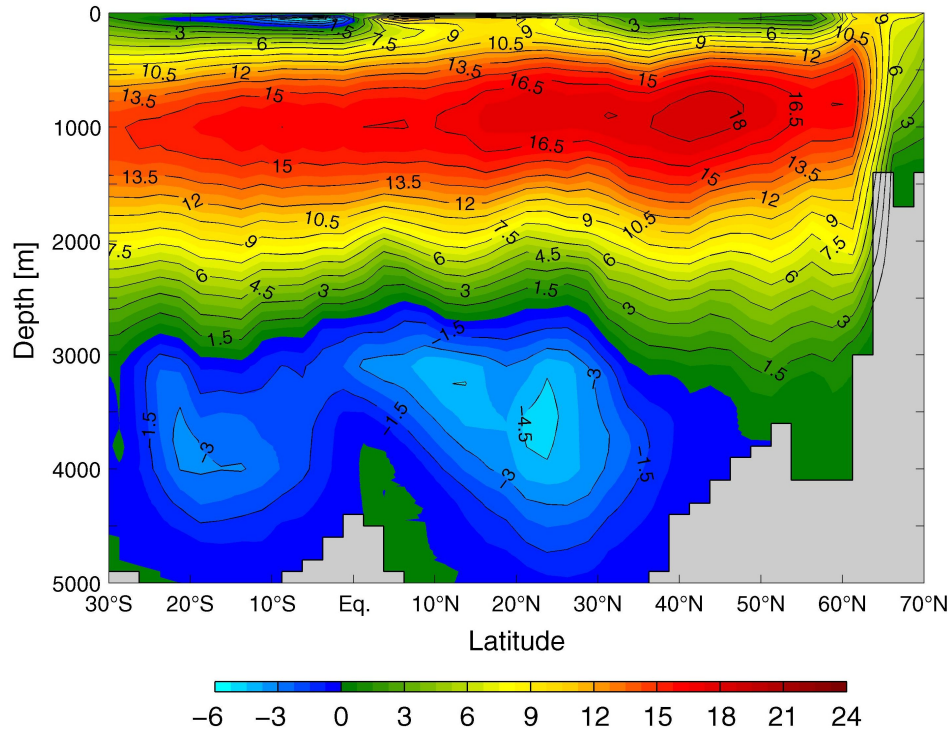
Figure 6.22: Modelled meridional overturning streamfunction in Sv ($1\,\mathrm{Sv} \equiv 10^6\ \mathrm{m^3 s^{-1}}$) in the Atlantic Ocean. Grey areas represent zonally integrated smoothed bathymetry.

lantic. The streamfunction is calculated as a cumulative sum of zonally integrated mass transports of the ocean at each latitude from surface to the particular depth. The zonally intregrated mass transport at a certain latitude derives from the zonally averaged meridional velocity component times the height of the ocean layer and the width of the ocean. Water flows along the stream lines. For instance, increasing positive values of MOC from surface to about 1,000m depth at mid latitudes of the Northern Hemisphere denote northward flowing water. With increasing depth the values of the MOC streamfunction decrease until a minimum at about 4,000m depth is reached. These waters move southward instead. In the Atlantic two major, a shallower and a deeper over-turning cell exist according to figure 6.22. One cell shows positive values, thus, clockwise volume transport and stretches from almost the surface to about 2,500 - 3,000m depth. The other expands from about 3,000m depth to the bottom of the ocean at latitudes south of 40°N. The shallower cell denotes the modelled equivalent of North Atlantic Deep Water (NADW) while Antarctic Bottom

Water (AABW), transporting Southern Ocean water into the Atlantic, is simulated by the deeper cell. An overturning maximum of 18.7 Sv (1 Sv $\equiv 10^6$ m^{3}s^{-1}) is found at 40°- 50°N and 1,000m depth and an export into the Southern Ocean across 30°S of 14.9 Sv. This results in an overturning ratio of 0.79, so only little recirculation occurs. A closer look at Figure 6.22 reveals that NADW is predominantly formed north of 60°N with about 10 Sv. The inflow of AABW into the Atlantic is much weaker than the outflow of NADW. At 30°S a value of less than 1 Sv is calculated by the model while the maximum counter-clockwise overturning of the bottom water cell reaches 4.7 Sv at 25°N.
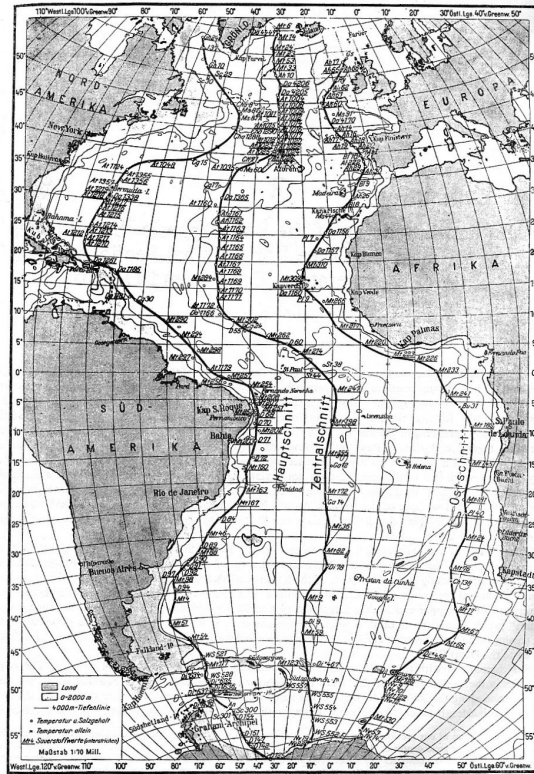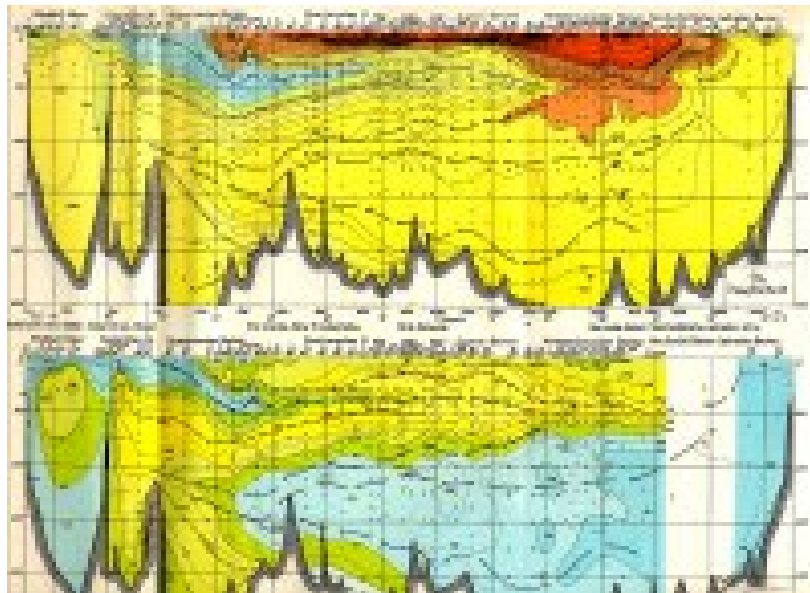
Figure 6.23: 1927-1929 Meteor Expedition, the first accurate hydrographic survey of the Atlantic from Wüst [1935]. Lower panel: Salinity and dissolved oxygen on the Hauptschnitt along the western side of the Atlantic.

Exercise 50 – **Ocean thermohaline circulation**

Consider a geostrophic flow (u, v)

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{6.142}$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \ . \tag{6.143}$$

Use the hydrostatic approximation

$$\frac{\partial p}{\partial z} = -g\rho \tag{6.144}$$

and equation (13.110) in order to derive the meridional overturning stream function $\Phi(y, z)$ as a fuction of density ρ at the basin boundaries! Φ is defined via

$$\Phi(y, z) = \int_0^z \frac{\partial \Phi}{\partial \tilde{z}} \, d\tilde{z} \tag{6.145}$$

$$\frac{\partial \Phi}{\partial \tilde{z}} = \int_{x_e}^{x_w} v(x, y, \tilde{z}) \, dx \quad \text{(zonally integrated transport)}, \tag{6.146}$$

where x_e and x_w are the eastward and westward boundaries in the ocean basin (think e.g. of the Atlantic Ocean). Units of Φ are $m^3 s^{-1}$. At the surface $\Phi(y, 0) = 0$.

Solution of Exercise 50: Ocean thermohaline circulation

$$\frac{\partial \Phi}{\partial z} = \int_{x_e}^{x_w} v(x, y, z) \, dx \tag{6.147}$$

$$= \frac{1}{\rho_0 f}\int_{x_e}^{x_w} \frac{\partial p}{\partial x} \, dx = \frac{1}{\rho_0 f}(p(x_w, y, z) - p(x_e, y, z)) \tag{6.148}$$

$$= -\frac{g}{\rho_0 f}\int_0^z (\rho(x_w, y, z') - \rho(x_e, y, z'))dz' \tag{6.149}$$

Exercise 51   –   Estimates of overturning

It is observed that water sinks in to the deep ocean in polar regions of the Atlantic basin at a rate of 15 Sv. (Atlantic basin: $80,000,000\,km^2 area \times 4\,km$ depth.)

1. How long would it take to 'fill up' the Atlantic basin?

2. Supposing that the local sinking is balanced by large-scale upwelling, estimate the strength of this upwelling. Hint: Upwelling $= area \times w$. Express your answer in $m\ y^{-1}$.

3. Compare this number with that of the Ekman pumping in (6.95)!

Solution of Exercise 51: Estimates of overturning

1. Timescale T to 'fill up' the Atlantic basin:

$$T = \frac{80 \cdot 10^{12}\,m^2 \cdot 4000\,m}{15 \cdot 10^6\,m^3 s^{-1}} = 2.13 \cdot 10^{10} s = 676\ years$$

2. Overturning is balanced by large-scale upwelling:

$$area \cdot w = 15 \cdot 10^6\,m^3 s^{-1}$$

$$w = 0.1875 \cdot 10^{-6} m\ s^{-1} = 5.9 \cdot 10^{-15} m\ y^{-1}.$$

3. Ekman pumping

$$w_E \simeq 32\ m\ y^{-1}.$$

Simple model of meridional overturning

It is instructive to derive a simple concept of the meridional overturning based on vorticity dynamics in the (y,z)-plane. The dynamical model in two dimensions reads

$$\frac{\partial}{\partial t} v = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} - fu - \kappa v \tag{6.150}$$

$$\frac{\partial}{\partial t} w = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{g}{\rho_0}(\rho - \rho_0) - \kappa w \tag{6.151}$$

with κ as parameter for Rayleigh friction. Using the continuity equation

$$0 = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{6.152}$$

one can introduce a streamfunction $\Phi(y, z)$ with $v = \partial_z \Phi$ and $w = -\partial_y \Phi$. The associated vorticity equation in the (y,z)-plane is therefore

$$\frac{\partial}{\partial t} \nabla^2 \Phi = -f \frac{\partial u}{\partial z} + \frac{g}{\rho_0} \frac{\partial \rho}{\partial y} - \kappa \nabla^2 \Phi \tag{6.153}$$

We can choose the ansatz[13] satisfying that the normal velocity at the boundary vanishes, $\Phi = 0$:

$$\Phi(y, z, t) = \Phi_{max}(t) \sin\left(\frac{\pi y}{L}\right) \times \sin\left(\frac{\pi z}{H}\right) \tag{6.155}$$

The parameters L and H dentote the meridional and depth extend (y goes from 0 to L, z from 0 to H). With the assumption that the term $-f \frac{\partial u}{\partial z}$ is absorbed into the viscous terms, and the integration

---

[13]In principle, a complete Galerkin approximation shall be applied

$$\Phi(y, z, t) = \Sigma_{k=1}^{\infty} \Sigma_{l=1}^{\infty} \Phi_{max}^{k,l}(t) \sin(\pi k y/L) \times \sin(\pi l z/H) \tag{6.154}$$

yielding a first order differential equation in time for $\Phi_{max}^{k,l}(t)$. For a different approach: [Maas, 1994], for an overview of simple climate models: [Olbers, 2001].

$\int_0^L dy \int_0^H dz$, we derive for the three remaining terms in (6.153):

$$\frac{d}{dt}\Phi_{max}\left(\frac{\pi^2}{L^2}+\frac{\pi^2}{H^2}\right)\int_0^L dy \sin\left(\frac{\pi y}{L}\right)\int_0^H dz \sin\left(\frac{\pi z}{H}\right) \;=\; 4LH\left(\frac{1}{L^2}+\frac{1}{H^2}\right)\frac{d}{dt}\Phi_{max}$$

$$\int_0^L dy \int_0^H dz \frac{g}{\rho_0}\frac{\partial\rho}{\partial y} \;=\; \frac{g}{\rho_0}\;H\;(\rho_{north}-\rho_{south})$$

$$\kappa\Phi_{max}\left(\frac{\pi^2}{L^2}+\frac{\pi^2}{H^2}\right)\int_0^L dy \sin\left(\frac{\pi y}{L}\right)\int_0^H dz \sin\left(\frac{\pi z}{H}\right) \;=\; \kappa\;4LH\left(\frac{1}{L^2}+\frac{1}{H^2}\right)\Phi_{max}$$

with $\rho_{north}=\rho(y=L)$, and $\rho_{south}=\rho(y=0)$, and the equation

$$\frac{d}{dt}\Phi_{max} \;=\; \frac{a}{\rho_0}(\rho_{north}-\rho_{south}) \;-\; \kappa\Phi_{max} \tag{6.156}$$

with $a=gLH^2/4(L^2+H^2)$.

This shows that the overturning circulation depends on the density differences on the right and left boxes. In the literature, (6.156) is simplified to a diagnostic relation

$$\Phi_{max} = \frac{a}{\rho_0\,\kappa}\;(\rho_{north}-\rho_{south}) \tag{6.157}$$

because the adjustement of Φ_{max} is quasi-instantaneous due to adjustment processes, e.g. Kelvin waves.

Here, we introduce a hemispheric (Stommel-type) or interhemispheric (Rooth-type) box model of the thermohaline circulation. The common assumption of these box models is that the oceanic overturning rate Φ can be expressed by the meridional density difference:

$$\Phi \;=\; -c\,(\alpha\Delta T - \beta\Delta S) \quad , \tag{6.158}$$

where α and β are the thermal and haline expansion coefficients, $c=a(\rho_0\kappa)^{-1}$, and Δ denotes the meridional difference operator applied to temperature T and salinity S, respectively. The
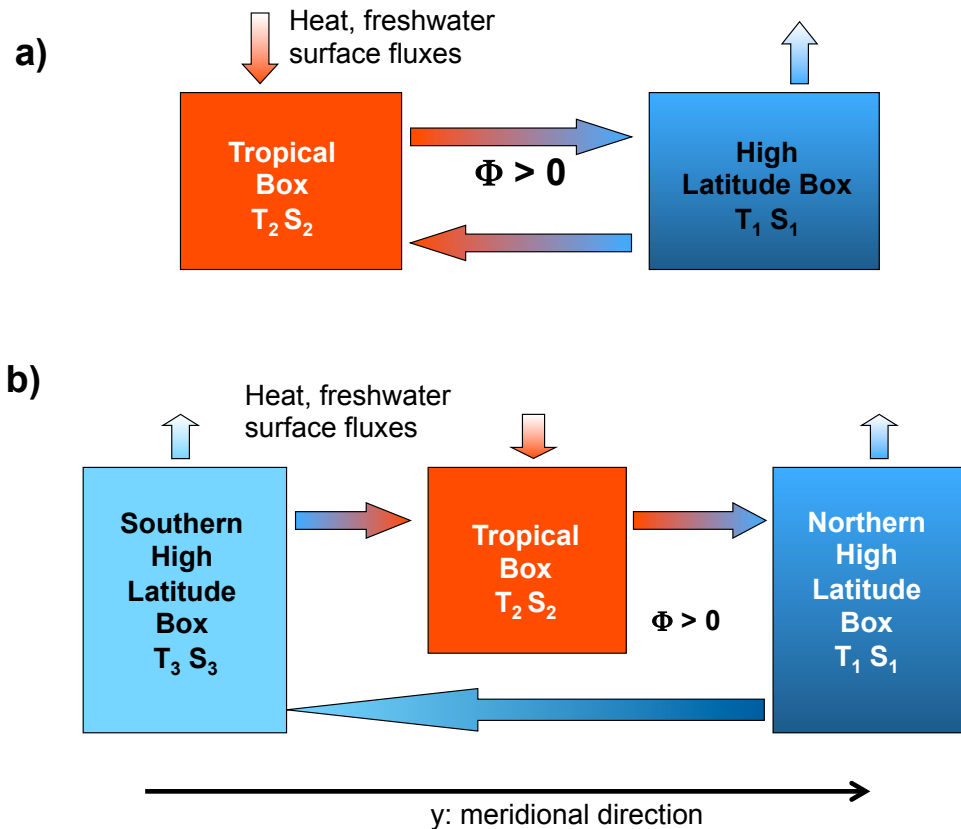
Figure 6.24: Schematic picture of the hemispheric two box model (a) and of the interhemispheric box model (b).

meriodional density differences are clearly dominated by temperature differences (Fig. 6.25a). In a single hemispheric view, the salinity difference breakes the temperature difference.

In the model of (Rooth, 1982) the Atlantic ocean is described over both hemispheres and the densities have to be taken in the North Atlantic and South Atlantic Ocean, respectively. In the interhemisph eric model the densities at high northern and southern latitudes are close, the pole-to-pole differences are caused by salinity differences (Fig. 6.25b).
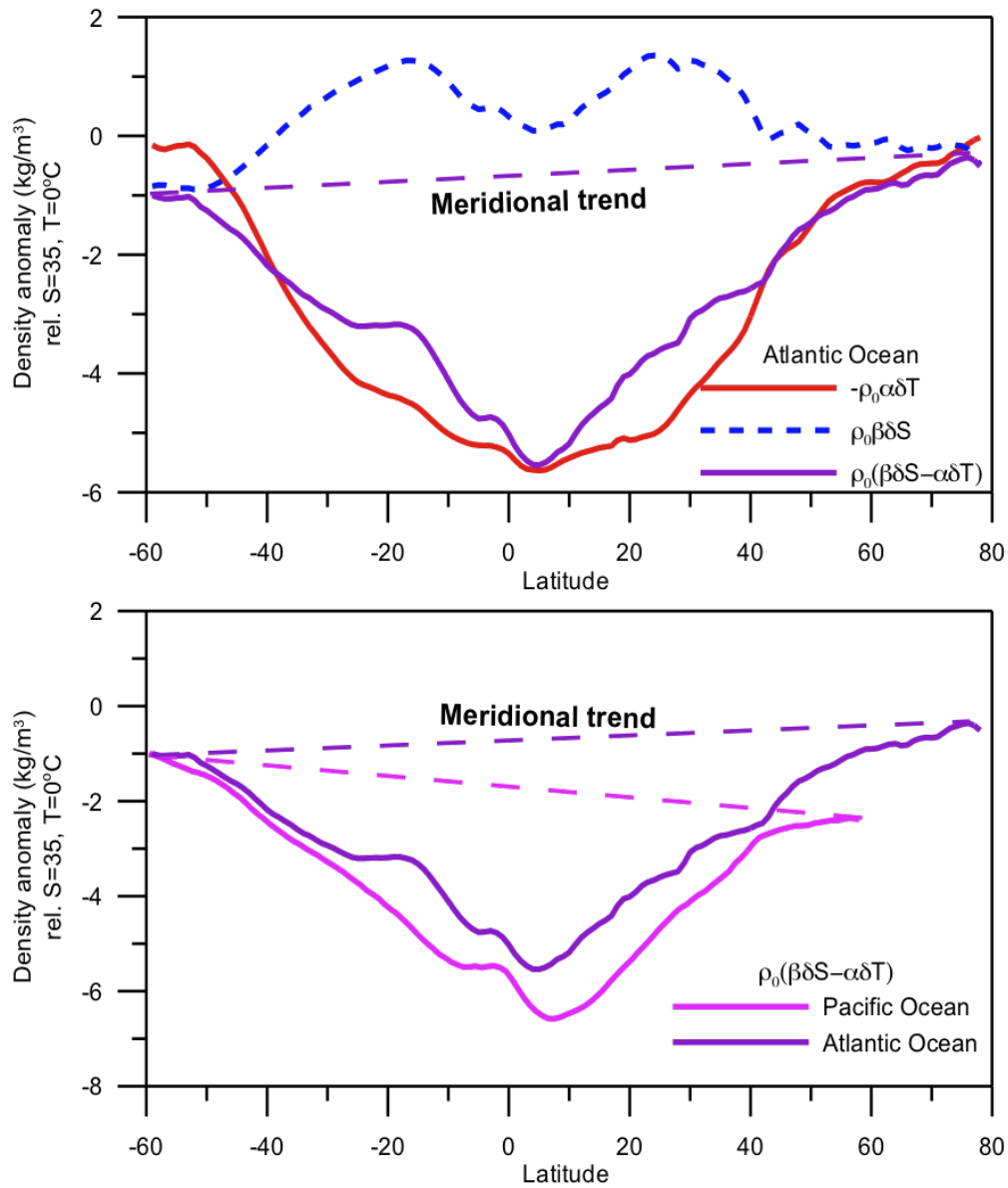
Figure 6.25: a) The Atlantic surface density is mainly related to temperature differences. b) But the pole-to-pole differences are caused by salinity differences.

Chapter 7

Simple Climate Models

7.1 Engery balance model

7.1.1 Zero-dimensional Model

Energy balance models (EBM) are highly simplified systems of the climate system. A zero-dimensional model of the radiative equilibrium of the Earth is

$$(1 - \alpha)S\pi R^2 = 4\pi R^2 \epsilon \sigma T^4 \tag{7.1}$$

where the left hand side represents the incoming energy from the Sun the right hand side represents the outgoing energy from the Earth, calculated from the Stefan-Boltzmann law assuming a constant radiative temperature, T, that is to be found, and S is the solar constant - the incoming solar radiation per unit area– about $1367 W m^{-2}$, α is the Earth's average albedo, measured to be 0.3. R is Earth's radius = 6.371×10^6 m, σ is the Stefan-Boltzmann constant = $5.67 \times 10^{-8} J K^{-4} m^{-2} s^{-1}$, and ϵ is the effective emissivity of earth (about 0.612). The geometrical constant πR^2 can be factored out, giving

$$(1 - \alpha)S = 4\epsilon \sigma T^4 \tag{7.2}$$

Solving for the temperature,

$$T = \sqrt[4]{\frac{(1-\alpha)S}{4\epsilon\sigma}} \tag{7.3}$$

This yields an average earth temperature of **288** K. This is because the above equation represents the effective radiative temperature of the Earth (including the clouds and atmosphere). The use of effective emissivity and albedo account for the greenhouse effect.

The average emissivity of the earth is readily estimated from available data. The emissivities of terrestrial surfaces are all in the range of 0.96 to 0.99 (except for some small desert areas which may be as low as 0.7). Clouds, however, which cover about half of the earth's surface, have an average emissivity of about 0.5 (which must be reduced by the fourth power of the ratio of cloud absolute temperature to average earth absolute temperature) and an average cloud temperature of about 258 K. Taking all this properly into account results in an effective earth emissivity of about 0.64 (earth average temperature 285 K). This simple model readily determines the effect of changes in solar output or change of earth albedo or effective earth emissivity on average earth temperature. It says nothing, however about what might cause these things to change. Zero-dimensional models do not address the temperature distribution on the earth or the factors that move energy about the earth. A logical next step is to introduce a latitude-dependent model.

Exercise 52   –   Energy balance

The EMB (7.1) determines the effect on average earth temperature of changes in solar constant or change of albedo or effective earth emissivity. Show: The percent change of the average amount of each parameter, considered independently, to cause a one degree Kelvin change in steady-state average earth temperature is Solar constant 1.4%, Albedo 3.3%, Effective emissivity 1.4% using (7.3).

7.1.2   One dimensional atmospheric energy balance model

The model considers a zonally and annually averaged circulation of the atmosphere and calculates surface fresh water fluxes and surface heat fluxes along with sea surface temperatures. The EBM treats the transport processes as diffusion. The balances of energy will be used to derive equations for the atmospheric temperature and fresh water flux. The thermodynamic equation (internal plus potential energy) in the atmosphere in isobaric coordinates reads

$$C_p \left[\partial_t T_a + \nabla \cdot (v\, T_a) + \partial_p(\omega T_a) \right] \;=\; \partial_p Q_R + Q_L + \partial_p Q_S + \frac{RT_a}{p}\omega \qquad (7.4)$$

where v and ∇ are the horizontal vector of wind and the gradient operator, T_a atmospheric temperature, p pressure, $\omega = \frac{d}{dt}p$ vertical wind, and C_p specific heat at constant pressure $(1004\ J\ kg^{-1}\ K^{-1})$. Q_R and Q_S are the radiative and sensible heat fluxes, respectively. Q_L denotes the latent heat release due to phase transitions in the air. This term includes condensation of water vapour $(c > 0)$, evaporation of cloud water $(c < 0)$, and evaporation in unsaturated air $(e > 0)$:

$$Q_L \;=\; L_v\,(c \,-\, e) \qquad (7.5)$$

where L_v is the latent heat of condensation $(L_v = 2.5 \cdot 10^6\ J\ kg^{-1})$.

The last term on the right hand side of equation (7.4) is related to the fact that because pressure decreases with height and air is a compressible fluid, air that rises expands (and air that sinks contracts). Air that expands does work against its surroundings and because of the first law of thermodynamics (conservation of energy) this work needs to be paid for (reduction in temperature). So internal energy is consumed in expanding the parcel of air outwards against the atmosphere (expanding air cools). Furthermore, we can assume an adiabatic process is one where no heat is exchanged with the surroundings. This is a reasonable approximation for typical rising air because other processes like conduction or radiative heat transfer are quite slow. In this context, the potential temperature is introduced. The potential temperature of a parcel of fluid at pressure

P is the temperature that the parcel would acquire if adiabatically brought to a standard reference pressure P_0, usually 1000 millibars. The potential temperature is denoted θ and, for air, is often given by

$$\theta = T \left(\frac{P_0}{P}\right)^{R/c_p},\tag{7.6}$$

where T is the current absolute temperature (in K) of the parcel, R is the gas constant of air, and c_p is the specific heat capacity at a constant pressure. $R/c_p = 0.286$ for air (meteorology). Potential temperature is a more dynamically important quantity than the actual temperature. This is because it is not affected by the physical lifting or sinking associated with flow over obstacles or large-scale atmospheric turbulence. A parcel of air moving over a small mountain will expand and cool as it ascends the slope, then compress and warm as it descends on the other side- but the potential temperature will not change in the absence of heating, cooling, evaporation, or condensation (processes that exclude these effects are referred to as dry adiabatic). Since parcels with the same potential temperature can be exchanged without work or heating being required, lines of constant potential temperature are natural flow pathways.[1]

Additionally to (7.4), the budget equations for the mass mixing ratio of water vapour q_v and cloud water q_w are used:

$$\partial_t q_v + \nabla \cdot (v\, q_v) + \partial_p(\omega\, q_v) = e - c + E \tag{7.7}$$

$$\partial_t q_w + \nabla \cdot (v\, q_w) + \partial_p(\omega\, q_w) = c - \wp \tag{7.8}$$

where $\wp$ denotes the formation of precipitation and E denotes the evaporation from the ground (ocean and land).

---

[1]The equation comes from the enthalpy form of the first law of thermodynamics can be written as: $dh = T\, ds + v\, dp$, where dh denotes the enthalpy change, T the temperature, ds the change in entropy, v the specific volume, and p the pressure. For adiabatic processes, the change in entropy is zero: $dh = v\, dp$. For approximately ideal gases, such as the dry air in the Earth's atmosphere, the equation of state, $pv = RT$ can be substituted by $dpv = R/vdT$ $\frac{dp}{p} = \frac{c_p}{R}\frac{dT}{T}$, where the $dh = c_p dT$ was used and both terms were divided by the product pv. Integrating yields 7.6.

The budget equations (7.4, 7.7) and (7.8) are now vertically integrated and zonally averaged. It is assumed that in the vertically integrated left hand side of (7.8) the first two terms vanish (stationarity, small horizontal transports). Furthermore, the vertically integrated last term on the right hand side of (7.4) is neglected. With $\omega = 0$ at the top and bottom, (7.8) reduces to

$$\int \frac{dp}{g} c = \int \frac{dp}{g} \wp \qquad , \tag{7.9}$$

where g is the gravitational acceleration ($9.81 \, m \, s^{-2}$). The net precipitation P on the ground ($p = p_0 = 10^5 Nm^{-2} = 1000 \, mb$) is defined as

$$P = \int \frac{dp}{g} (\wp - e) \quad . \tag{7.10}$$

This yields the vertically integrated balances for the mixing ratio of water vapour and atmospheric temperature:

$$\begin{aligned}
\int \frac{dp}{g} \partial_t (C_p T_a) + \int \frac{dp}{g} \nabla \cdot (C_p v T_a) &= Q_R^{top} - Q_R^{bottom} + L_v P + Q_S^{bottom} \\
\int \frac{dp}{g} \partial_t (L_v q_v) + \int \frac{dp}{g} \nabla \cdot (L_v v q_v) &= L_v (E - P) \quad .
\end{aligned} \tag{7.11}$$

The one dimensional atmosphere EBM prognoses the vertically integrated mixing ratio of water vapour and atmospheric temperature along with (7.11).[2] This yields one vertically integrated energy equation:

$$C \, \partial_t T_A + \int \frac{dp}{g} \nabla \cdot (C_p v T_a) + \int \frac{dp}{g} \nabla \cdot (L_v v q) = Q_R^{top} - F_{oa} \tag{7.12}$$

where $F_{oa} = Q_R^{bottom} - L_v E - Q_S^{bottom}$ denotes the ocean-atmosphere heat flux calculated by bulk formulas. The net radiation on top of the atmosphere Q_R^{top} is the difference between net solar

---

[2]To evaluate the effective change of the vertically integrated humidity and temperature in equations (7.11), the height distribution of humidity and temperature must be taken into account using an empirical relation between the lapse rate and surface temperature β_1, β_2: $C \, \partial_t T_A = (C_p \beta_1 + L_v \beta_2) \, \partial_t T_A$

radiation and net outgoing longwave radiation Q_{LW}^{top}. In a further approximation, the longwave radiation Q_{LW}^{top} is described by a linear law:

$$
\begin{aligned}
Q_{LW}^{top} &= A + B \; T_A \quad \text{with} \\
A &= 213.2 \; W \, m^{-2} \quad \text{and} \quad B = 2.2 \; W \, m^{-2} K^{-1} .
\end{aligned}
\tag{7.13}
$$

For Q_S, one needs a climate-dependent formulation of the planetary albedo α, which is parameterized in terms of the surface air temperature:

$$
\alpha(T_A) = 0.42 \; - \; 0.20 \cdot \tanh\left[0.052 \left(T_A - 276.15\right)\right] \qquad .
\tag{7.14}
$$

This parameterization incorporates high albedos of snow and ice in terms of the surface temperature (T_A in Kelvin). With equation (7.12) the surface temperature T_A is calculated prognostically, while the fresh water flux for the ocean surface is given by the right hand side of equation (7.11) by evaluating the left hand side of the water vapour budget.

Meridional Transports

The transport parameterizations are based on diffusion. The mechanism of heat and moisture transport in middle and high latitudes by baroclinic instability is the most important mechanism in the atmospheric energy balance model. Consider an atmospheric condition with isotherms coincident with latitude circles. A cold anomaly, which could be thought of as a cold air outbreak from the North American continent, results in a changed surface heat flux from the ocean. In the region of strong temperature gradient, cyclones (low pressure) and anticyclones (high pressure on the northern hemisphere) are formed. These traveling weather systems move north-eastward defining the major storm track. The scale over which this process is important is about 1000 km.

This process is the main source of meridional heat transport in middle and high latitudes. The balances of heat and moisture (7.11) are averaged over a length scale of synoptic scale disturbances of O(1000) km and a time scale longer than the life time of such disturbances (e.g. two weeks). The variables can be splitted into a large-scale, long-term quantities ($\overline{T_a}, \overline{q_v}, \overline{v}$) and the deviations

(T'_a, q'_v, v'). The moments $\overline{v'T'_a}$ and $\overline{v'q'_v}$ are connected mainly with transient processes in the atmosphere. The transients act as diffusion in a statistical sense bringing warm and moist air poleward due to individual high and low pressure contributions.

The eddy fluxes, in a statistically steady state of the atmosphere, scale as:

$$\overline{v'T'_a} \sim \left(\frac{\partial \overline{T_a}}{\partial y} \right) \tag{7.15}$$

where T', v' are the perturbation of potential temperature and meridional velocity. The vertical integrated sensible eddy heat transport can be calculated in terms of the surface temperature gradient T_A

$$\int \frac{dp}{g} \; \overline{v'T'_a} \; = \; - \; K_s \; \left(\frac{\partial T_A}{\partial y} \right) \quad . \tag{7.16}$$

where K_s is tuned to reproduce the current climate.

The latent eddy heat transport is parameterized as

$$\overline{v'q'_v} \; = \; rh(p) \; \frac{\partial q_s}{\partial \overline{T_a}}(\overline{T_a}, p) \; \overline{v'T'_a} \quad , \tag{7.17}$$

where rh is the relative humidity and q_s the saturation water vapour. The relative humidity is prescribed. For the latent heat transport (7.17), the relative humidity and $\frac{\partial q_s}{\partial T}$ strongly decrease with height. Therefore, the surface values for the latent heat transport is a good choice in the vertical integrated model:

$$\int \frac{dp}{g} \; \overline{v'q'_v} \; = \; -K_l \; rh(p_0) \; \frac{\partial q_s}{\partial T_A}(T_A, p_0) \; \left(\frac{\partial T_A}{\partial y} \right) \quad . \tag{7.18}$$

As for the sensible heat transport, the coefficient K_l can be tuned that (7.18) reproduces the latent eddy heat transports of current climate. The eddy activity is greatly enhanced over the ocean surfaces as opposed to over land surfaces. In the Northern Hemisphere, two major storm tracks

exists extending northeast across the Atlantic and Pacific oceans from the east coast of the major continents. It is along these tracks that the majority of eddy heat and vorticity transport takes place. In the Southern Hemisphere the transport is relatively homogeneous in the zonal direction.[3] In the boxmodel in section 7.2, we assume that the atmospheric heat transport across the box boundaries are completely by transient eddies.

Exercise 53   –   Analytical EBM

Starting from (7.12), we are interested in the vertically integrated energy equation where we include the atmosphere and ocean in one temperature T and describe the heat transport (sensible, latent and ocean) as diffusion:

$$C_p \, \partial_t T \; + \; k \, \partial_y^2 T \; = \; (1 - \alpha) Q_S^{top} - (A + B \, T) \tag{7.19}$$

If α is chosen as parameter, we can easily solve (7.19) with the ansatz with a global component and a latitude component

$$T(y, t) \;\; = \;\; T_0(t) + T_1(t) \cdot \cos(by) \tag{7.20}$$

$$Q_S^{top} \;\; = \;\; Q_0 + Q_1 \cdot \cos(by) \tag{7.21}$$

with $y = a\varphi$ and $b = 2/a$.

Task: Solve the Energy balance model (7.19) for constant α.

---

[3]The heat transport by stationary baroclinic waves is larger in winter on the northern hemisphere when the land-sea contrast is most pronounced. Green **?** argued that stationary eddies are more transient phenomena which repeatedly occur at the same location. This happens due to fixed topographic effects providing perturbations upon which baroclinic waves can grow. These phenomena relating to stationary eddies are ultimately driven by the large scale baroclinicity of the atmosphere. Therefore, standing eddies could be parameterized as transient eddies.

Solution

Hint: For constant α, one can separate the dynamics for T_0 and T_1:

$$C_p d_t T_0 = (1 - \alpha)Q_0 - (A + BT_0) \tag{7.22}$$

$$C_p d_t T_1 = (kb^2 + B)T_1 + (1 - \alpha)Q_1 \tag{7.23}$$

$$\text{because} \quad \int_{-90°}^{90°} \cos(2\varphi) \, d\varphi = 0$$

The solution of an inhomogenous linear differential equation is the solution of the homogenous differential equation plus a particular solution of the inhomogenous case. $y' + uy = v$, then the solution of the homogenous system is $y_h = C \exp(-ut)$, and a particular solution $y_p = v/u$ independent on t, and therefore $y = y_h + y_p$ as the solution. Note that a particular solution y_p can be also found through variation of constant $C(t)$.

Exercise 54   –   Analytical EBM: Ice-albedo feedback

Based in exercise 53, one can introduce a climate-dependent formulation of the planetary albedo α on the global temperature:

$$\alpha(T) = \alpha_0 - \alpha_1 \cdot T_0 \tag{7.24}$$

Tasks:

a) Solve the Energy balance model (7.19) for the case $\alpha(T_0)$ as in (7.24).

b) Show that the stability of the solution depends on $B - \alpha_1 Q_0$!

c) explain the ice-albedo effect through this solution!

7.2   Interhemispheric box model

7.2.1   Model description

Here we introduce an interhemispheric box model of the deep ocean circulation to study the feedbacks in the climate system. Like in the model of Rooth [1982] the Atlantic Ocean is described over both hemispheres. The box model consists of four oceanic and three atmospheric boxes, as indicated in Fig. 13.6. The ocean boxes represent the Atlantic Ocean from $80°N$ to $60°S$. The indices of the temperatures T, the salinities S, the surface heat fluxes H, the atmospheric heat fluxes F, the radiation terms R as well as later on the volumes bear on the different boxes (N for the northern, M for the tropical, D for the deep and S for the southern box).

The discrete boxes are utterly homogeneous, which implies that the temperatures and the salinities everywhere within one box are alike. The climate model is based on mass and energy considerations. Emphasis is placed on the overturning flow Φ of the ocean circulation.
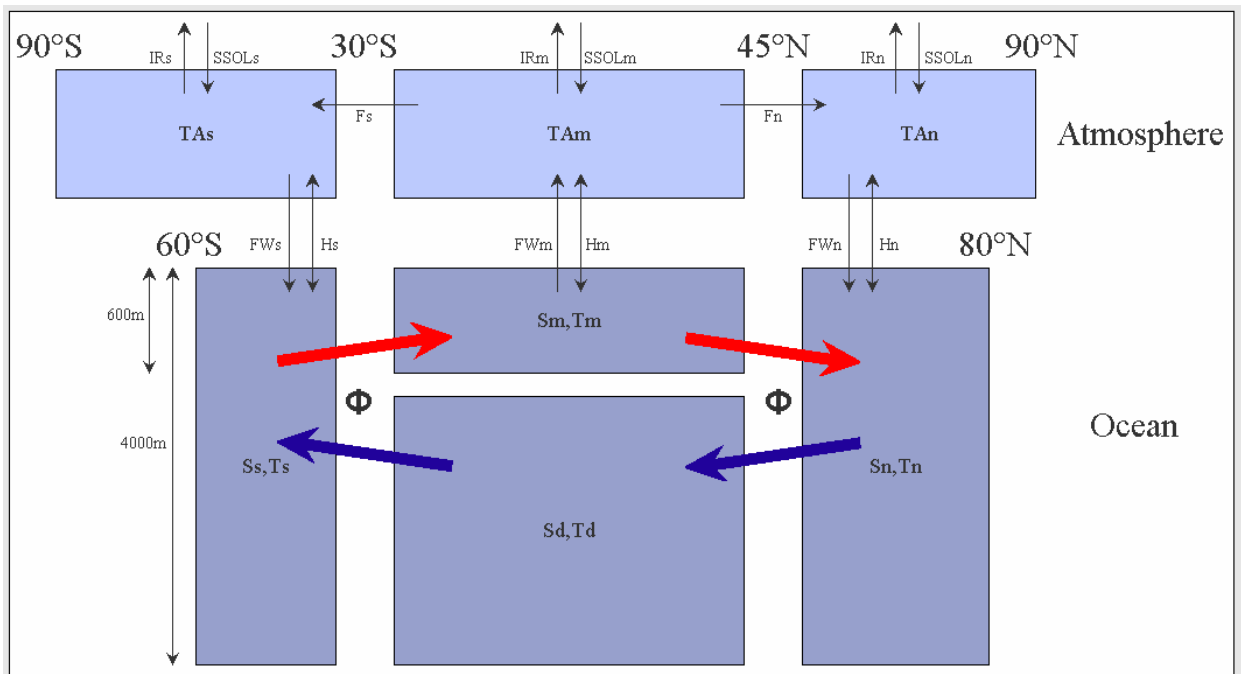
Figure 7.1: Schematic illustration of the Climate-Box-Model

The prognostic equations for the temperatures of the ocean boxes read

$$\frac{d}{dt}T_N = -\left(T_N - T_M\right)\frac{\Phi}{V_N} + \frac{H_N}{\rho_0 c_p dz_2}, \tag{7.25}$$

$$\frac{d}{dt}T_M = -\left(T_M - T_S\right)\frac{\Phi}{V_M} + \frac{H_M}{\rho_0 c_p dz_1}, \tag{7.26}$$

$$\frac{d}{dt}T_S = -\left(T_S - T_D\right)\frac{\Phi}{V_S} + \frac{H_S}{\rho_0 c_p dz_2} \quad \text{and} \tag{7.27}$$

$$\frac{d}{dt}T_D = -\left(T_D - T_N\right)\frac{\Phi}{V_D} \tag{7.28}$$

where ρ_0 denotes a reference density for saltwater and c_p the specific heat capacity of water. The factors dz_i and V_i indicate the depths and volumes of the discrete ocean boxes, respectively. The first terms in the equations are proportional to the overturning flow Φ and represent the advective transport between the boxes. The second terms (except for the deep box) represent the surface heat fluxes coupling the ocean and atmosphere. The overturning flow is assumed to be proportional to the density gradients of the oceans boxes after Stommel [1961]. Like in Rahmstorf [1996] the northern and the southern box will be taken into account for this, which leads to the equation for the calculation of the overturning flow

$$\Phi = c\left[-\alpha\left(T_N - T_S\right) + \beta\left(S_N - S_S\right)\right] \tag{7.29}$$

The constants α and β represent the thermal and the haline expansion coefficients in the equation of state. c is an adjustable parameter which is set to produce present-day overturning rates. This form of the overturning is also explained in section 6.7.[4]

The surface heat fluxes can be simplified according to Haney [1971]:

$$H_i = Q_{1_i} - Q_2\left(T_i - T_{A_i}\right) \tag{7.30}$$

---

[4]For other scaling laws: [Maas, 1994]. In his model, the dynamics bears similarities with the Lorenz system.

Analogue to (13.52) to (7.28) the prognostic differential equations for the salinities consist out of two components. One of those is again the advective part, caused by the interconnection between the boxes and the other one is the influence of the freshwater fluxes between the ocean and the atmosphere. The latter is again only for the boxes near the surface, thus the equations are

$$\frac{d}{dt} S_N = - (S_N - S_M) \frac{\Phi}{V_N} - S_{ref} \frac{(P - E)_N}{dz_N}, \tag{7.31}$$

$$\frac{d}{dt} S_M = - (S_M - S_S) \frac{\Phi}{V_M} + S_{ref} \frac{(P - E)_M}{dz_M}, \tag{7.32}$$

$$\frac{d}{dt} S_S = - (S_S - S_D) \frac{\Phi}{V_S} - S_{ref} \frac{(P - E)_S}{dz_S}, \tag{7.33}$$

$$\frac{d}{dt} S_D = - (S_D - S_N) \frac{\Phi}{V_D}. \tag{7.34}$$

The reference salinity S_{ref} is a characteristic average value for the entire Atlantic Ocean, and the freshwater fluxes are denoted as precipitation minus evaporation (P-E). These freshwater fluxes are calculated by the divergence of the latent heat transport in the atmosphere and are assumed to be proportional to the meridional moisture gradient explained below.

The atmospheric energy-balance-model (EBM) calculates the heat fluxes between the ocean and atmosphere, as well as horizontal latent and sensible heat transports as diffusion following Chen et al. [1995]. The EBM contains sensible and latent heat transports, radiation R_i, as well as the surface heat fluxes H_i between the atmosphere and the ocean. The atmospheric temperatures T_{A_i} follow the prognostic equations

$$c_2 \frac{d}{dt} T_{A_N} = \frac{\partial (F_{s_N} + F_{l_N})}{\partial y} + R_N - H_N, \tag{7.35}$$

$$c_2 \frac{d}{dt} T_{A_M} = \frac{\partial (F_{s_S} + F_{l_S})}{\partial y} + R_M - H_M, \tag{7.36}$$

$$c_2 \frac{d}{dt} T_{A_S} = \frac{\partial (F_{s_S} + F_{l_S})}{\partial y} + R_S - H_S. \tag{7.37}$$

c_2 is related to the specific heat of air. The sensible F_{s_i} and latent F_{l_i} heat transport are described in dependence of the meridional gradient of the surface temperature T_A and moisture q

$$F_s = K_s \frac{\partial T_A}{\partial y} \tag{7.38}$$

$$F_l = K_l \left(\frac{\partial q}{\partial y} \right). \tag{7.39}$$

K_s and K_l are empirical parameters, which must be adjusted to generate realistic values for sensible and latent heat transports. The radiation terms R_i in (7.35) to (7.37) consist of an incoming solar shortwave S_i and an outgoing infrared longwave I_i part. The extraterrestrial solar radiation is not absorbed entirely, and a latitude-dependent average albedo α_i is introduced to account for the reflectance. The outgoing infrared radiation I_i is calculated through a linear formula of Budyko [1969]. Thus, the equation for the net radiation balance is

$$R_i = S_i - I_i = S_{sol,i} (1 - \alpha_i) - (A + B T_{A_i}). \tag{7.40}$$

In this model, one can even include thw effect for changes in the the greenhouse gases (by multiplying $A + B T_{A_i}$ with a factor γ) and changes in the solar constant (by changing $S_{sol,i}$) which is left to the reader (see also the exercises).

The model calculates the freshwater fluxes from the divergence of the latent heat transport ($P - E \sim \partial F_l / \partial y$). The integration of the system is implemented with an Euler-forward scheme. The time step is $1/100$ of a year to ensure the stability of the system according to the Courant-Friedrichs-Levy-Criterion (CFL-Criterion, Courant et al. [1928][5]).

---

[5]For an English translation, refer to Courant et al. [1967].

7.2.2   Run the model

Here, we will use this box model using R[6]. As a tutorial: Download and install the R version for your operating system (http://cran.r-project.org).

Furthermore, it is recomended to use R studio (http://www.rstudio.com/), which provides a user interface for R. Perturbation experiments are done for the four ocean boxes. First the function sevenbox.r has to be defined, then the script must be run selecting the perturbations in the different boxes:

```
source('sevenbox.r')
sevenbox(1)  # for the southern box
sevenbox(3)  # for the northern box
```

The code creates png files of model output. The coding follows the names in Fig. 13.6, and the temperature (13.52, 7.26, 7.27 , 7.28) and salinity (7.31, 7.32, 7.33 , 7.34) budgets, respectively. Here is some part of the time step loop:

```
#-------------------------------------------------------------------------
# Euler forward for ocean temperature
#-------------------------------------------------------------------------

Tnln = Tnl + dts * ((Hfnl)/(rcz2)-(Tnl-Tml)*phi/Vnl);

Tmln = Tml + dts * ((Hfml)/(rcz1)-(Tml-Tsl)*phi/Vml);

Tsln = Tsl + dts * ((Hfsl)/(rcz2)-(Tsl-Td)*phi/Vsl);

Tdn = Td + dts * (-(Td-Tnl)*(phi/Vd));

#-------------------------------------------------------------------------
# Euler forward for Salinity
#-------------------------------------------------------------------------

Snln = Snl + dts * (-(Snl-Sml)*phi/(Vnl)-Sref*(FWFAN/Vnl)-Sref*F4/Vnl);

Smln = Sml + dts * (-(Sml-Ssl)*phi/Vml+Sref*FWFAN/Vml+Sref*FWFAS/Vml);
```

---

[6]The box model is available in four versions: one in R, one in matlab, and two Fortran versions. The program-examples in fortran require a compiler which supports several features from recent versions of the Fortran-standard (Fortran 2003 and Fortran 2008) and is explained in detail in Chirila and Lohmann [2014]. The reason to include fortran is that it is the most used computer language in climate research and that it is faster. Furthermore, a regular fortran version and a version using object oriented programming is given. This section is more for the specialists in the course. Matlab is a commercial program, but quite often used in climate sciences. The program sevenbox.m is similar to the R-code.

```
Ssln = Ssl + dts * (-(Ssl-Sd)*phi/Vsl-Sref*(FWFAS/Vsl)-Sref*F3/Vnl);

Sdn = Sd + dts * (-(Sd-Snl)*(phi/Vd));

#----------------------------------------------------------------------
```

One particular package is Shiny which provides a Gui web application easy to use. Here are the steps:

unpack the data from the online version

open R

```
#go to this directory (setwd):
setwd('/Users/glohmann/Vorlesungen/PhDCourses/CourseNL/Rprogr/boxgui/')
#load Script
source ('run_ui.R')
#run Script
run_ui()
# for multicore: provide the numbers of processors, e.g. run_ui(8).


#  or do the following:
library(shiny)
runApp('/Users/glohmann/model/Boxmodel_GUI')
```

Fig. 7.2 illustrates how the model works.

Sevenbox

Simulation length [years]

1000

Start Simulation

This may take some time!

☐ Change Perturbations
☐ Change heat cap. and heat fluxes
☐ Change solar CO_2 effect
☐ Change add. fresh water fluxes

Temperature   Salinity   Ocean Flux

Sevenbox

Simulation length [years]

1000

Start Simulation

This may take some time!

Save results

☑ Change Perturbations

Choose which box should be perturbed:

north

Perturbations

-0.1; -0.2

Devide multiple pertubations by semicolon!

☐ Change heat cap. and heat fluxes
☐ Change solar CO_2 effect
☐ Change add. fresh water fluxes

Temperature   Salinity   Ocean Flux

Sevenbox

Simulation length [years]

1000

Start Simulation
This may take some time!

Save results

☑ Change Perturbations
Choose which box should be perturbed:

north ▾

Perturbations

-0.1; -0.2

Devide multiple pertubations by semicolon!

☐ Change heat cap. and heat fluxes
☐ Change solar CO$_2$ effect
☐ Change add. fresh water fluxes

Temperature Salinity Ocean Flux

Sevenbox

Simulation length [years]

1000

Start Simulation
This may take some time!

Save results

☑ Change Perturbations
Choose which box should be perturbed:

north ▾

Perturbations

-0.1; -0.2

Devide multiple pertubations by semicolon!

☐ Change heat cap. and heat fluxes
☐ Change solar CO$_2$ effect
☐ Change add. fresh water fluxes

Temperature Salinity Ocean Flux
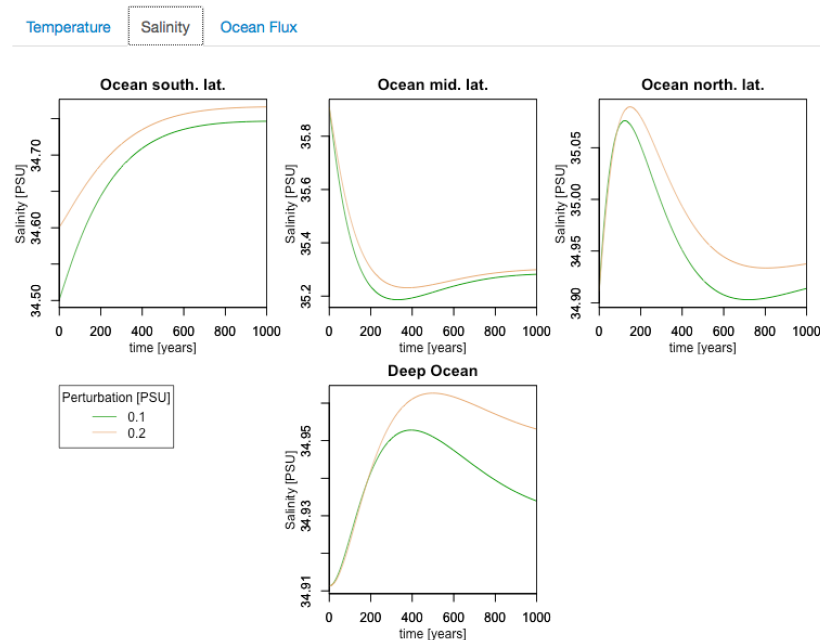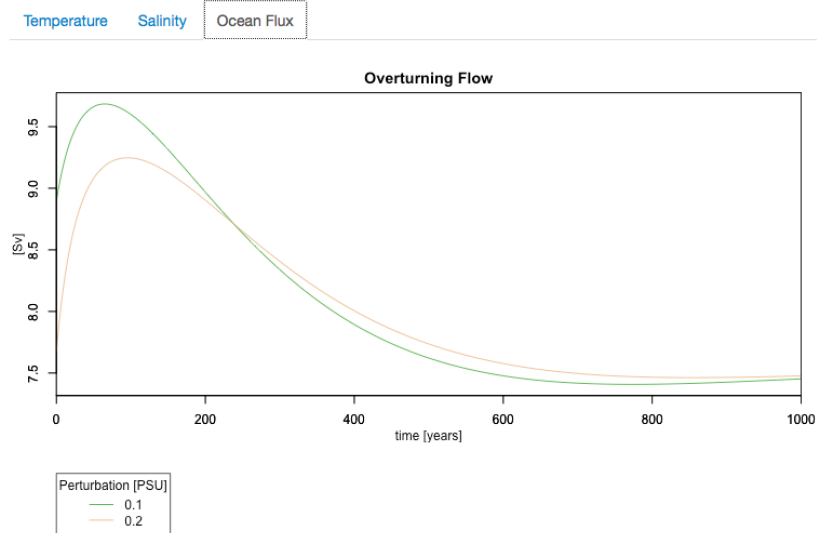
Figure 7.2: Output of the climate box model illustrating the procedure.

7.2.3 Run the box model in Fortran90*

The box model is available in four versions: one in R, one in matlab, and two fortran versions. The program-examples in fortran require a compiler which supports several features from recent versions of the Fortran-standard (Fortran 2003 and Fortran 2008) and is explained in detail in section **??**. The reason to include fortran is that it is the most used computer language in climate research and that it is faster. Furthermore, a regular fortran version and a version using object oriented programming is given. This section is more for the specialists in the course.

In this document, we describe (briefly) how to install a version of the GNU Fortran Compiler (`gfortran`) which meets these requirements. `gfortran` is part of the GNU Compiler Collection (`gcc`), which provides a suite of compilers for various programming-languages for all the major PC platforms (`Linux`, `Windows`, and `OS X`).

In the following sections, we discuss each platform separately. Note that, in all cases, you have the options of installing a pre-compiled package, or of compiling from source. However, we only emphasize the first method here, for simplicity.

Disclaimers/Notes:

- To keep this document short, we only describe the basic tools, necessary for compiling programs at the command-prompt/terminal. Of course, more user-friendly Integrated Development Environments (IDEs) exist for all platforms.

- An alternative set of compilers available for free[7] in `Linux` is offered by Intel:

  http://software.intel.com/en-us/non-commercial-software-development

  This is not discussed here (see link for details).

Instructions for `GNU/Linux` (focusing on Ubuntu)   In this section, we describe two methods for installing `gfortran-4.8` using the package-manager in Ubuntu.

---

[7]For non-commercial users.

Ubuntu 12.04 (Precise)   Because gcc-4.8 was released quite recently (March 22, 2013), most distributions did not yet have this package in the default-repositories at the time of this writing (June 25, 2018). However, Ubuntu Precise (12.04) users can add a repository for this, as in:

```
sudo add-apt-repository ppa:ubuntu-toolchain-r/test
sudo apt-get update
sudo apt-get install gfortran-4.8
```

Note that, when using this method, the compiler to be invoked will be named gfortran-4.8.

Ubuntu (other releases)   Users of other Ubuntu releases can add the latest release of GCC as follows:

```
sudo apt-get install gcc-snapshot
# can add the two lines below to ~/.bashrc,
# to automatize the compiler-selection
export PATH=$PATH:/usr/lib/gcc-snapshot/bin/
export LD_LIBRARY_PATH=$LD_LIBRARY_PATH:/usr/lib/gcc-snapshot/lib/
```

An overview of the gcc/gfortran versions available for each distribution is also available here: http://gcc.gnu.org/wiki/GFortranDistros.

Instructions for `MS Windows`   The easiest way to get GCC on Windows is to use the packages provided at the website www.equation.com:

1. go to www.equation.com, then (in the menu on the left) navigate to "Programming Tools" → "Fortran, C, C++"

2. scroll to the section "DOWNLOAD: OFFICIAL RELEASES"

3. download a "Self-Extracting File" (either the 32bit or the 64bit version, depending on your version of Windows)

4. start the installation program

5. accept the license agreement

6. choose a path for installation (or leave the default, i.e. `c:\gcc`)

7. click "Install" and wait for the process to complete; a new entry in the Windows Start-menu will be created, containing useful links to the manuals of the compilers, etc.

8. click "Finish" to close the installation-program, and re-login into your user-account

Instructions for `Apple OSX` A good overview of the possible alternatives for installing `gfortran-4.8` on OSX is available here:

`http://scicomp.stackexchange.com/questions/2469/how-should-i-install-a-fortran-compiler-on-a-mac-os-x-10-x-x-4`

And here is some Fortran90-code of the time stepping:

```fortran
!-----------------------------------------------------------------------
! Euler forward for ocean temperature
!-----------------------------------------------------------------------

      Tnln = Tnl + dts * ((Hfnl)/(rcz2)-(Tnl-Tml)*phi/Vnl)


      Tmln = Tml + dts * ((Hfml)/(rcz1)-(Tml-Tsl)*phi/Vml)


      Tsln = Tsl + dts * ((Hfsl)/(rcz2)-(Tsl-Td)*phi/Vsl)

      Tdn = Td + dts * (-(Td-Tnl)*(phi/Vd))


!-----------------------------------------------------------------------
! Euler forward for Salinity
!-----------------------------------------------------------------------

      Snln = Snl + dts * &
     & (-(Snl-Sml)*phi/(Vnl)-Sref*(FWFAN/Vnl)-Sref*F4/Vnl)

      Smln = Sml + dts * &
     & (-(Sml-Ssl)*phi/Vml+Sref*FWFAN/Vml+Sref*FWFAS/Vml)
```

```fortran
    Ssln = Ssl + dts * &
  & (-(Ssl-Sd)*phi/Vsl-Sref*(FWFAS/Vsl)-Sref*F3/Vnl)

    Sdn = Sd + dts * (-(Sd-Snl)*(phi/Vd))


!----------------------------------------------------------------------
! Euler forward for atmosphere temperature
!----------------------------------------------------------------------

    Tanln = Tanl + dts * (((cos45/(r*(Sin(90.0*Degree)-&
  & Sin(45*Degree))))*(FSN+FLN)+(Qsolarnl*(1.-alpn)-CFnl*(A+B*Tanl))+&
  & (frf1*(-(Q1nl-Q2nl*(Tnl-Tanl)))))/(cpa*betanl))

    Tamln = Taml + dts * ((-(1./(r*(Sin(30.0*Degree)+&
  & Sin(45*Degree))))*(cos30*FTS+cos45*FTN)+(Qsolarml*(1.-alpm)-&
  & CFml*(A+B*Taml))+(frf2*(-(Q1ml-Q2ml*(Tml-Taml)))))/(cpa*betaml))

    Tasln = Tasl + dts * (((cos30/(r*(Sin(90.0*Degree)-&
  & Sin(30*Degree))))*(FSS+FLS)+(Qsolarsl*(1.-alps)-CFsl*(A+B*Tasl))+&
  & (frf3*(-(Q1sl-Q2sl*(Tsl-Tasl)))))/(cpa*betasl))

!----------------------------------------------------------------------
```

7.2.4 Model scenarios

Paleoclimatic evidence suggests (e.g. Dansgaard et al. [1993]) that some past climate shifts were associated with changes in North Atlantic Deep Water (NADW) formation. Deep water formation in the Greenland-Iceland-Norwegian Sea and in Labrador Sea drive the large-scale ocean circulation imposing strong northward heat transport. This makes the northern North Atlantic about 4 K warmer than corresponding latitudes in the Pacific and is responsible for the mild climate of Western Europe. Variations in NADW circulation therefore have the potential to cause significant climate change in the North Atlantic region.

Numerical simulations by Manabe and Stouffer [1993] showed, for the North Atlantic, that between two and four times the preindustrial CO_2 concentration, a threshold value is passed and the thermohaline circulation ceases completely. One other example of early Holocene rapid climate change is the '8200 yr BP' cooling event recorded in the North Atlantic region possibly induced by freshwater. One possible explanation for this dramatic regional cooling is a shutdown in the formation of deep water in the northern North Atlantic due to freshwater input caused by catastrophic drainage of Laurentide lakes (e.g., Barber et al. [1999]; Lohmann [2003]). After the end of the last glacial, freshwater entered into the Atlantic Ocean (Fig. 9.9) and may have affected the ocean circulation.

Exercise 55 – Investigations with the box-model

1. In the regions of deep water formation in the North Atlantic, relatively small amounts of fresh water added to the surface can stabilize the water column to the extent that convection can be prevented from occurring. Such interruption decreases the poleward oceanic mass transport Φ. Furthermore, this perturbation of the meridional transport can be amplified by positive feedbacks: a weaker northward salt transport brings less dense water to high latitudes, which further reduces the high-latitude density. Discuss the case where the initial conditions in salinity at different latitudes is changed. Show this scenario in the box model!

2. Comment on the scenario of climate change as shown in the cinema movie The Day After Tomorrow: link to the website or go to the trailer.

3. Which feedbacks are acting for global warming? You can change the long wave radiation. A doubling of pCO_2 is equivalent to an additional forcing of 4 Wm^{-2}. For this you have to modify the net radiation balance (7.40) through reduction in the outgoing longwave radiation (parameter γ). Additional radiative forcing may come from increased tracer gas concentrations in the atmosphere. Please evaluate the hydrological cycle and atmospheric heat transports! What is the change in the overturning rate?

4. Change the ocean heat capacity by a factor of 10 and describe the change in the response to warming induced by 90

5. The initial values of the model represent averages for present-day climate conditions. Determine the effect of the parameter c in the numerical example (representing a different long wave radiation) Can you derive a glacial climate? The glacial climate was 3 K colder in the tropics.

6. Calculate the ocean heat transport in the model and compare it with the following estimate!

$$H = \int_{bottom}^{top} \rho_0 v T dz \qquad (7.41)$$

$$= -c_p \int_{bottom}^{top} \frac{\partial \Phi}{\partial z} T dz \qquad (7.42)$$

$$= c_p \int_{bottom}^{top} \Phi \frac{\partial T}{\partial z} dz \qquad (7.43)$$

$$= c_p \int_{T(bottom)}^{T(top)} \Phi dT \qquad (7.44)$$

where $\Phi = \rho_0 \Phi_{MOC}$ with Φ_{MOC} being the volume transport. Therefore, the heat transport can be estimated in terms of the mass transport in temperature layers:

$$H = c_p \underbrace{(T(top) - T(bottom))}_{15K} \underbrace{\Phi_{max}}_{20 \cdot 10^9 kg/s} \qquad (7.45)$$

which is about $1.2\ PW\ (PW = 10^{15}W)$.

7. Question for specialists: The coupled model shall be used to investigate the sensitivity of the system with respect to stochastic weather perturbations reflecting unresolved effects of the atmospheric transports modeled as white noise. How will the atmospheric noise influence the stability of the system?

Exercise 56 – Interhemispheric Box model on the web

Repeat the exercise 55 using the R-Shiny App on `https://paleosrv2.awi.de/`. The username is *student* and the Password is *EbJir5ow* !

7.3   Weather and climate: Stochastic climate model

The daily observed maximum and minimum temperatures is often compared to the "normal" temperatures based upon the 30-year average. Climate averages provide a context for something like "this winter will be wetter (or drier, or colder, or warmer, etc.) than normal. It has been said "Climate is what you expect. Weather is what you get."

What is the difference between weather and climate? This can be also answered by an example/a methaphor in the football league. Predicting the outcome of the next game is difficult (weather), but predicting who will end up as German champion is unfortunately relatively easy (climate). In this section, I will give a general approach to the mean and fluctuations in the climate system. Indeed, the Brownian motion approach is a helpful analogue for weather and climate.

7.3.1   Brownian motion

The Roman Lucretius's scientific poem On the Nature of Things (ca. 60 BC) has a remarkable description of Brownian motion of dust particles[8]. Jan Ingenhousz had described the irregular motion of coal dust particles on the surface of alcohol in 1785. Nevertheless Brownian motion is traditionally regarded as discovered by the botanist Robert Brown in 1827. It is believed that Brown was studying pollen particles floating in water under the microscope. He then observed minute particles within the vacuoles of the pollen grains executing a jittery motion. By repeating the experiment with particles of dust, he was able to rule out that the motion was due to pollen particles being 'alive', although the origin of the motion was yet to be explained.

See the film: https://en.wikipedia.org/wiki/Brownian_motion#/media/

---

[8]He uses this as a proof of the existence of atoms: "Observe what happens when sunbeams are admitted into a building and shed light on its shadowy places. You will see a multitude of tiny particles mingling in a multitude of ways... their dancing is an actual indication of underlying movements of matter that are hidden from our sight... It originates with the atoms which move of themselves [i.e. spontaneously]. Then those small compound bodies that are least removed from the impetus of the atoms are set in motion by the impact of their invisible blows and in turn cannon against slightly larger bodies. So the movement mounts up from the atoms and gradually emerges to the level of our senses, so that those bodies are in motion that we see in sunbeams, moved by blows that remain invisible." Although the mingling motion of dust particles is caused largely by air currents, the glittering, tumbling motion of small dust particles is, indeed, caused chiefly by true Brownian dynamics.

File:Brownian_motion_large.gif.
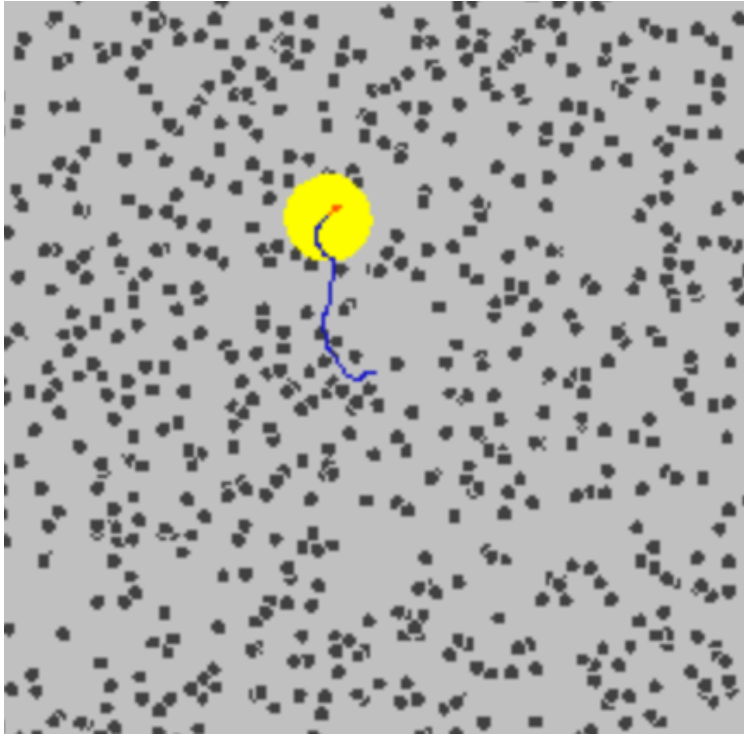
Figure 7.3: Snapshot of a movement of a Brownian particle.

The first person to describe the mathematics behind Brownian motion was Thorvald N. Thiele in 1880 in a paper on the method of least squares. This was followed independently by Louis Bachelier in 1900 in his PhD thesis "The theory of speculation", in which he presented a stochastic analysis of the stock and option markets. However, it was Albert Einstein's (in his 1905 paper) and Marian Smoluchowski's (1906) independent research of the problem that brought the solution to the attention of physicists, and presented it as a way to indirectly confirm the existence of atoms and molecules. The confirmation of Einstein's theory constituted empirical progress for the kinetic theory of heat. In essence, Einstein showed that the motion can be predicted directly from the kinetic model of thermal equilibrium. The importance of the theory lay in the fact that it confirmed the kinetic theory's account of the second law of thermodynamics as being an essentially statistical law.

Brownian motion: Statistical description

Einstein's argument was to determine how far a Brownian particle travels in a given time interval. Classical mechanics is unable to determine this distance because of the enormous number of bombardments a Brownian particle will undergo, roughly of the order of 1021 collisions per second. Thus Einstein was led to consider the collective motion of Brownian particles.
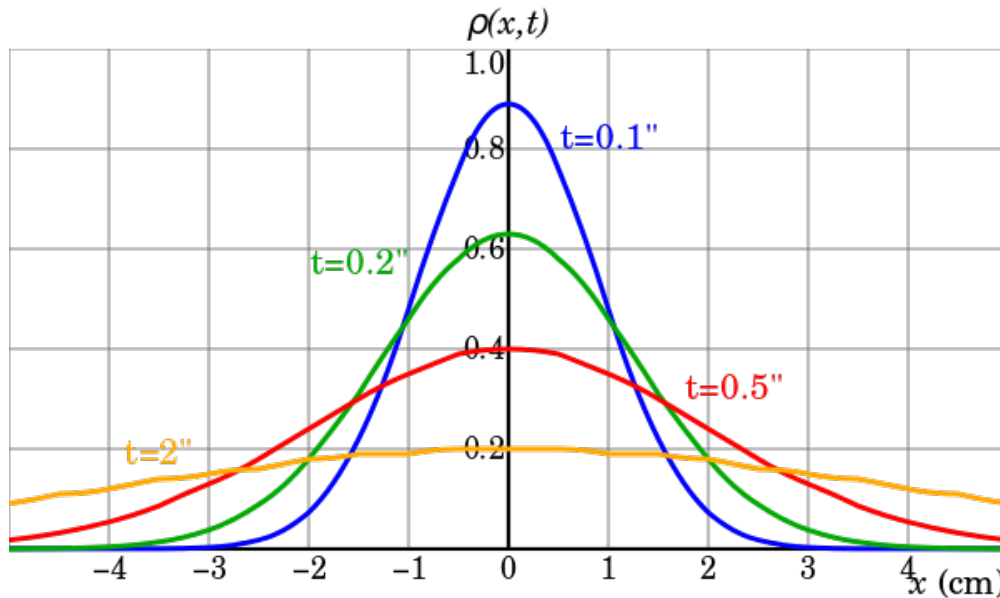
Figure 7.4: The characteristic bell-shaped curves of the diffusion of Brownian particles. The distribution begins as a Dirac delta function, indicating that all the particles are located at the origin at time t=0, and for increasing times they become flatter and flatter until the distribution becomes uniform in the asymptotic time limit.

He regarded the increment of particle positions in unrestricted one dimensional $x-$ domain as a random variable (Δ or x, under coordinate transformation so that the origin lies at the initial position of the particle) with some probability density function $\phi(\Delta)$. Further, assuming conservation of particle number, he expanded the density (number of particles per unit volume) change

in a Taylor series:

$$\rho(x, t + \tau) = \rho(x, t) + \tau \frac{\partial \rho(x)}{\partial t} \tag{7.46}$$

$$= \int_{-\infty}^{+\infty} \rho(x + \Delta, t) \cdot \phi(\Delta) \, \mathrm{d}\Delta \tag{7.47}$$

$$= \rho(x, t) \cdot \int_{-\infty}^{+\infty} \phi(\Delta) \, d\Delta + \frac{\partial \rho}{\partial x} \cdot \int_{-\infty}^{+\infty} \Delta \cdot \phi(\Delta) \, d\Delta$$

$$+ \frac{\partial^2 \rho}{\partial x^2} \cdot \int_{-\infty}^{+\infty} \frac{\Delta^2}{2} \cdot \phi(\Delta) \, \mathrm{d}\Delta \quad + \quad \dots \tag{7.48}$$

$$= \rho(x, t) \cdot 1 + 0 + \frac{\partial^2 \rho}{\partial x^2} \cdot \int_{-\infty}^{+\infty} \frac{\Delta^2}{2} \cdot \phi(\Delta) \, \mathrm{d}\Delta \quad + \quad \dots \tag{7.49}$$

The integral in the first term is equal to one by the definition of probability, and the second and other even terms (i.e. first and other odd moments) vanish because of space symmetry. What is left gives rise to the following relation:

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} \cdot \int_{-\infty}^{+\infty} \frac{\Delta^2}{2\,\tau} \cdot \phi(\Delta) \, \mathrm{d}\Delta + \textbf{higher order even moments}$$

Where the coefficient before the Laplacian, the second moment of probability of displacement Δ, is interpreted as mass diffusivity D :

$$D = \int_{-\infty}^{+\infty} \frac{\Delta^2}{2\,\tau} \cdot \phi(\Delta) \, \mathrm{d}\Delta$$

Then the density of Brownian particles ρ at point x at time t satisfies the diffusion equation:

$$\frac{\partial \rho}{\partial t} = D \cdot \frac{\partial^2 \rho}{\partial x^2},$$

Assuming that N particles start from the origin at the initial time $t = 0$, the diffusion equation has

the solution

$$\rho(x, t) = \frac{N}{\sqrt{4\pi Dt}} e^{-\frac{x^2}{4Dt}}.$$

This expression allowed Einstein to calculate the moments directly. The first moment is seen to vanish, meaning that the Brownian particle is equally likely to move to the left as it is to move to the right. The second moment is, however, non-vanishing, being given by

$$\overline{x^2} = 2\,D\,t.$$

This expresses the mean squared displacement in terms of the time elapsed and the diffusivity. From this expression Einstein argued that the displacement of a Brownian particle is not proportional to the elapsed time, but rather to its square root. His argument is based on a conceptual switch from the "ensemble" of Brownian particles to the "single" Brownian particle: we can speak of the relative number of particles at a single instant just as well as of the time it takes a Brownian particle to reach a given point.

This can be formalized as follows. The Wiener process is a continuous-time stochastic process with stationary independent increments. The Wiener process W_t is characterized by three facts:

- $W_0 = 0$
- W_t is almost surely continuous
- W_t has independent increments with normal distribution $W_t - W_{t_0} \sim N(0, t - t_0)$. $N(\mu, \sigma^2)$ denotes the normal distribution with expected value μ and variance σ^2. The condition that it has independent increments means that if then and are independent random variables.

The Wiener process can be constructed as the scaling limit of a random walk, or other discrete-time stochastic processes with stationary independent increments. It can be denoted as

$$var(W_t) = \overline{dW_t^2} = 2\sigma^2 t \tag{7.50}$$

where $\overline{dW_t^2}$ is the mean square displacement of a Brownian particle in time t (t_0 is set to zero).[9] The so-called diffusion constant $D = \sigma^2$ is related to the mean free path λ and the average time between collisions τ:

$$2D = \frac{\lambda^2}{\tau} \quad . \tag{7.51}$$

The time evolution of the position of the Brownian particle itself is best described using Langevin equation, an equation which involves a random force field representing the effect of the thermal fluctuations of the solvent on the particle. The displacement of a particle undergoing Brownian motion is obtained by solving the diffusion equation under appropriate boundary conditions and finding the root mean square of the solution. This shows that the displacement varies as the square root of the time (not linearly).

Exercise 57   –   Brownian motion on a computer

Imagine a so-called red-noise process

$$\frac{dx}{dt} = -\lambda x + \xi \, . \tag{7.52}$$

1. Calculate the model using the following R code:

```
#brownian motion, multiple particle
#forward modelling

Nparticle<-1000 #number of particles
T<-  1000   #integration time in time units
h<-  0.5   #step size in time units

beta<-0.00001 #friction term
lambda<-1 #noise term

N<-T/h
```

---

[9] A heuristic helpful interpretation of the stochastic differential equation is that in a small time interval of length dt, the stochastic process changes its value by an amount that is normally distributed with variance $2\sigma^2 dt$ and is independent of the past behavior of the process. This is so because the increments of a Wiener process are independent and normally distributed.
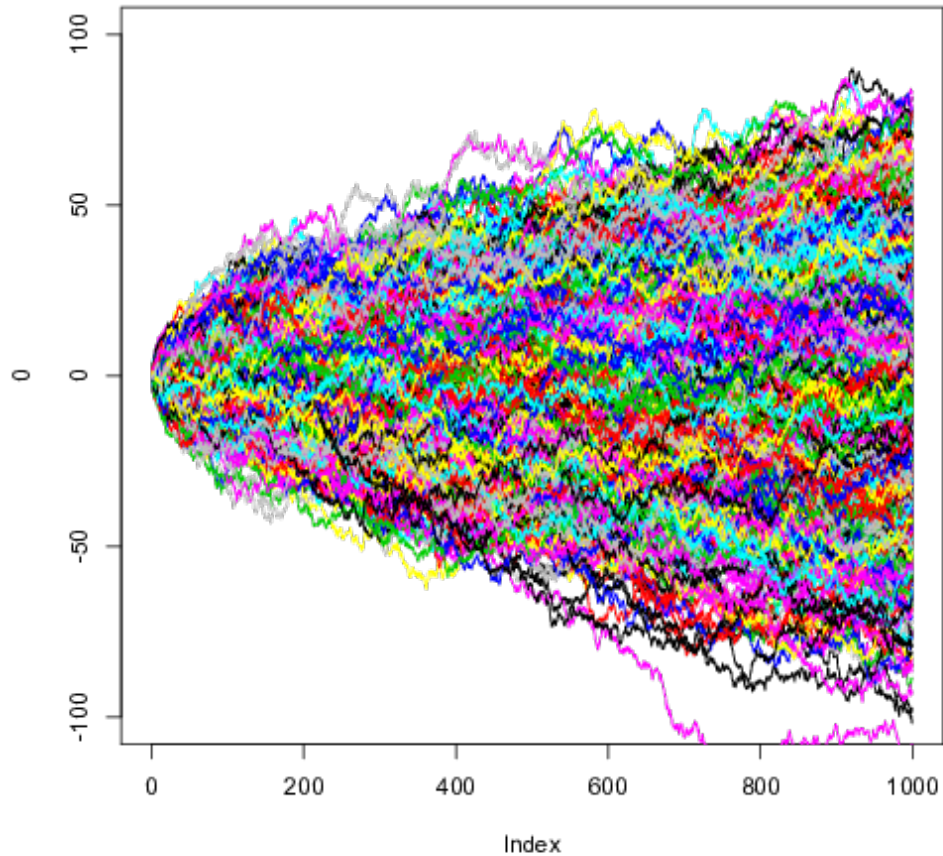
Figure 7.5: Numerical solution of the brownian motion, multiple particles. See exercise 57 for details.

```
t<-(0:(N-1))*h

x<-matrix(0,Nparticle,N) # Initial condition, all = 0


for (i in 1:(N-1))
{
  x[,i+1]<-x[,i]*(1-beta*h)+ rnorm(Nparticle)*sqrt(h)

}


plot(0,xlim=c(0,T),ylim=c(-100,100),type="n")
for (i in 1:N) lines (t,x[i,],col=i)
```

```
#analyse the densities
h<-matrix(0,N,40)

for (i in 1:(N-1)) h[i,]<-hist(x[,i],breaks=c((-20:20)*10),freq=FALSE,
              ylim=c(0,0.04))$counts

filled.contour(t,(-19:20)*10-5,h,color.palette=rainbow,xlab="time",
              ylab="space")
```

2. Show that the displacement varies as the square root of the time (not linearly).

7.3.2 Stochastic climate model

In a stochastic framework of climate theory one may use an appropriate stochastic differential equation (Langevin equation)

$$\frac{d}{dt}x(t) = f(x) + g(x)\xi, \tag{7.53}$$

where $\xi = \frac{d}{dt}W(t)$ is a stationary stochastic process and the functions $f, g : \mathbf{R}^n \to \mathbf{R}^n$ describe the climate dynamics. The properties of the random force are described through its distribution and its correlation properties at different times. The process ξ is assumed to have a Gaussian distribution of zero average,

$$< \xi(t) >= 0 \tag{7.54}$$

and to be δ-correlated in time,

$$< \xi(t)\xi(t + \tau) >= \delta(\tau) \tag{7.55}$$

where δ is the delta function defined by

$$\int_{\mathbf{R}} f(x)\,\delta(x - x_0)\,dx = f(x_0) \quad . \tag{7.56}$$

The brackets indicate an average over realizations of the random force.[10] For a Gaussian process only the average and second moment need to be specified since all higher moments can be expressed in terms of the first two. Note that the dependence of the correlation function on the time difference τ assumes that ξ is a stationary process. ξ is called a white-noise process (for the colors of noise: https://en.wikipedia.org/wiki/Colors_of_noise). In general, the stochastic processes can be also described by the probablity distributions (3.19) which will be considered later.

Additionally, there might be an external forcing $F(x, t)$ which is generally time-, variable-, and space-dependent. In his theoretical approach, Hasselmann [1976] formulated a linear stochastic climate model

$$\frac{d}{dt}x(t) = Ax + \sigma\xi + F(t) \quad , \tag{7.57}$$

with system matrix $A \in \mathbf{R}^{n \times n}$, constant noise term σ, and stochastic process ξ. Many features of the climate system can be well described by (7.57), which is analogous to the Ornstein-Uhlenbeck process in statistical physics [Uhlenbeck and Ornstein, 1930]. Notice that $\sigma\xi$ represents a stationary random process. The relationship derived above is identical to that describing the diffusion of a fluid particle in a turbulent fluid. In a time-scale separated system, during one slow-time unit the fast uninteresting variables y perform many 'uncorrelated' events (provided that the fast dynamics are sufficiently chaotic). The contribution of the uncorrelated events to the dynamics of the slow interesting variables x is as a sum of independent random variables. By the weak central limit theorem this can be expressed by a normally distributed variable. Note, in the absence of any feedback effects Ax, the climate variations would continue to grow indefinitely as the Wiener process.

---

[10]Formally: $\xi(t)$ is a random variable, i.e. $\xi(t)(\alpha)$ with different realizations due to random variable α. The expectation $< \xi(t) >$ is thus the mean over all $\alpha :< \xi(t)(\alpha) >_\alpha$. Using the ergodic hypothesis, the ensemble average $\langle \rangle$ can be expressed as the time average $\lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt$ of the function. Almost all points in any subset of the phase space eventually revisit the set. (https://en.wikipedia.org/wiki/Ergodic_theory

Numerical integration of the Langevin equation

One can numerically integrate such a nonlinear Langevin equation with flow $f(x)$ using a simple Euler-Maruyama method with a fixed time step Δt :

$$x(t + \Delta t) = x(t) + f(x)\Delta t + g(x)\sqrt{\Delta t}\Delta W_n \tag{7.58}$$

The variables ΔW_n are known as increments of the Wiener process; they are Gaussian numbers generated in an uncorrelated fashion, for example by using a pseudo-random number generator in combination with the Box-Müller algorithm.

```
% calculate sde in matlab\\
%
th = 1;
mu = 1.2;
sig = 0.3;
dt = 1e-2;
t= 0:dt:20;
x = zeros(1,length(t)); % Allocate output vector, set initial condition
rng(1);                  % Set random seed
for i = 1:length(t)-1
    x(i+1) = x(i)+th*(mu-x(i))*dt+sig*sqrt(dt)*randn;
end
figure;
plot(t,x);
```

and this is for the analytical solution:

```
th = 1;
mu = 1.2;
sig = 0.3;
dt = 1e-2;
t = 0:dt:20;      % Time vector
x0 = 0;           % Set initial condition
rng(1);           % Set random seed
ex = exp(-th*t);
x = x0*ex+mu*(1-ex)+sig*ex.*cumsum([0 sqrt(diff(exp(2*th*t)-1)).
          *randn(1,length(t)-1)])/sqrt(2*th);
figure;
plot(t,x);
```
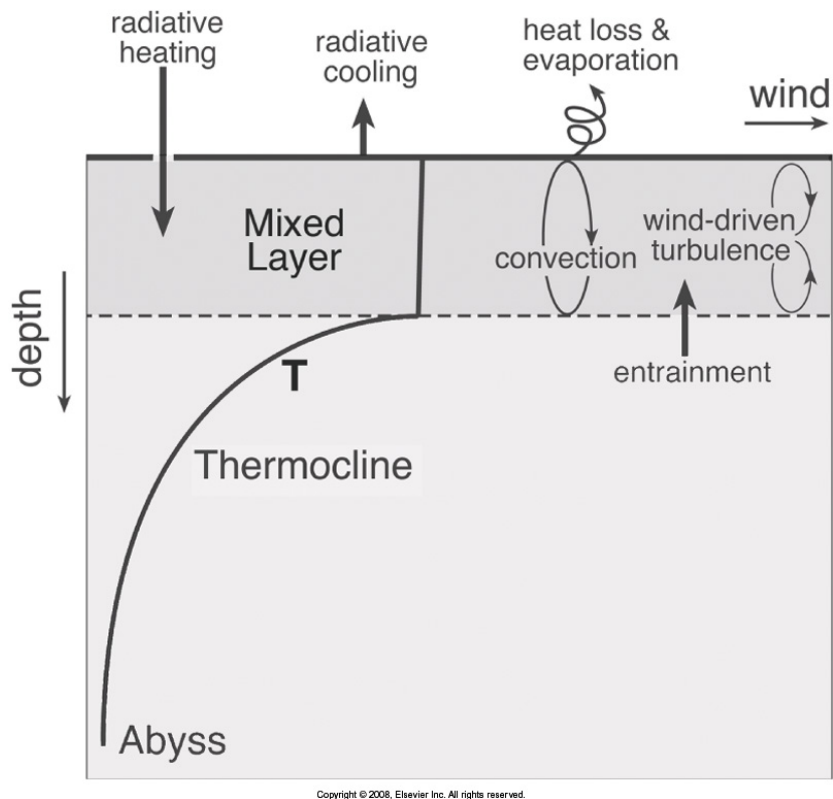
Figure 7.6: Schematic picture of mixed layer in the ocean.

Exercise 58 – **Stochastic Climate Model**

Imagine that the temperature of the ocean mixed layer of depth h (Fig. 7.6) is governed by

$$\frac{dT}{dt} = -\lambda T + \frac{Q_{net}}{\gamma_O},$$
(7.59)

where coefficient γ_O is given by the heat capacity time density times mixed layer depth $c_p \rho h$. $(h = 100m; c_p = 4.2 \cdot 10^3 Jkg^{-1}K^{-1}; \rho = 1023kgm^{-3})$. λ is the typical damping rate of a temperature anomaly. Observations show that sea surface temperatures a typically damped at a rate of $15 Wm^{-2}K^{-1}$.

1. Calculate the typical time scale $1/\lambda$.

2. Calculate the stochastic climate model using the R code

```
# Stochastic climate model/Ornstein-Uhlenbeck/Red Noise: Brown.R
T<-  5000   #integration time in time units
h<-  0.1   #step size in time units
X0<- 10     #inital value
beta<-0.05 #friction term
lambda<-1 #noise term
N<-T/h
t<-(0:(N-1))*h

x<-vector()
x[1]<-X0

for (i in 1:(N-1)) {
  x[i+1]<-x[i]*(1-beta*h)+ rnorm(1)*sqrt(h)
}

plot(t,x,type="l")
hist(x,freq=FALSE, col="gray")
```

From the online malterial, please see the browngui directory: BrownianMotion.zip See Figure 7.8.

3. Do the same, but for many Brownian particles in a potential (cf. Fig. 7.9).

```
# Brownian motion, multiple particle: Brown_mult.R
# forward modelling

#the function dy/dt<-f(y,a,b,c,d)
f<-function(y,a,b,c,d)
{ return(d*y^3+c*y^2+b*y-a)     }

#constants
Ca<-10
a<-1
b<- 0.8
c<- 0
d<- -0.001

Nparticle<-1000 #number of particles
T<-  500   #integration time in time units
h<-  0.5   #step size in time units
N<-T/h
t<-(0:(N-1))*h

x<-matrix(10,Nparticle,N) # Initial condition, all = 0
# Initial condition,

for (i in 1:(N-1)) {
  x[,i+1]<- x[,i]+h*f(x[,i],a,b,c,d)  + Ca*rnorm(Nparticle)*sqrt(h)
}

ama2=max(x,2)
ami=min(x,-2)
ama=max(ama2,-ami)
plot(0,xlim=c(0,T),ylim=c(ami,ama),type="n")
for (i in 1:10) lines (t,x[i,],col=i)

#analyse the densities
h<-matrix(0,N,40)
for (i in 1:(N-1)) { h[i,]<-hist(x[,i],breaks=
      c(-20:20)*ama/10,freq=FALSE,ylim=c(0,0.04))$counts    }
hstat<-matrix(0,N)
for (i in N/2:(N-1)) hstat[]<-h[i,]+hstat[]
hstat[]<-hstat[] *2/Nparticle/N
#plot(t,hstat[],type="l")
plot(table(hstat[]), type = "h", col = "red")

# to show the time evolution, 1, 2, 4, 8, ....  time step
 op <- par(mfrow = c(3, 2))
plot(h[1,]/Nparticle,type="l")
plot(h[2,]/Nparticle,type="l")
plot(h[4,]/Nparticle,type="l")
plot(h[8,]/Nparticle,type="l")
plot(h[N/2,]/Nparticle,type="l")
```

```
plot(h[N-1,]+h[N-2,]/Nparticle/2,type="l")

filled.contour(t,(-19:20)*ama/10-ama/20,h,
      color.palette=rainbow,xlab="time",ylab="space")
```

4. Calculate the stationary density from the numerical example analytically using $\int f(y)dy$.
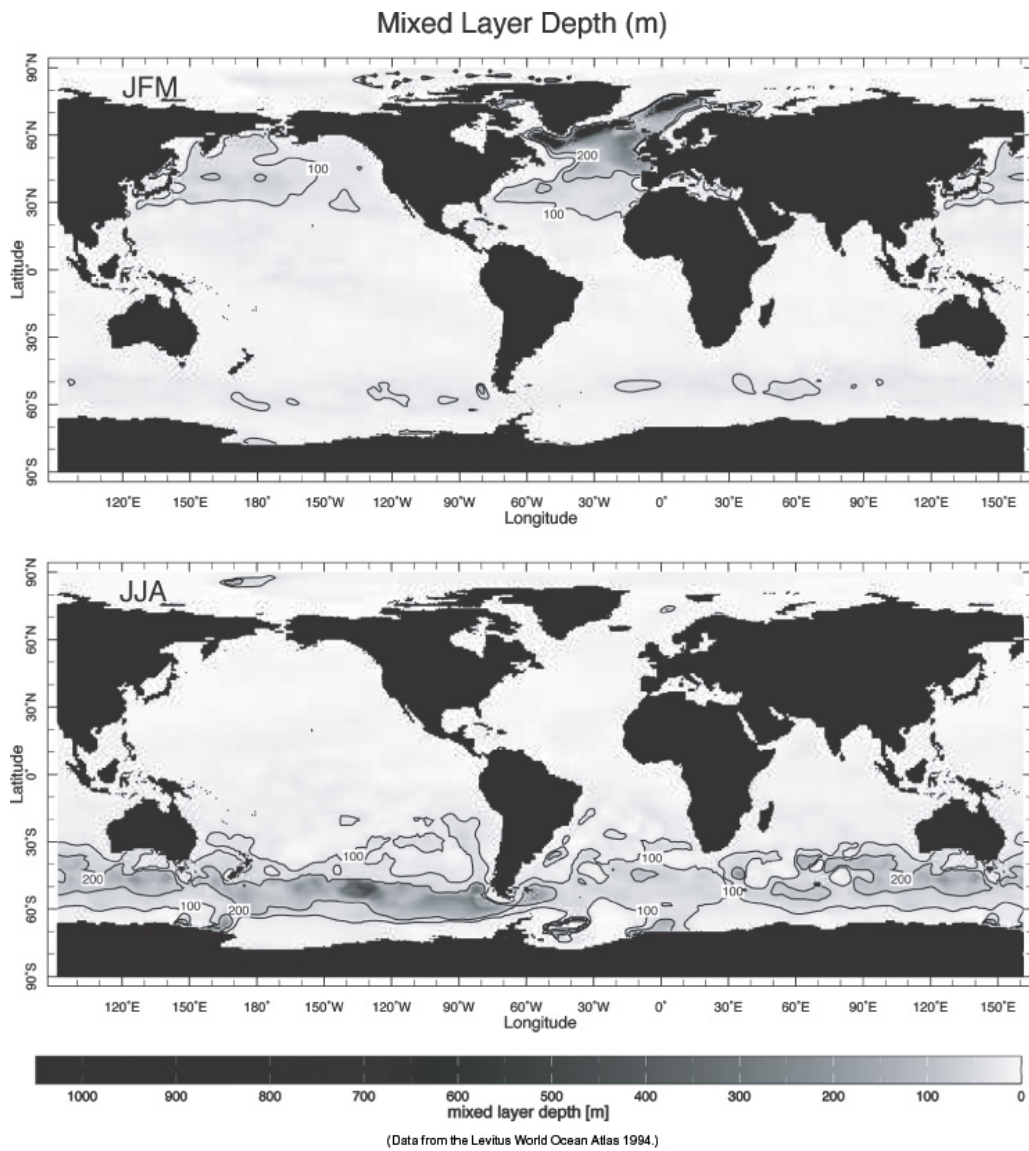
Figure 7.7: Mixed layer in the ocean distribution. Task: Describe the distribution of the seasonal mixed layer depth variations!
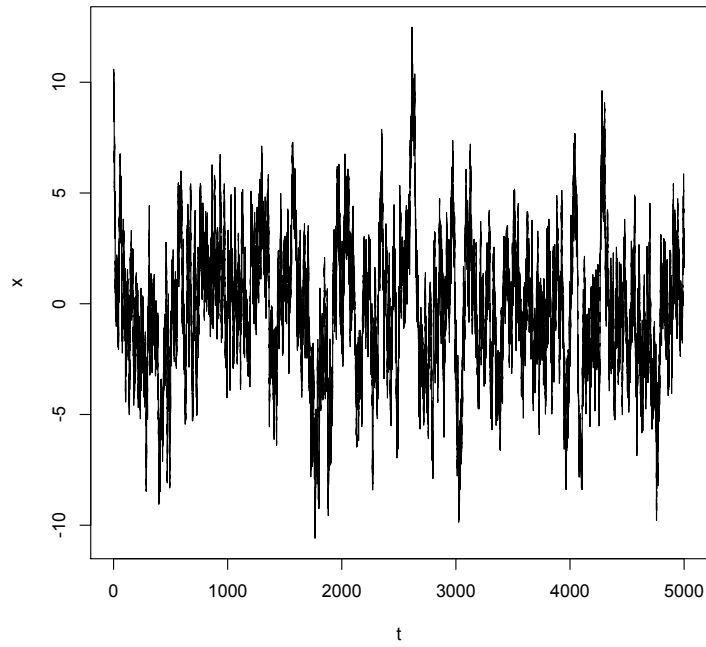
Figure 7.8: Stochastic Climate model, see (13.43).
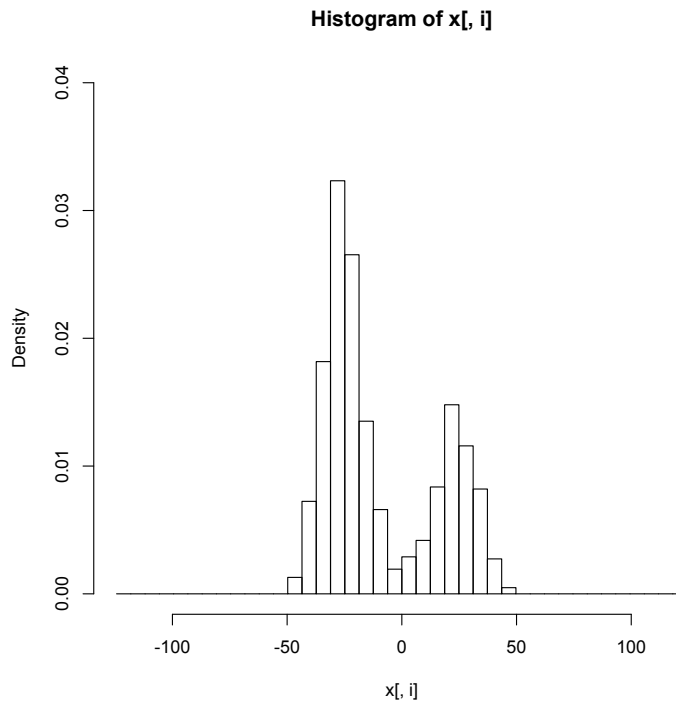
Figure 7.9: Histogram: Stochastic Climate model in potential

An important example is the equation for geometric Brownian motion

$$\mathrm{d}X_t = \mu X_t \,\mathrm{d}t + \sigma X_t \,\mathrm{d}W_t. \tag{7.60}$$

which is the equation for the dynamics of the price of a stock in the Black Scholes options pricing model of financial mathematics. For an arbitrary initial value X_0 the above SDE has the analytic solution (https://en.wikipedia.org/wiki/Geometric_Brownian_motion):

$$X_t = X_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t\right). \tag{7.61}$$

which is shown in Fig. 7.10.
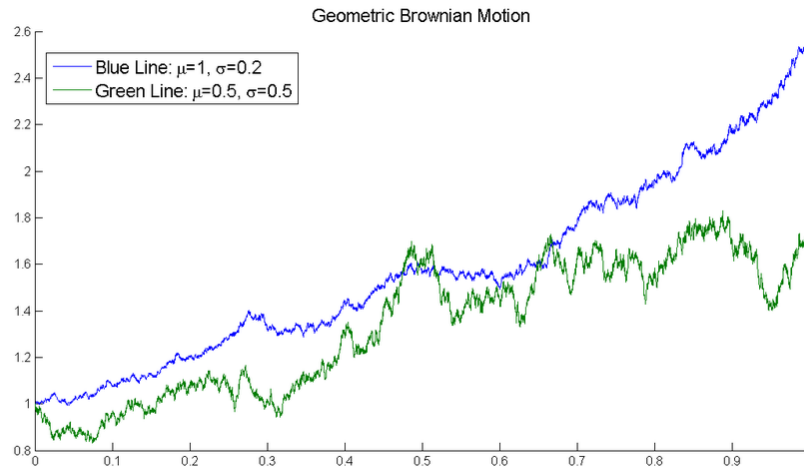
Figure 7.10: Two sample paths of Geometric Brownian motion, with different parameters. The blue line has larger drift, the green line has larger variance.

Exercise 59   –   Stochastic Stock market Model

1. Solve equation (7.61) in a similar way as exercise 58!

2. Why is X_t always positive?

3. Calculate the stationary density from the numerical example analytically using $\int f(y)dy$.

Exercise 60 – **Spectrum of Stochastic Climate Model**

Imagine that the temperature of the ocean mixed layer of depth h is governed by

$$\frac{dT}{dt} = -\lambda T + \frac{Q_{net}}{\gamma_O}, \tag{7.62}$$

where coefficient γ_O is given by the heat capacity $c_p\rho h$, and λ is the typical damping rate of a temperature anomaly. The air-sea fluxes due to weather systems are represented by a white-noise process $Q_{net} = \hat{Q}_\omega e^{i\omega t}$ where $\hat{Q}_\omega$ is the amplitude of the random forcing at frequency ω and $\hat{Q}^*$ is the complex conjugate. Remember that Q_{net} can be described through its distribution and its correlation properties: a Gaussian distribution of zero average $< Q_{net} >= 0$ and δ-correlated in time $< Q_{net}(t)Q_{net}(t+\tau) >= \delta(\tau)$ The brackets indicate an average over realizations of the random force. The spectrum of a process x is defined as

$$S(\omega) := \langle \hat{x}\hat{x}^* \rangle = \widehat{Cov_x(\tau)} = \int_R \exp(i\omega\tau)Cov_x(\tau)d\tau \tag{7.63}$$

1. Calculate $S_Q(\omega)$ and describe why Q_{net} is called a white noise process.

2. Solve Eq. 7.62 for the temperature response $T = \hat{T}_\omega e^{i\omega t}$ and hence show that:

$$\hat{T}_\omega = \frac{\hat{Q}_\omega}{\gamma_O (\lambda + i\omega)} \tag{7.64}$$

3. Show that it has a spectral density $\hat{T}_\omega\hat{T}_\omega^*$ is given by:

$$\hat{T}\hat{T}^* = \frac{\hat{Q}\hat{Q}^*}{\gamma_O^2 (\lambda^2 + \omega^2)} \tag{7.65}$$

and the spectrum

$$S(\omega) =< \hat{T}\hat{T}^* >= \frac{1}{\gamma_O^2 (\lambda^2 + \omega^2)}. \tag{7.66}$$

The brackets $< \cdots >$ denote the ensemble mean. Make a sketch of the spectrum using a

log-log plot and show that fluctuations with a frequency greater than λ are damped.

4. Calculate the spectrum of a regular oszillation with noise. How does the spectrum changes when you rectify the signal?

```
a<-sin(2*pi*(1:5000)/20)+0.5*rnorm(5000)/10
plot(a,type="l",xlim = c(0,2*pi*20),xlab='time (kyrs)',ylab='forcing')
b<-pmax(-0.1,a)  # rectify the signal
plot(b,type="l",col="red",xlim=c(0,2*pi*20),
  xlab='time (kyrs)',ylab='climate')
sa<-spectrum(a,spans=10,
  main="Spectrum of forcing (spans=10)",col="blue")
sb<-spectrum(b,spans=10, col="red")
plot(sa,col="blue",main="Spectrum of the rectified signal (spans=10)")
plot(sb,add = TRUE, col = "red")
```
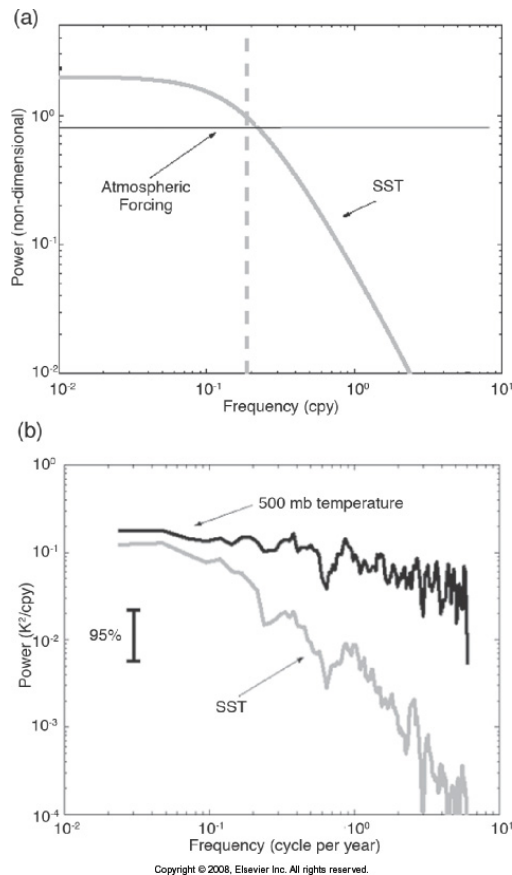
Figure 7.11: Powerspectrum of atmospheric temperature and sea surface temperature. Here $1/\lambda = 300$ days from equation (13.43).

Exercise 61   –   Climate sensitivity and variability in the Stochastic Climate Model

As in exercise 60, imagine that the temperature of the ocean mixed layer of depth h is governed by

$$\frac{dT}{dt} = -\lambda T + Q_{net} + f(t), \tag{7.67}$$

where the air-sea fluxes due to weather systems are represented by a white-noise process with zero average $< Q_{net} >= 0$ and δ-correlated in time $< Q_{net}(t)Q_{net}(t+\tau) >= \delta(\tau)$. The function $f(t)$ is a time dependent deterministic forcing. Assume furthermore that $f(t) = c \cdot u(t)$ with $u(t)$ as unit step or the so-called Heaviside step function and solve (13.48). What is the relationship of the dissipation (through λ) and the fluctuations (through the spectrum $S(\omega)$) ?

Solution

Since $Q(t)$ is a stochastic process, it has to be solved for the moments. Because $< Q_{net} >= 0$, $< T(t) >$ can be solved using the Laplace transform:

$$
\begin{aligned}
< T(t) > &= \mathcal{L}^{-1}\{F(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{< T(0) >}{s+\lambda} + \frac{c}{s} \cdot \frac{1}{s+\lambda}\right\} & (7.68) \\
&= T(0) \cdot \exp(-\lambda t) + \frac{c}{\lambda}(1 - \exp(-\lambda t)) & (7.69)
\end{aligned}
$$

because we have $< T(0) >= T(0)$. As equilibrium response, we have

$$\Delta T = \lim_{t \to \infty} < T(t) >= \frac{c}{\lambda}. \tag{7.70}$$

The fluctuation can be characterized by the spectrum (exercise 60)

$$S(\omega) =< \hat{T}\hat{T}^* >= \frac{1}{\lambda^2 + \omega^2}. \tag{7.71}$$

and therefore, the spectrum and the equilibrium response are closely coupled (fluctuation-dissipation theorem).

For some energy considerations, it is usefull to re-write equation (13.48) as

$$C \frac{dT}{dt} = -\lambda_C T + f_C \,, \tag{7.72}$$

with $C = c_p \rho dz$ as the heat capacity of the ocean. For a depth of 200 m of water distributed over the globe, $C = 4.2 \cdot 10^3 \, W s \, kg^{-1} K^{-1} \times 1000 \, kg \, m^{-3} \times 200 \, m = 8.4 \cdot 10^8 \, W s \, m^{-2} \, K^{-1}$. The temperature evolution is

$$T(t) = T(0) \cdot \exp(-\lambda_c/C \, t) + \frac{f_C}{\lambda_C} (1 - \exp(-\lambda_C/C \, t)) \tag{7.73}$$

The left hand side of (7.72) represents the heat uptake by the ocean, which plays a central role in the transient response of the system to a perturbation (7.73).

Typical changes in f_C are $4 \, W m^{-2}$ for doubling of CO_2, $\lambda_C = 1 - 2 \, W m^{-2} K^{-1}$. The typical time scale for a mixed layer ocean is $C/\lambda_C = 13 - 26$ years . Please note that the climate system is simplified by a slab ocean with homogenous temperature and heat capacity. This is an approximation as the heat capacity should vary in time as the perturbation penetrates to deeper oceanic levels.

The equilibrium temperature change ΔT is

$$\Delta T = \frac{\Delta f_C}{\lambda_C} = \frac{c}{\lambda} \tag{7.74}$$

with values of $\Delta T = 2 - 4$ K. The term $CS = \frac{1}{\lambda_C}$ is called climate sensitivity to a radiative forcing Δf_C:

$$\Delta T = CS \, \cdot \, \Delta f_C \quad . \tag{7.75}$$

In the literature, the concept of climate sensitivity is quite often used as the equilibrium temperature increase for a forcing Δf_C related to doubling of CO_2.

Exercise 62 – **Stochastic differential equation**

Tasks:

1. Simulate the velocity evolution of one particle which is determined by the following stochastic $dv/dt = -b * v + k * dW(t)/dt$

2. What happens if you change the timestep?

3. Simulate the ensemble of multiple particles, plot the time evolution of the v-Distribution

4. Test the ergodic theorem: time average = ensemble average

Solution

```
#brownian motion, one particle

T<-   5000     #integration time in time units
h<-   0.1    #step size in time units
X0<- 10       #inital value

beta<-0.05 #friction term
lambda<-1 #noise term

N<-T/h
t<-(0:(N-1))*h

x<-vector()
x[1]<-X0

for (i in 1:(N-1))
{
  x[i+1]<-x[i]*(1-beta*h)+ rnorm(1)*sqrt(h)
}

plot(t,x,type="l")
# dev.print(postscript, file="random.ps")
#hist(x)
#hist(x,freq=FALSE, col="gray")
```

Solution brownian motion, multiple particle

```
#brownian motion, multiple particle
#forward modelling

#the function dy/dt<-f(y,a,b,c,d)
# double well potential
```

```r
f<-function(y,a,b,c,d)
{
  return(d*y^3+c*y^2+b*y-a)
}




#constants
Ca<-10  # noise

a<-1
b<- 0.8
c<- 0
d<- -0.001

# to do: caculate the stationary density analytically:  int 2/Ca f(y) dy

Nparticle<-1000 #number of particles
T<-  1000   #integration time in time units
h<-  0.5    #step size in time units

N<-T/h
t<-(0:(N-1))*h

x<-matrix(10,Nparticle,N) # Initial condition, all = 0
#x<-matrix(rnorm(Nparticle)*10,Nparticle,N) # Initial condition,

for (i in 1:(N-1))
{
  x[,i+1]<- x[,i]+h*f(x[,i],a,b,c,d)   + Ca*rnorm(Nparticle)*sqrt(h)
}

ama2=max(x,2)
ami=min(x,-2)
ama=max(ama2,-ami)

plot(0,xlim=c(0,T),ylim=c(ami,ama),type="n")  # frame
#plot(0,xlim=c(0,T),ylim=c(-100,100),type="n") # with fixed ylim

# plot the realizations in different colors
for (i in 1:10) lines (t,x[i,],col=i)


#analyse the densities: time evolution
h<-matrix(0,N,40)
#for (i in 1:(N-1)) h[i,]<-hist(x[,i],breaks=c((-20:20)*10),
#             plot=FALSE)$counts

for (i in 1:(N-1)) h[i,]<-hist(x[,i],breaks=c(-20:20)*ama/10,
         freq=FALSE,ylim=c(0,0.04))$counts

#hstat<-matrix(0,N)
```

```
#for (i in N/2:(N-1)) hstat[]<-h[i,]+hstat[]
#hstat[]<-hstat[] *2/Nparticle/N
#plot(t,hstat[],type="l")
 plot(table(hstat[]), type = "h", col = "red")


 op <- par(mfrow = c(3, 2))
plot(h[1,]/Nparticle,type="l")
plot(h[2,]/Nparticle,type="l")
plot(h[4,]/Nparticle,type="l")
plot(h[8,]/Nparticle,type="l")
plot(h[N/2,]/Nparticle,type="l")
plot(h[N-1,]+h[N-2,]/Nparticle/2,type="l")


#filled.contour(t,(-19:20)*10-5,h,color.palette=rainbow,xlab="time",
#               ylab="space")

filled.contour(t,(-19:20)*ama/10-ama/20,h,color.palette=rainbow,
               xlab="time",ylab="space")

#dev.print(postscript, file="/tmp/out.ps")
# system("lpr -Pps3 /tmp/out.ps")
```

Chapter 8

Waves in the climate system

8.1 Shallow water dynamics

One of the most understood dynamics are the tidal equation or shallow water dynamics (e.g. Gill [1982]). The equations are derived from depth-integrating the Navier-Stokes equations, in the case where the horizontal length scale is much greater than the vertical length scale.Under this condition, conservation of mass implies that the vertical velocity of the fluid is small. The variables u and v denote zonal and meridional perturbation flow velocity, and η the height perturbation. The pressure in the vertically homogenous ocean is $p = g\rho(H + \eta)$. The dynamics is as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x} \tag{8.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y} \tag{8.2}$$

where $x = R\lambda, y = R\cos\varphi$ denote eastward distance and distance from the equator, respectively. The equation for the conservation of mass

$$\frac{\partial}{\partial t}\Big(\rho\,(H+\eta)\Big) + \frac{\partial}{\partial x}\Big(u\rho\,(H+\eta)\Big) + \frac{\partial}{\partial y}\Big(v\rho\,(H+\eta)\Big) = 0$$

and since the density is constant it reads

$$\frac{\partial}{\partial t}\eta + u\frac{\partial}{\partial x}\eta + v\frac{\partial}{\partial y}\eta + \frac{\partial}{\partial x}(Hu) + \frac{\partial}{\partial y}(Hv) = 0 \qquad . \qquad (8.3)$$

Lagrangian invariant of the shallow water dynamics*

The dynamical system (8.1,8.2,8.3) has the Lagrangian invariant

$$D_t\Big(\frac{\nabla^2\psi + f}{H+\eta}\Big) = D_t q = 0 \tag{8.4}$$

where $\nabla^2\psi = \partial_x v - \partial_y u$ is the relative vorticity and ψ the streamfunction. The dynamical system (8.1,8.2,8.3) has integral invariants in domains ξ where the fluxes are zero or cancel, e.g. in periodic domains. One such invariant is the energy

$$E = \frac{1}{2}\int\Big((H+\eta)(u^2+v^2) + g\eta^2\Big)d\xi \tag{8.5}$$

and for any scalar functions f(q) of potential vorticity q, another class of integral invariants has the form

$$S = \frac{1}{2}\int (H+\eta)f(q)d\xi \tag{8.6}$$

When function f is the square function $\sim q^2$, this invariant is called potential enstrophy.

Shallow water dynamics: linear model

We now simplify the system to a linear model. Ignoring bulk advection (u and v are small) in (8.1,8.2,8.3), and assuming the wave height is a small proportion of the mean height $(\eta << H)$, we have:

$$\partial_t u \quad = \quad f\,v \quad - \quad g\,\partial_x \eta \tag{8.7}$$

$$\partial_t v \quad = \quad -f\,u \quad - \quad g\,\partial_y \eta \tag{8.8}$$

$$\partial_t \eta \quad = \quad -\partial_x(Hu) \quad - \quad \partial_y(Hv) \quad . \tag{8.9}$$

Skew-Hermetian propertiy of the linear shallow water dynamics*

The dynamical system (8.7,8.8,8.9) can be rewritten in a more compact form (using the non-dimensional values).

$$\partial_t W + \mathbf{L} W = 0 \tag{8.10}$$

With $W = (u, v, \eta)$ and the operaor

$$\mathbf{L} = \begin{pmatrix} 0 & -f & \partial_x \\ f & 0 & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \quad . \tag{8.11}$$

The x and t dependences can be separated in form of zonally propagating waves $\exp(ikx - i\omega t)$. W can therfore be writen as

$$W(x, y, t) = \begin{pmatrix} \hat{u}(y) \\ \hat{v}(y) \\ \hat{\eta}(y) \end{pmatrix} \exp(ikx - i\omega t) = \hat{W}\,\exp(ikx - i\omega t) \tag{8.12}$$

This leads to an eigenvalue problem

$$-i\omega \hat{W}(k, y) + \hat{\mathbf{L}}\hat{W}(k, y) = 0 \tag{8.13}$$

where

$$\hat{\mathbf{L}} = \begin{pmatrix} 0 & -f & ik \\ f & 0 & \partial_y \\ ik & \partial_y & 0 \end{pmatrix} \quad . \tag{8.14}$$

The adjoint of $\hat{\mathbf{L}}$ with respect to the inner product is the operator $\hat{\mathbf{L}}^+$ (transpose and conjugate):

$$\hat{\mathbf{L}}^+ = \begin{pmatrix} 0 & f & -ik \\ -f & 0 & \partial_y \\ -ik & \partial_y & 0 \end{pmatrix} \quad . \tag{8.15}$$

The operator $\hat{\mathbf{L}}$ is skew-Hermetian, as the adjoint of $\hat{\mathbf{L}}$ is $\hat{\mathbf{L}}^+ = -\hat{\mathbf{L}}$ (8.15). For two arbitrary vector functions W_1, W_2 one can define a scalar product:

$$(\hat{\mathbf{L}}W_1, W_2) = \int_{-\infty}^{\infty} \hat{\mathbf{L}}W_1 \cdot W_2^* dy = (W_1, -\hat{\mathbf{L}}W_2) = (W_1, \hat{\mathbf{L}}^+ W_2) \tag{8.16}$$

with the symbol * being the conjugate.

The skew-Hermetian property dictates that the eigenvalues of $\hat{\mathbf{L}}$ are purely imaginary, so that we have a mathematical basis for looking for wave-like solutions. In addition, the eigenfunctions form a complete orthogonal set for the functions W satisfying $(W, W) < \infty$. This is because $\hat{\mathbf{L}}$ is normal:

$$\hat{\mathbf{L}}\hat{\mathbf{L}}^+ = \hat{\mathbf{L}}^+\hat{\mathbf{L}} \quad . \tag{8.17}$$

Furthermore, $\hat{\mathbf{L}}$ in (8.14) belongs to the unitary group $U(3)$, forming a compact connected Lie group and has the special property $\det(\mathbf{L}) = 0$.

These considerations provide the mathematical framework for wave studies. Analytical work is presented in section 8.5 in the case of equatorial wave dynamics. The dynamical system (8.7,8.8,8.9) contains already the zoo of waves. Here, we give a short description. In the exercises, these waves are numerically solved.

Exercise 63   –   Energy conservation

Show that the dynamical system (8.1,8.2,8.3) has integral invariants in domains $\boldsymbol{\xi}$ where the fluxes are zero or cancel, e.g. in periodic domains. One such invariant is the energy

$$E = \frac{1}{2} \int \Big((H + \eta)(u^2 + v^2) + g\eta^2 \Big) d\boldsymbol{\xi} \tag{8.18}$$

and for any scalar functions f(q) of potential vorticity q, another class of integral invariants has the form

$$S = \frac{1}{2} \int (H + \eta) f(q) d\boldsymbol{\xi} \tag{8.19}$$

8.2 Planetary waves on the computer

Rossby (or planetary) waves are giant meanders in high-altitude winds that are a major influence on weather. They are easy to observe as (usually 4-6) large-scale meanders of the jet stream. When these loops become very pronounced, they detach the masses of cold, or warm, air that become cyclones and anticyclones and are responsible for day-to-day weather patterns at mid-latitudes. Their emergence is due to shear in rotating fluids, so that the Coriolis force changes along the sheared coordinate.[1] In planetary atmospheres, they are due to the variation in the Coriolis effect with latitude. The waves were first identified in the Earth's atmosphere by Rossby [1939]. The terms "barotropic" and "baroclinic" Rossby waves are used to distinguish their vertical structure. Barotropic Rossby waves do not vary in the vertical, and have the fastest propagation speeds. The baroclinic wave modes are slower, with speeds of only a few centimetres per second or less (atmosphere).

Oceanic Rossby waves are thought to communicate climatic changes due to variability in forcing, due to both the wind and buoyancy. Both barotropic and baroclinic waves cause variations of the sea surface height, although the length of the waves made them difficult to detect until the advent of satellite altimetry [Chelton and Schlax, 1996]. Baroclinic waves also generate significant displacements of the oceanic thermocline, often of tens of meters. Satellite observations have revealed the stately progression of Rossby waves across all the ocean basins, particularly at low- and mid-latitudes. These waves can take months or even years to cross a basin like the Pacific.

The first order equations of motion into an appropriate wave equation is cumbersome, namely because the two-dimensional geometry of the spherical surface is non-Euclidean (the Coriolis effect depends on the latitude). It can be shown [Müller et al., 1994; Müller and O'Brien, 1995;

---

[1]The dynamics in an inertial reference frame, e.g. with a coordinate system fixed at the Sun, would not have a Coriolis force, but would certainly observe Rossby wave propagation. In the inertial system, the near-equatorial motion is seen to be faster than off the equator. Zero vorticity in the rotating Earth's coordinate system corresponds to a basic flow with non-zero vorticity flow (zonal velocity $U = R\Omega \cos \varphi$) (φ: latitude) in the inertial reference frame [Müller and Maier-Reimer, 2000]. Linearizing the dynamics in the non-rotating system around the basic state U yields exactly Matsuno's wave equations taking the partial substantial derivative with advection U. Therefore, the effect of Earth's rotation is formally equivalent to a shear flow system. The mean flow energy is supplied by the Earth's rotation.

Müller and Maier-Reimer, 2000; Gerkema et al., 2008] that tidal theory differs from the plain

waves because it accounts consistently for the globe's sphericity. If Cartesian coordinates are cho-

sen with $f = \beta y$ then the dynamics reduces to the Matsuno equation as discussed in section

8.
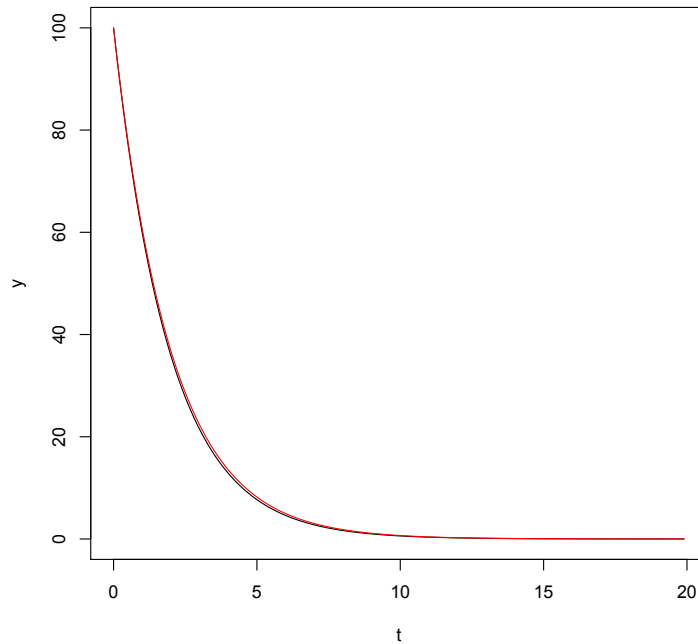
Figure 8.1: Euler forward method in exercise 64.

```
# ODE1.R
#demonstration of Euler forward method in 1st order ODE: dy/dt=A*y

#constants
A<-  -0.5  #growth / decay rate
T<-  20    #integration time in time units
dt<-  .1    #step size in time units
Y0<- 100   #inital value


n<-T/dt         #number of time steps (time / timestep)
t<-(0:(n-1))*dt  #create a vector of discrete timesteps
y<-vector()     #define an empty vector for the state variable y(t)
y[1]<-Y0        #assign initial value


for (i in 1:(n-1))
{
```

```
   y[i+1]<-y[i]+dt*A*y[i]
}

plot(t,y,type="l") #plot the result against time

#additionaly plot the analytical solution in red
lines(t,Y0*exp(A*t),col="red")
```

Exercise 64   –   Euler numerical scheme

Demonstration of the Euler forward scheme on the first order ODE: dy/dt=A*y

- Describe one physical process which can be described with this ODE

- Write the analytic solution for this ODE

- Write the ODE as finite differences

- Open the program ODE1.R and try roughly to understand the code (where is the integration?)

- Run the code and compare the numerical results with the analytic ones.

Exercise 65   –   Numerical solution of 1D Diffusion

- What is the differential equation for the 1D Diffusion

- Open DiffusionEulerForward.R

- Identify / extract the finite difference scheme which is used approximate the 2nd derivative

- Run the program and play with the parameters

```
#Diffusion_EulerForward.R
# 1D diffusion equation, explicit scheme

#Constants
L.X<-50 #width of lattice
L.T<-5  #length of time
dx <- 1    #space step
dt <- 0.1   #time step
D<-1            #Diffusion coefficent


N.x<-L.X/dx + 2   #number of space boxes + 2 boundary boxes
N.t<-L.T/dt       #number of time boxes

u<-matrix(0,N.t,N.x)    #grid
#temporary vector which stores the state of of one timestep:
u.temp<-rep(0,N.x)
```
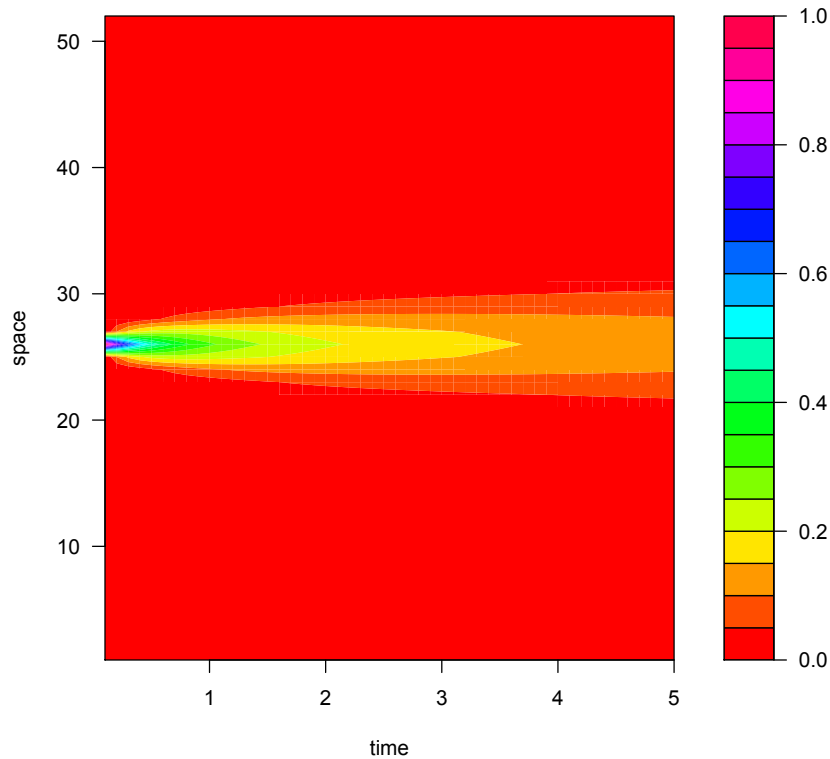
Figure 8.2: Numerical solution of 1D diffusion in exercise 65.

```
#Set the starting and boundary condition, here one value in the middle:
u[1,N.x/2]<-1


for (n in 1:(N.t-1))
{
  for (j in 2:(N.x-1))
  {
    u.temp[j]<-u[n,j]+D*dt/(dx^2)*(u[n,j+1]-2*u[n,j]+u[n,j-1])
  }
  u[n+1,]<-u.temp
}


filled.contour((1:N.t)*dt,(1:N.x)*dx,u,
        color.palette=rainbow,xlab="time",ylab="space")
```

Exercise 66   –   Numerical solution of shallow-water gravity waves

- open shallow1D.R

- Identify the lines of the code in which the momentum equation and in which the continuum
  equation are solved.

- Run the program. Which type of waves do you see?

- Change the constants of water depth H, gravity g, describe your observations!

- Can you roughly estimate the phase speed of the waves?

```
#shallow1D.R
ni<-200 #number of grid cells
nt<-20000 #number of time steps

ia.0<-1:ni
ia.m1<-c(ni,1:(ni-1))
ia.p1<-c(2:ni,1)

g<-0.1 #9.81 m/s^2
dx<-1e5 #gridcell 10km
dt<-100 #timstep 1 second
H<-1e3 #1km depth

u<-rep(0,ni) #speed at each point
h<-rep(0,ni) #pertubation at each point
u.new<-vector()
h.new<-vector()

#h[31:50,1]<--0.5  #one pertubation in the middle
#h[51:70,1]<-0.5   #one pertubation in the middle
h[50:90]<-sin(0:40/2*pi/20)

#1st step euler forward
#momentum equation:
u.new[ia.0]<-u[ia.0]-g*dt/2/dx*(h[ia.p1]-h[ia.m1])
#Continuity eq. horizontal divergence:
h.new[ia.0]<-h[ia.0]-H*dt/2*((u[ia.p1]-u[ia.m1])/dx)

#from step 3 on use Leapfrog
for (n in 2:(nt-1))
{
  u.old<-u
  h.old<-h
  h<-h.new
  u<-u.new

  u.new[ia.0]<-u.old[ia.0]-g*dt/dx*(h[ia.p1]-h[ia.m1])
  h.new[ia.0]<-h.old[ia.0]-H*dt*((u[ia.p1]-u[ia.m1])/dx)
```

```r
  # modulo operator, smoothing every 10 time steps
  if ((n%%10)==0)
  { u.new[ia.0]<-(u.new[ia.0]+u[ia.0])/2
          h.new[ia.0]<-(h.new[ia.0]+h[ia.0])/2
        }

  # modulo operator: plotting
  if ((n%%101)==0)
  {par(ask = TRUE)   # to make a break
  plot(h,type="l",lwd=2,ylim=c(-1,1))
  }
}

############
```
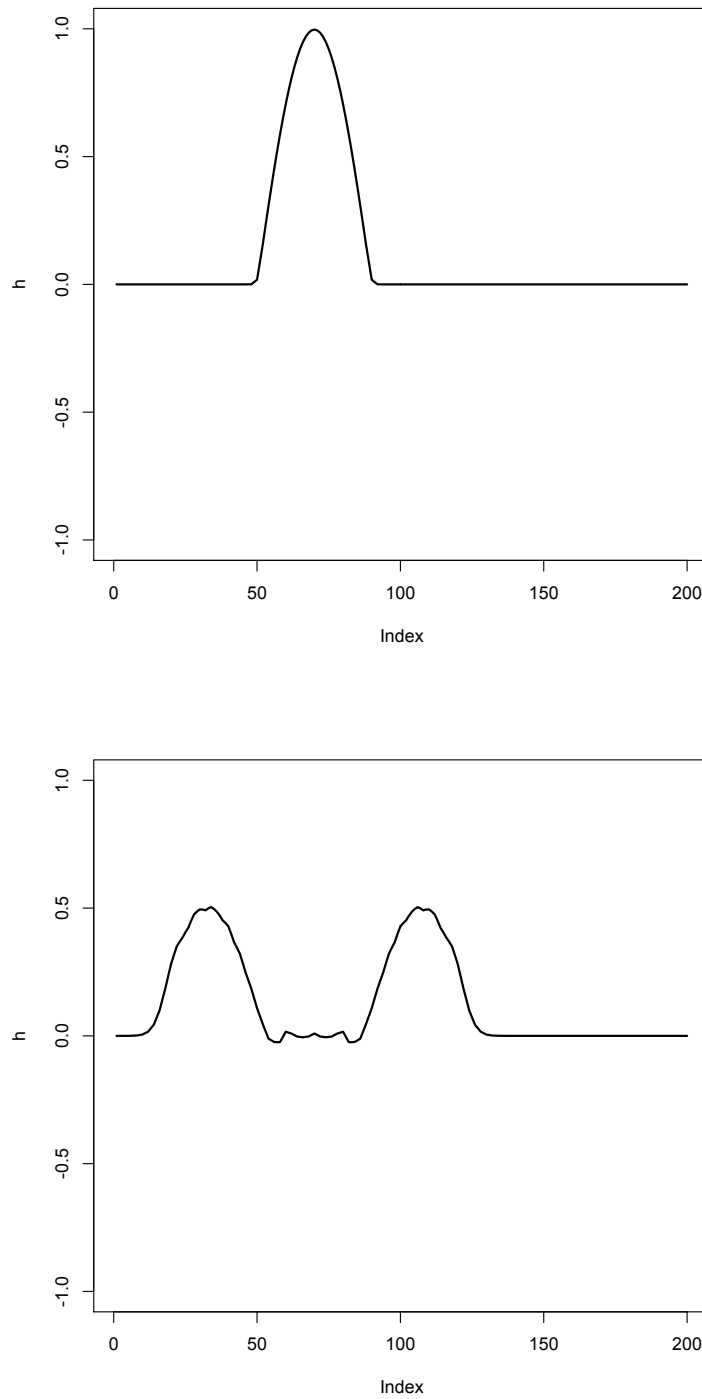
Figure 8.3: Numerical solution of 1D shallow water equation in exercise 66. Upper panel: initial condition. Lower panel: time snap shot.

Exercise 67   –   Numerical solution of the shallow water equation

Study the wave dynamics on a water Earth, for simplicity the metric terms are neglected.

- open shallow2D_rossby.R

- Identify the lines of the code in which the momentum equation and in which the continuum

  equation are solved.

- Run the program: Which type of waves do you see?

- Change the constants of water depth H, gravity g, describe your observations!

- Can you roughly estimate the phase speed of the waves?

```
#
# shallow2D_rossby.R
#
#This is just a definition of a function to plot vectorplots
par.uin<-function()
 {  u <- par("usr")
    p <- par("pin")
    c(p[1]/(u[2] - u[1]), p[2]/(u[4] - u[3]))
 }

quiver<-function(lon,lat,u,v,scale=1,length=0.05,maxspeed=200, ...)
  {  ypos <- lat[col(u)]
     xpos <- lon[row(u)]
     speed <- sqrt(u*u+v*v)
     u <- u*scale/maxspeed
     v <- v*scale/maxspeed
     matplot(xpos,ypos,type="p",cex=0,xlab="lon",ylab="lat", ...)
     arrows(xpos,ypos,xpos+u,ypos+v,length=length*min(par.uin()))
  }

#Program starts here
#Shallow water 2D,cyclic boundary conditions + Coriolis term

nn<- 50
ni<- 2*nn+1    #number of gridcells in one direction
nt<-10000 #number of timesteps

#The physical constants
g<-0.1 #low gravity, 0.1 m/s^2
dx<-1e5 #gridcell 10km
#dx=400e3 # 400 km
dy<-dx/2  # double resolution in meridional direction
dt<-1000 #timstep 1000 second
H<-1e3 #1km depth
Omega<-1e-4

#define three index vectors.. the middle one,
```

```r
#one shifted one cell to the left, and one to the right
#(including the periodic boundary conditions)
ia.0<-1:ni
ia.m1<-c(ni,1:(ni-1))
ia.p1<-c(2:ni,1)
u<-matrix(0,ni,ni) #speed at each point
v<-matrix(0,ni,ni) #speed at each point
h<-matrix(0,ni,ni) #pertubation at each point
f<-matrix(0,ni,ni) #pertubation at each point

lat<-c(-nn:nn)*90/nn
weight<-sin(lat*pi/180)
lon<-c(-nn:nn)*180/nn

f<-rep(weight*2*Omega,each=ni)  # Coriolis parameter
dim(f)<-c(ni,ni)
filled.contour(f)

u.new<-u
h.new<-h
v.new<-v

#Inital condition: One smooth blobs at each side of the "equator"(sin)
idit=nn/5*2
inix=ni-idit-1
iniy=ni-2*idit-1
endx=ni-1
endy=ni-1
endy2=2*idit+1
h[inix:endx,iniy:endy]<-sin(0:20/2*pi/10)*t(sin(0:40/2*pi/20))
h[inix:endx,1:endy2]<-sin(0:20/2*pi/10)*t(sin(0:40/2*pi/20))

#equator to study the Kelvin wave:
ii=idit+1
iy=nn-10
iy2=nn+10
h[1:ii,iy:iy2]<- -sin(0:20/2*pi/10)*t(sin(0:20/2*pi/10))

#Inital condition: One smooth blobs at each side of the "equator"(sin)
#h[60:80,60:80]<-sin(0:20/2*pi/10)*t(sin(0:20/2*pi/10))
#h[30:50,80:100]<-sin(0:20/2*pi/10)*t(sin(0:20/2*pi/10))

#1st step euler forward
u.new[ia.0,ia.0]<-u[ia.0,ia.0]-g*dt/2/dx*(h[ia.p1,ia.0]-h[ia.m1,ia.0])
v.new[ia.0,ia.0]<-v[ia.0,ia.0]-g*dt/2/dy*(h[ia.0,ia.p1]-h[ia.0,ia.m1])
h.new[ia.0,ia.0]<-h[ia.0,ia.0]
       -H*dt/2*((u[ia.p1,ia.0]-u[ia.m1,ia.0])/dx
       + (v[ia.0,ia.p1]-v[ia.0,ia.m1])/dy)

#Divide the screen in two parts
# par(mfcol=c(1,2))
#par(mfcol=c(2,1))
```

```r
#Leapfrog from the third step on
for (n in 3:(nt-1))
{
  u.old<-u
  v.old<-v
  h.old<-h
  h<-h.new
  u<-u.new
  v<-v.new
  u.new[ia.0,ia.0]<-u.old[ia.0,ia.0]
     -g*dt/dx*(h[ia.p1,ia.0]-h[ia.m1,ia.0])+dt*f*v
  v.new[ia.0,ia.0]<-v.old[ia.0,ia.0]
     -g*dt/dy*(h[ia.0,ia.p1]-h[ia.0,ia.m1])-dt*f*u
  h.new[ia.0,ia.0]<-h.old[ia.0,ia.0]
     -H*dt*((u[ia.p1,ia.0]-u[ia.m1,ia.0])/dx
     + (v[ia.0,ia.p1]-v[ia.0,ia.m1])/dy)

#plot every 50th image
  if ((n %% 50) == 0) {
      #quiver(lon,lat,u,v,scale=200,maxspeed=1.5,length=3)
      #image(lon,lat,h,zlim=c(-1,1),col=rainbow(200)) # color coated
      persp(h/3, theta = 0, phi = 40, scale = FALSE, ltheta = -120,
     shade = 0.6, border = NA, box = FALSE,zlim=c(-0.3,0.3))
     }
  }
```
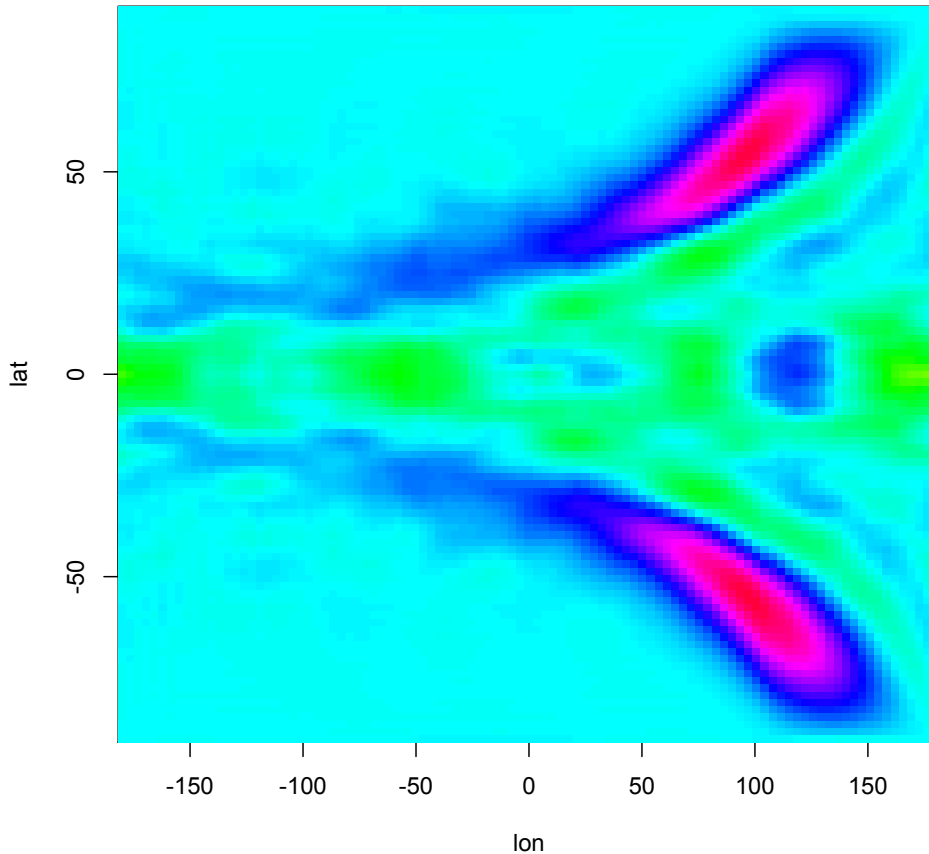
Figure 8.4: Global Rossby and Kelvin wave signatures in the exercise 67.

8.3 Plain waves

The analysis of the spherical version of the tidal problem is complicated because the Coriolis effect depends on the latitude and in general we do not have plain waves with sinus and cosinus base functions.[2] However, because of its simplicity, we will study the plain wave theory here. In this approach, the Coriolis parameters f and β are taken as **fixed parameters** in the equations. Then, the wave equations can be reduced to plain waves with eigenfunctions $\sim \exp(ikx + ily - i\omega t)$.

---

[2]This approximation may be questioned because the trapped character of the Rossby waves is no included, which is however, observed and simulated (Fig. 8.4). This shows a general problem in perturbation theory: The concept of manipulations in the differential equations (e.g., by neglecting terms) is not entirely free from ambiguities, and may lead to a undesirable transition in the solutions of the system. The type of solutions shall be of the form of the observed (macroscopic) functions and a proper framework of approximations is required (section 3.5).

8.3.1   Inertial Waves

From the equations (8.7,8.8,8.9), we drop the term $\partial_x \eta$, $\partial_y \eta$, and $f = f_0 = const.$ (no pressure gradients and constant f). Then, air or water mass moving with speed v subject only to the Coriolis force travels in a circular trajectory called an 'inertial circle'. Since the force is directed at right angles to the motion of the particle, it will move with a constant speed, and perform a complete circle with frequency f. The magnitude of the Coriolis force also determines the radius of this circle:

$$R = v/f .$$ (8.20)

On the Earth, a typical mid-latitude value for f is $10^{-4} s^{-1}$; hence for a typical atmospheric speed of 10 m/s the radius is 100 km, with a period of about 14 hours. In the ocean, where a typical speed is closer to 10 cm/s, the radius of an inertial circle is 1 km. These inertial circles are clockwise in the Northern Hemisphere (where trajectories are bent to the right) and anti-clockwise in the Southern Hemisphere. If the rotating system is a parabolic turntable, then f is constant and the trajectories are exact circles. On a rotating planet, f varies with latitude and the paths of particles do not form exact circles. Since the parameter f varies as the sine of the latitude, the radius of the oscillations associated with a given speed are smallest at the poles and increase toward the equator (Fig. 8.5).

$$\frac{\partial u}{\partial t} - f_0 v \;=\; 0$$ (8.21)

$$\frac{\partial v}{\partial t} + f_0 u \;=\; 0$$ (8.22)

yields

$$\frac{\partial^2 u}{\partial t^2} = -f_0^2 u \quad .$$ (8.23)

Figure 8.5: Schematic representation of inertial circles of air masses in the absence of other forces, calculated for a wind speed of approximately 50 to 70 m/s. Note that the rotation is exactly opposite of that normally experienced with air masses in weather systems around depressions.

The solution is

$$u(t) = u(0)\sin(f_0 t) \tag{8.24}$$

$$v(t) = u(0)\cos(f_0 t) \tag{8.25}$$

which is known as intertial movement and can be observed in drifting buoys (upper panel Fig. 8.6). The water parcels move around a circle of radius of $u(0)/f_0$ in a clockwise direction (anticyclonically) with a period $2\pi/f_0$.

Exercise 68   –   Inertial waves

- Derive the solution of (8.21, 8.22). Since the force is directed at right angles to the motion of the particle, it will move with a constant speed, and perform a complete circle with frequency f. Show that the magnitude of the Coriolis force determines a radius R of this circle. Hint: A typical mid-latitude value for f is $10^{-4}s^{-1}$; a typical atmospheric speed of $10m/s$, in the ocean a typical speed is closer to $10cm/s$.
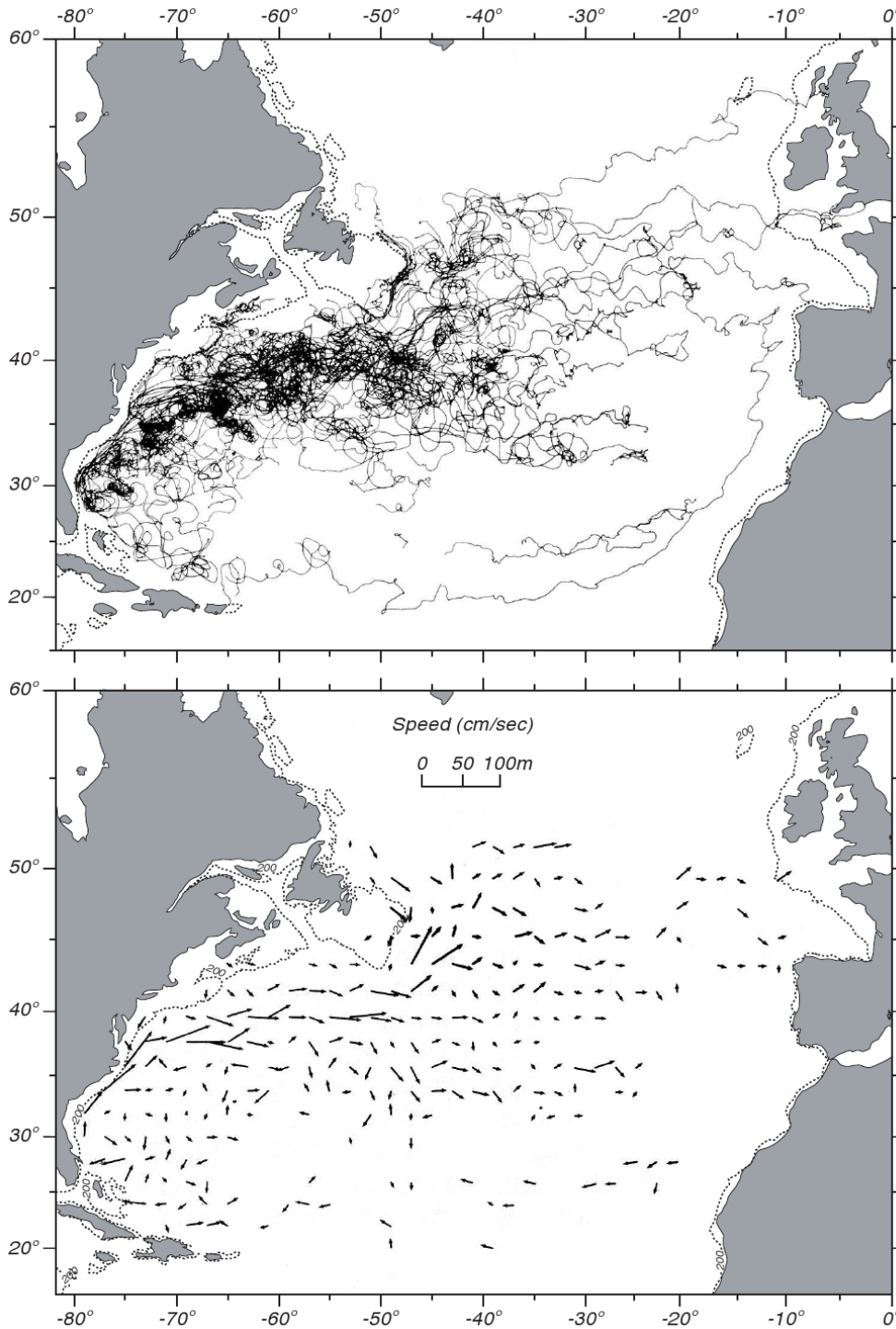
Figure 8.6: **Top:** Tracks of 110 drifting buoys deployed in the western north Atlantic. **Bottom:** Mean velocity of currents in $2° \times 2°$ boxes calculated from tracks above. Boxes with fewer than 40 observations were omitted. Length of arrow is proportional to speed. Maximum values are near $0.6ms$ in the Gulf Stream near $37°$N $71°$W. After Richardson (1981).

- Provide the solution for the coordinates $x(t), y(t)$.
- Show that the dynamics in the inertial coordinate system reduces to

$$u_{in}(t) = 0 \qquad\qquad (8.26)$$

$$v_{in}(t) = u(0) \cos \Omega t \qquad\qquad (8.27)$$

The trajectory in the inertial frame is a straight line. The length of the line is twice the diameter of the inertial circle and the frequency of the oscillation is one-half that observed in the rotating frame.

8.3.2   Gravity Waves

Shallow-water gravity waves are defined through their dynamics without the effect of the Earth's rotation, i.e. $f = 0$:

$$\frac{\partial^2 \eta}{\partial t^2} = gH \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \eta \tag{8.28}$$

With the ansatz

$$\eta = \exp(ikx + ily - i\omega t) \tag{8.29}$$

ω is given by

$$\omega(k, l) = \pm \sqrt{gH \left(k^2 + l^2 \right)}, \tag{8.30}$$

where k and l are the zonal and meridional wavenumbers. Since there is no preferred direction in the (x,y) coordinate, we simply drop the y-dependence and introduce the phase speed

$$c = \omega / k = \pm \sqrt{gH} \quad . \tag{8.31}$$

In the limit $\beta \to 0$ i.e. $f = f_0 = const.$, the dynamics consists of gravity waves with

$$\omega^2 = f_0^2 + (ck)^2 \tag{8.32}$$

Output from a shallow water equation model of water in a bathtub. The water experiences five splashes which generate surface gravity waves that propagate away from the splash locations and reflect off the bathtub walls. (https://en.wikipedia.org/wiki/Shallow_water_equations#/media/File:Shallow_water_waves.gif)

Exercise 69   –   **Baroclinic shallow-water gravity waves**

In case we have a layed ocean, we consider the so-called baroclinic dynamics with the modified gravity $g' = \frac{\rho_1 - \rho_2}{\rho_1}$ using the densities $\rho_{1,2}$. Task: Derive the baroclinic dynamics using the shallow water equations for 2 different layers and subtract the equations from each other!

Exercise 70   –   **Shallow-water waves**

We consider tidal equation on the $\beta-$plane. This fluid dynamical system is described as

$$\partial_t u = f\,v - g\,\partial_x \eta \tag{8.33}$$

$$\partial_t v = -f\,u - g\,\partial_y \eta \tag{8.34}$$

$$\partial_t \eta = -\partial_x(Hu) - \partial_y(Hv) \quad . \tag{8.35}$$

The variables u and v denote zonal and meridional perturbation flow velocity, and η the height perturbation.

- Derive the dispersion relationships $\omega(k)$ for the cases:

    a) In the limit $\beta \to 0$, i.e. $f \to f_0$

    b) $c \to \infty$.

    c) For infinite Rossby radius $a = \sqrt{c/(2\beta)}$.

    d) When filtering out gravity waves by eliminating the time derivative in (8.35), (u, v) in (8.33, 8.34) can be taken as plane waves proportional to $\exp(ikx + ily)$, where l denotes the meridional wave number. Derive the dispersion relationships $\omega(k)$ for the so-called non-divergent Rossby waves.

- Provide typical values of $\omega(k)$ for M,N=1,2,3 and the atmosphere and ocean.

8.3.3   Extratropical Rossby Waves

From the equations (8.7,8.8,8.9), we drop the term $\partial_t \eta$ and introduce the stream function ψ through

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \tag{8.36}$$

such that (8.9) is fulfilled. Taking $\frac{\partial}{\partial y}$ of (8.7) and subtract $\frac{\partial}{\partial x}$ of (8.8) elimintates the η term as in section 4.4:

$$\frac{\partial}{\partial t} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi = -\beta \frac{\partial \psi}{\partial x} \tag{8.37}$$

With the ansatz

$$\psi = \exp(ikx + ily - i\omega t) \tag{8.38}$$

and assumption that β is just a parameter, ω is given by

$$\omega(k, l) = -\frac{\beta k}{k^2 + l^2}, \tag{8.39}$$

where k and l are the zonal and meridional wavenumbers. Again, β is used as a parameter (also called Rossby parameter) and is not expressed in terms of y:

$$\beta = \frac{df}{dy} = \frac{1}{R}\frac{d}{d\varphi}(2\Omega \sin \varphi) = \frac{2\Omega \cos \varphi}{R} \tag{8.40}$$

where φ is the latitude, Ω is the angular speed of the Earth's rotation, and R is the mean radius of the Earth. The wave speed $c = \omega/k = -\beta \left(k^2 + l^2 \right)^{-1}$. The feature that the phase speed is faster at low latitudes can be also seen in Fig. 8.4 using the full dynamics.

More information about Rossby waves: https://youtu.be/6UCiRIc0nK0

Rossby waves and extreme weather: https://youtu.be/MzW5Isbv2A0

Exercise 71   –   Rossby waves

Consider the vorticity equation

$$\frac{D}{Dt}[(\zeta + f)/h] = 0 \tag{8.41}$$

with h =const., u and v are the velocity components.

1. Assume a mean flow with constant zonal velocity U

$$u = U = const > 0 \tag{8.42}$$

and a varying north-south component

$$v = v(x, t) \tag{8.43}$$

which gives the total motion a wave-like form around a reference latitude where the wave is trapped. Derive the associated vorticity equation and linearize the vorticity equation by dropping all non-linear terms!

2. With the ansatz

$$v(x, t) = A \cos[(kx - \omega t)] \tag{8.44}$$

determine the disperion relation $\omega(k)$, group velocity $\frac{\partial \omega}{\partial k}$, and the phase velocity $c = \omega/k$.

3. Derive the wavelength $L = 2\pi/k$ of the stationary wave given by $c = 0$.

4. A typical wavelength is 6000 km, a typical U is 15 m/s. Does the wave propagate from east to west or opposite?

8.4   Kelvin waves

8.4.1   Coastal Kelvin waves

A Kelvin wave is a wave in the ocean or atmosphere that balances the Coriolis force against a topographic boundary such as a coastline. If one assumes that the Coriolis coefficient f is constant along the right boundary conditions, $u = 0$, and the zonal wind speed is set equal to zero, then the equations become the following:

$$\frac{\partial \eta}{\partial t} = -H \frac{\partial v}{\partial y} \tag{8.45}$$

$$\frac{\partial v}{\partial t} = -g \frac{\partial \eta}{\partial y} \tag{8.46}$$

and therefore

$$\frac{\partial^2 \eta}{\partial t^2} = gH \; \frac{\partial^2}{\partial y^2} \eta \tag{8.47}$$

The solution to these equations yields the following phase speed: $c^2 = gH$ and $\omega = \pm cl$, which is the same speed as for shallow-water gravity waves without the effect of Earth's rotation. We see that η and v have also an x-dependence

$$\eta(x, y, t) \;=\; \tilde{\eta}(x) \exp(ily - i\omega t) \tag{8.48}$$

$$v(x, y, t) \;=\; \tilde{v}(x) \exp(ily - i\omega t) \quad . \tag{8.49}$$

Using (8.46), we obtain

$$-i\omega \, \tilde{v}(x) = -gil \, \tilde{\eta}(x) \quad \text{and therefore } \tilde{v}(x) = \frac{g}{\omega} l \, \tilde{\eta}(x) = \pm \frac{g}{c} \, \tilde{\eta}(x) \tag{8.50}$$

$$\text{From the u-momentum equation} \quad \frac{\partial \eta}{\partial x} = \frac{f}{g} v \tag{8.51}$$

$$\text{we obtain therefore} \quad \frac{\partial \tilde{\eta}}{\partial x} = \pm \frac{f}{c} \tilde{\eta} \tag{8.52}$$

where only the minus sign provides a useful solution (not blowing up). The solution has an exponential decay of $\tilde{\eta}(x) = \exp(-x/L_r)$ on the scale of the Rossby radius $L_r = c/f$. The wave has a trapped character along the boundary. It is important to note that for an observer traveling with the wave, the coastal boundary (maximum amplitude) is always to the right in the Northern Hemisphere and to the left in the Southern Hemisphere, i.e. these waves move equatorward/southward on a western boundary and poleward/northward on an eastern boundary. Thus, the waves move cyclonically around an ocean basin.

On the black board: A Coastal Kelvin Wave moving northward along the coast is deflected to the right, but the coast prevents the wave from turning right and instead causes water to pile up on the coast. The pile of water creates a pressure gradient directed offshore and a geostrophic current directed northward.

On the northern hemisphere: The Kelvin wave always travels with the wall on its right side (anti-clockwise). The wave amplitude decreases exponentially away from the wall. The wave is trapped along the wall by rotation. Rotation does not affect the particle motion and wave propagation; only traps the wave to the coastline.

8.4.2   Equatorial Kelvin waves

Analogous we have Equatorial Kelvin waves: assume $v = 0$, then the equations become the following:

$$\frac{\partial \eta}{\partial t} = -H \frac{\partial u}{\partial y} \tag{8.53}$$

$$\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial y} \tag{8.54}$$

and therefore again

$$\frac{\partial^2 \eta}{\partial t^2} = gH \quad \frac{\partial^2}{\partial y^2}\eta \tag{8.55}$$

The solution to these equations yields the phase speed: $c^2 = gH$ and $\omega = ck$, which is the same speed as for shallow-water gravity waves without the effect of Earth's rotation. We see that η and u have also an x-dependence

$$\eta(x, y, t) = \tilde{\eta}(y)\exp(ikx - i\omega t) \tag{8.56}$$

$$u(x, y, t) = \tilde{u}(y)\exp(ikx - i\omega t) \quad . \tag{8.57}$$

Using (8.54), we obtain

$$-i\omega\,\tilde{u}(y) = -gik\,\tilde{\eta}(y) \quad \text{and therefore } \tilde{u}(y) = \frac{g}{\omega}k\,\tilde{\eta}(y) = \frac{g}{c}\,\tilde{\eta}(y) \tag{8.58}$$

$$\text{From the v-momentum equation} \quad \frac{\partial \eta}{\partial y} = \frac{\beta y}{g}u \tag{8.59}$$

$$\text{we obtain therefore} \quad \frac{\partial \tilde{\eta}}{\partial y} = -\frac{\beta y}{c}\tilde{\eta}\,. \tag{8.60}$$

The solution is $\tilde{\eta}(x) = \exp(-\beta y^2/c)$ with the scale of the Rossby radius $L_r = \sqrt{c/\beta}$. The wave has a trapped character along the equator.

A feature of a Kelvin wave is that it is non-dispersive, i.e., the phase speed of the wave crests is equal to the group speed of the wave energy for all frequencies. This means that it retains its shape in the alongshore direction over time. In the ocean these waves propagate along coastal boundaries (and hence become trapped in the vicinity of the coast itself) on a scale of about 30 km.

Equatorial Kelvin waves are a special type of Kelvin wave that balances the Coriolis Force in

the northern hemisphere against its southern hemisphere counterpart. This wave always propagates eastward and only exists on the equator. Equatorial Kelvin Waves propagating in the thermocline have wave speeds slow enough to give a Rossby Radius of Deformation that is on the order of 250 km and thus they appear to be trapped close to the equator.

8.5 Equatorial waves: Theory of Matsuno*

We consider the equations (8.7,8.8,8.9) on the equatorial β−plane. In the equatorial region, the fluid dynamical system is described as

$$\partial_t u = \beta y\, v - g\, \partial_x \eta \tag{8.61}$$

$$\partial_t v = -\beta y\, u - g\, \partial_y \eta \tag{8.62}$$

$$\partial_t \eta = -\partial_x (H u) - \partial_y (H v) \quad . \tag{8.63}$$

We non-dimensionalize the system through the parameters listed in Table 8.1. In the non-dimensional form (and dropping the stars in Table 8.1), the system reads then

$$\partial_t u = y\, v - \partial_x \eta \tag{8.64}$$

$$\partial_t v = -y\, u - \partial_y \eta \tag{8.65}$$

$$\partial_t \eta = -\partial_x u - \partial_y v \quad . \tag{8.66}$$

Introducing the new variables

$$q = \eta + u \tag{8.67}$$

$$r = \eta - u \tag{8.68}$$

Parameter	description	formula	typical values
H	equivalent height		
g	reduced gravity		
R	Earth's radius		$6.371 \cdot 10^6 \, m$
Ω	Earth's rotation rate	$2\pi \, day^{-1}$	$7.272 \cdot 10^{-5} \, s^{-1}$
M	zonal wave number		$0, \pm 1, \pm 2, \ldots$
N	mode number		$0, 1, 2, \ldots$
φ	latitude		
f	Coriolis parameter	$2\Omega \sin \varphi$	
β	β-term	$2\Omega/R$	$2.0 \cdot 10^{-11} \, m^{-1} s^{-1}$
c	barotropic phase speed of pure gravity wave	$\sqrt{g\bar{H}}$	atmosphere: $2000 \, ms^{-1}$ ocean: $200 \, ms^{-1}$
c	baroclinic phase speed of pure gravity wave	$\sqrt{g\bar{H}}$	atmosphere: $20 - 80 \, ms^{-1}$ ocean: $2 \, ms^{-1}$
a	meridional wave guide (Rossby radius)	$\sqrt{\frac{c}{2\beta}}$	atmosphere: $6.6 \cdot 10^5 \, m$ ocean: $6.6 \cdot 10^4 \, m$
t^*	time	$t\sqrt{2\beta c}$	
x^*	eastward distance	x/a	
y^*	meridional distance	y/a	
ω^*	frequency	$\omega/\sqrt{2\beta c}$	
k^*	zonal wave vector	Ma/R	

Table 8.1: List of parameters for the Matsuno equations.

yields

$$\partial_t q \;\; = \;\; -\partial_x q \;\; - \;\; \left[\partial_y - \frac{y}{2}\right] v \tag{8.69}$$

$$\partial_t v \;\; = \;\; -\frac{1}{2}\left[\partial_y + \frac{y}{2}\right] q \;\; - \;\; \frac{1}{2}\left[\partial_y - \frac{y}{2}\right] r \tag{8.70}$$

$$\partial_t r \;\; = \;\; +\partial_x r \;\; - \;\; \left[\partial_y + \frac{y}{2}\right] v \;\;\;\; . \tag{8.71}$$

The dynamics (8.69,8.70,8.71) describe wave propagation in an inhomogenous and anisotropic medium. Zonal wave dynamics differ significantly from those in meridional direction. The primary source of inhomogenity is due to the Coriolis force. The x and t dependences can be separated in form of zonally propagating waves $\exp(ikx - i\omega t)$. The eigenfunctions in $y-$direction are related to parabolic cylinder functions $\left(\text{or Hermite polynomials with weight } \exp(-y^2)\right)$. The Hermite polynomials are defined as

$$He_n(y) = (-1)^n e^{y^2/2} \frac{d^n}{dy^n} e^{-y^2/2} \tag{8.72}$$

The first Hermite polynomials are

$$He_0(y) \;\; = \;\; 1 \tag{8.73}$$

$$He_1(y) \;\; = \;\; x \tag{8.74}$$

$$He_2(y) \;\; = \;\; y^2 - 1 \tag{8.75}$$

$$He_3(y) \;\; = \;\; y^3 - 3y \tag{8.76}$$

$$He_4(y) \;\; = \;\; y^4 - 6y^2 + 3 \tag{8.77}$$

To display the Hermite polynomials:

```
# for a read.me: http://cran.r-project.org/doc/manuals/R-intro.pdf
# gemerate a list of normalized Hermite polynomials of orders 0 to 10
install.packages("orthopolynom")
normalized.p.list <- hermite.he.polynomials(5, normalized=TRUE ) # a list
print(normalized.p.list)  # display the polynomials
```
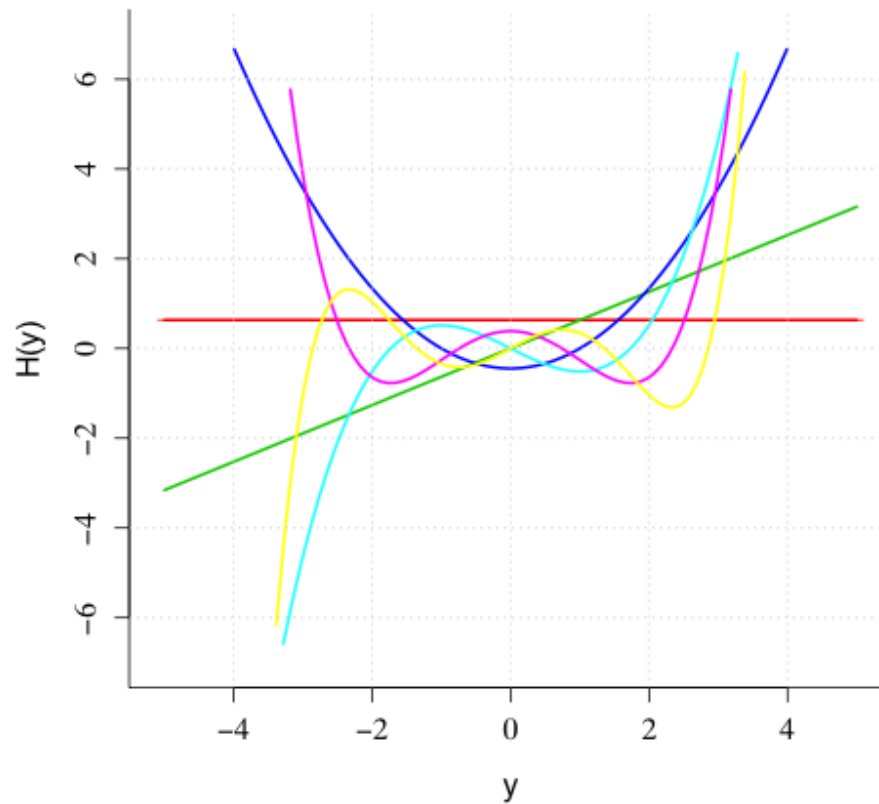
Figure 8.7: Hermite polynomials to degree 5.

```
H=normalized.p.list
ticks=seq(from=-5, to=5, by=10)
plot(H[[1]], xlim=c(-5,5),ylim=c(-7,7),col="red",ylab="H(y)",xlab="y")
for (i in 1:6) {lines(H[[i]],xlim=c(-5,5),ylim=c(-7,7),col =i+1, lwd = 2)}
grid(nx=NULL,col="lightgray",lty ="dotted",lwd=2,equilogs=TRUE)
dev.copy(png,'Hermite.png')
dev.off()
```

They satisfy following recursion relationship:

$$\left[\partial_y + \frac{y}{2}\right] D_N = N \, D_{N-1} \quad ; \qquad \left[\partial_y - \frac{y}{2}\right] \, D_N = -D_{N+1} \qquad . \quad (8.78)$$

The operators $[\partial_y \pm \frac{y}{2}]$ annihilate or excite one quantum of mode index number N and are called lowering and raising ladder operators in quantum mechanics. A basic feature of $D_N \sim \exp(-y^2)$ is that significant wave amplitudes are trapped in a wave guide centered at the latitude φ_0, similar to the equator-centered Yoshida guide [Gill, 1982].

The Fourier modes $\hat{\xi}_N(t) := (\hat{q}_{N-1}, \hat{v}_N, \hat{r}_{N+1})$ correspond to order $N > 0$ and wave vector k. The prognostic equations for the Fourier modes are first order in time

$$\frac{d}{dt} \, \hat{\xi}_N \;\; = \;\; A_N(k) \; \hat{\xi}_N \qquad . \qquad (8.79)$$

and are described by 3×3 matrices $A_N(k)$

$$A_N(k) \;=\; \begin{pmatrix} -ik & 1 & 0 \\ -N/2 & 0 & 1/2 \\ 0 & -(N+1) & ik \end{pmatrix} \qquad . \qquad (8.80)$$

Matrix $A_N(k)$ describes the dynamics of one Rossby and two gravity waves with eigenfrequencies ω (eigenvalue of $A = i\omega$) satisfying

$$\omega^3 - \omega \left(\frac{2N+1}{2} + k^2\right) - \frac{k}{2} \;=\; 0 \quad . \qquad (8.81)$$

The sum of the eigenfrequencies in (8.81) is zero due to $trace\,(A_N) = 0$ and

$$\sum_{l=1}^3 \omega_l \;=\; \lim_{T \to \infty} \frac{1}{T} \int_0^T dt \;\; trace\,(A_N) \;=\; 0 \quad . \qquad (8.82)$$

For $N = 0$, the system matrix A_0 is specified to be

$$
A_0(k) = \begin{pmatrix} \boxed{ik} & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & -1 & ik \end{pmatrix} \quad .
$$
(8.83)

The different signs of the $\cdot_{11}$−elements in (8.80) and (8.83) originate from the requirement that the corresponding eigenmode $q_{N=0}$ in (8.83) is integrable [Gill, 1982]. This mode with $v = r = 0$ is called equatorial Kelvin wave which propagates eastward without dispersion:

$$
\omega = k \quad .
$$
(8.84)

The dynamics of the Kelvin wave is decoupled from the Yanai wave dynamics described by the second and third eigenvectors of matrix (8.83). The Yanai wave, also known as mixed planetary-gravity wave in the literature [Gill, 1982], has a quadratic relation

$$
\omega^2 - k\omega - 1/2 = 0 \quad .
$$
(8.85)

Dispersion curves for the Rossby/gravity (8.81), Kelvin (8.84), and Yanai (8.85) waves are shown in Fig. 8.8 as a function on zonal wave vector $k = Ma/R$ and mode number N. The figure depicts eastward propagating Kelvin and westward propagating Rossby modes. Gravity waves can propagate east- and westward. The Yanai wave behaves as a gravity wave for $k \geq 0$ and as a Rossby wave for $k < 0$. Note that (8.81) is invariant under $\omega \rightarrow -\omega, k \rightarrow -k$, which is a consequence of (8.82). Dispersion diagrams like Fig. 8.8 can be found in standard text books of geophysical fluid dynamics showing the upper [Gill, 1982] or right [Holton, 2004] part of Fig. 8.8, respectively.

The equatorial zone essentially acts as a waveguide, causing disturbances to be trapped in the vicinity of the equator. For the first baroclinic mode in the ocean, a typical phase speed would be about $2.8 m/s$, causing an equatorial Kelvin wave to take 2 months to cross the Pacific Ocean
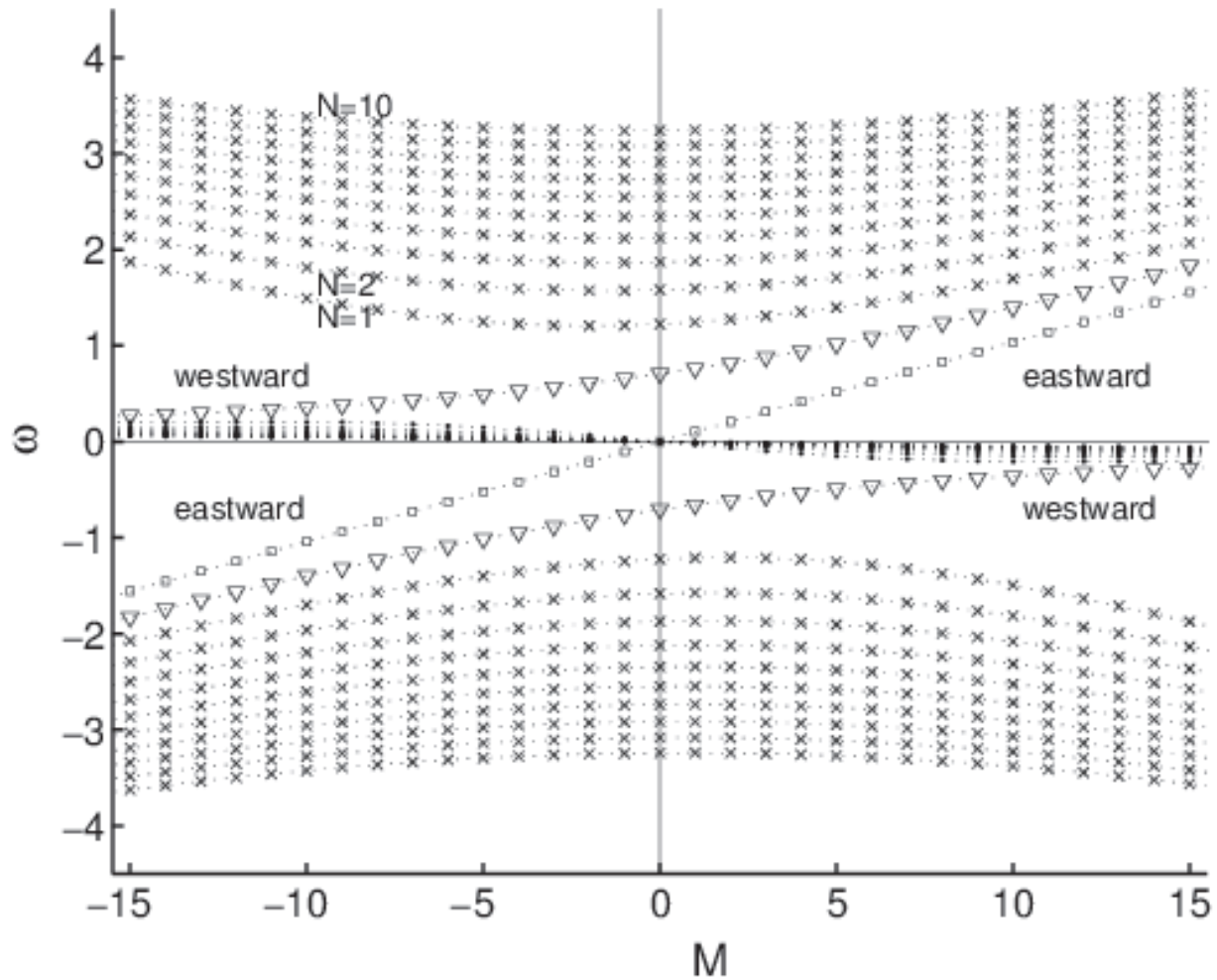
Figure 8.8: Dispersion relation for equatorial waves. Curves show dependence of frequency on zonal wave number M for mode numbers $N \leq 10$. Kelvin waves propagate eastward, Rossby waves (●) westward, while gravity waves (×) exist for both directions. Yanai waves (▽) behave Rossby-like for $M < 0$ and gravity-like for $M \geq 0$.

between New Guinea and South America; for higher ocean and atmospheric modes, the phase speeds are comparable to fluid flow speeds. Why is the Kelvin wave trapped? When the motion at the equator is to the east, any deviation toward the north is brought back toward the equator because the Coriolis force acts to the right of the direction of motion in the Northern Hemisphere, and any deviation to the south is brought back toward the equator because the Coriolis force acts to the left of the direction of motion in the Southern Hemisphere. Note that for motion toward

the west, the Coriolis force would not restore a northward or southward deviation back toward the equator; thus, equatorial Kelvin waves are only possible for eastward motion (as noted above). Both atmospheric and oceanic equatorial Kelvin waves play an important role in the dynamics of El Niño-Southern Oscillation, by transmitting changes in conditions in the Western Pacific to the Eastern Pacific [Gill, 1982]. This can be also studied in exercise 67.

It is instructive to look for approximations in tidal theory. One can simplify the solution of the Matsuno theory, or simplify the equations (8.7,8.8,8.9) which will be done in section 8.3.

When filtering out gravity waves by eliminating the time derivative in (8.63), (u, v) in (8.61, 8.62) is equivalent to $c \to \infty$. The evolution equation reduces to

$$\partial_t(\partial_y u - \partial_x v) = \beta u \tag{8.86}$$

with plane waves proportional to $\exp(ikx + ily)$. Then, non-divergent Rossby waves with $\omega = -\beta k/(k^2 + l^2)$ are retained only. The trapped character of the waves vanishes with infinite Rossby radius $a = \sqrt{c/(2\beta)}$, a measure of the wave guide geography.

Exercise 72   –   Shallow-water dynamics: eigenfunctions

- Show that the eigenfunctions in $y-$direction, which are are related to parabolic cylinder functions $\left(\text{or Hermite polynomials with weight } \exp(-y^2)\right)$, satisfy following recursion relationship:

$$\left[\partial_y + \frac{y}{2}\right] D_N = N\, D_{N-1} \quad ; \qquad \left[\partial_y - \frac{y}{2}\right] D_N = -D_{N+1} \qquad . (8.87)$$

The operators $[\partial_y \pm \frac{y}{2}]$ annihilate or excite one quantum of mode index number N and are called lowering and raising ladder operators in quantum mechanics.

$$D_N(y) = \frac{1}{\pi} \int_0^\pi \sin(N\Theta - y\sin\Theta))$$

- Show that the functions are orthogonal, i.e.

$$\int_{-\infty}^{\infty} dy\, D_N(y) D_M(y) = \delta_{NM}\, N!\sqrt{2\pi}$$

- The dynamics in an inertial reference frame, e.g. with a coordinate system fixed at the Sun, would not have a Coriolis force (and thus $f = 0$), but would certainly observe Rossby wave propagation. How can this be reconciled?

  (Hint: In the inertial system, the near-equatorial motion is seen to be faster than off the equator. Zero vorticity in the rotating Earth's coordinate system corresponds to a basic flow $U = R\Omega \cos\varphi$ with non-zero vorticity flow.)

Exercise 73   –   Shallow-water dynamics: A different approach

We may seek travelling-wave solutions of the form

$$\left\{u, v, \eta\right\} = \left\{\hat{u}(y), \hat{v}(y), \hat{\eta}(y)\right\} e^{i(kx-\omega t)} \tag{8.88}$$

Please check the following arguments.

- Subsitituting this exponential form into the three equations (8.61,8.62,8.63), and eliminating u, and η leaves us with an eigenvalue equation for $\hat{v}(y)$

$$-\frac{\partial^2 \hat{v}}{\partial y^2} + \left(\frac{\beta^2}{c^2}\right)\hat{v} = \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega}\right)\hat{v}. \tag{8.89}$$

- Recognizing this as the Schrödinger equation of a quantum harmonic oscillator of frequency β/c, we know that we must have

$$\left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega}\right) = \frac{\beta}{c}(2n+1), \quad n \geq 0 \tag{8.90}$$

for the solutions to tend to zero away from the equator. For each integer n, this last equation provides a dispersion relation linking the wavenumber k to the angular frequency ω.

- In the special special case $n = 0$ the dispersion equation reduces to

$$(\omega + ck)(\omega^2 - ck\omega - c\beta) = 0, \tag{8.91}$$

   but the root $\omega = -ck$ has to be discarded because we had to divide by this factor in eliminating u, η.

- The remaining pair of roots correspond to the Yanai or mixed Rossby-gravity mode whose group velocity is always to the east and interpolates between two types of $n > 0$ modes: the higher frequency Poincare gravity waves whose group velocity can be to the east or to the west, and the low-frequency equatorial Rossby waves whose dispersion relation can be approximated as

$$\omega = \frac{-\beta k}{k^2 + \beta(2n + 1)/c} \quad . \tag{8.92}$$

8.6   Spheroidal Eigenfunctions of the Tidal Equation*

Laplace's tidal equations, governing the small amplitude dynamics of a shallow fluid on a rotating sphere, are the fundamental linear problems of large-scale geophysical fluid dynamics. Originally formulated by Laplace, its general solution and, in particular, its full dispersion relation are still not known. The current understanding of the problem rests essentially on the equatorial β-plane approximation [Matsuno, 1966] and extensive numerical studies [Longuet-Higgins, 1968]. In this framework the system has been found to exhibit Rossby waves, Yanai waves, and gravity waves, including the Kelvin wave. In spite of its elegance and fundamental significance for the terrestrial climate problem, the β-plane concept is not entirely free from ambiguities. Although the Matsuno wave equation appears to be a well-posed problem, its dispersion relation admits an unphysical, westward propagating "Kelvin" mode. This has to be ruled out a posteriori [Matsuno, 1966]. Furthermore, the β-plane concept does not yield a physically meaningful nonrotating limit. And

finally, the Matsuno equation is invariant under meridional translations. Yanai waves should thus be observed at all latitudes and every latitude defines the center of a waveguide. Such properties are not physically realizable. On the other hand, the analysis of the spherical version of the tidal problem is complicated. First, the manipulation of the first order equations of motion into an appropriate wave equation is cumbersome, namely because the two-dimensional (2D) geometry of the spherical surface is non-Euclidean. Second, as shown below, the basic wave operator in the various forms of the tidal problem is the spheroidal wave equation. For this equation, infinity is an irregular singular point precluding the establishment of recurrence relations similar to those for functions of the hypergeometric type [Flammer, 1957]. The transformation of differential operations in physical space into some simpler algebra in wave number space is thus impossible. Geometrical difficulties are largely simplified by a systematic application of tensor analysis in 2D Riemann space. Here, index notation will be used with indices m, n,... running from 1 to 2 and a semicolon denoting covariant differentiation. For details of the notation and the form of the geometrical tensors in spherical, geophysical coordinates see [Townsend et al., 1992]. The linearized equations of motion of a shallow fluid on a rotating sphere are shown in [Townsend et al., 1992] to assume the form

$$\partial_t r + j^n{}_{;n} = 0,$$

$$\partial_t j_n + \epsilon_{mn} f j^m + c^2 \partial_n r = 0,$$

where $j_n = R v_n$ is the effective momentum density, R the constant equilibrium mass per unit area, and $f = 2\Omega \sin \varphi$ using the latitude dependent Coriolis parameter. For the potential vorticity z, defined by

$$Rz = \epsilon^{mn} v_{n;m} - fr/R,$$

the linearized equations of motion imply the relation

$$R\partial_t z + f^n v_n = 0,$$

where f^n is the contravariant gradient of the Coriolis parameter. Furthermore, the gradient of the divergence of a vector on the spherical surface is given by

$$j^a{}_{;an} = g^{ab}(j_{n;ab} - \epsilon_{na}\epsilon^{rs}j_{r;sb} - G_{ambn}j^m),$$

where g^{ab} is the metric and G_{ambn} the Riemannian [Townsend et al., 1992]. Using this identity and the potential vorticity equation it is fairly straightforward to derive the system

$$R^2\left[(\partial_t^2 + f^2 - c^2\Delta)\partial_t + c^2\epsilon^{ab}f_a\partial_b\right]\partial_t z = -c^2(\Delta f)\partial_t^2 r, \tag{8.93a}$$

$$\left[(\partial_t^2 + f^2 - c^2\Delta)\partial_t + c^2\epsilon^{ab}f_b\partial_a\right]r = -2R^2 f\partial_t z, \tag{8.93b}$$

from the equations of motion, where Δ denotes the 2D Laplacian in spherical coordinates. If Cartesian coordinates (x, y) are chosen with $f = \beta y$, i.e., in particular $\Delta f = 0$, the first of these equations reduces to the Matsuno equation. To obtain (8.93a) in the spherical case, eliminate r from the equations of motion

$$(\partial_t^2 + f^2 - c^2\underline{\Delta})\partial_t j_n + c^2\epsilon_{na}\left[\partial^a(f_b j^b) + f^a j^b{}_{;b}\right] = 0, \tag{8.94}$$

with

$$\underline{\Delta}j_n = g^{ab}j_{n;ab} - a^{-2}j_n,$$

where a denotes the Earth's radius. Using

$$f^n\underline{\Delta}j_n = \Delta(f^n j_n) - (\Delta f)j^n{}_{;n}$$

.

scalar multiplication of (8.94) with the contravariant gradient of the Coriolis parameter yields (8.93a). Equation (8.94) as well as the coupled nature of the system (8.93) demonstrate that the

tidal equation is inherently a 20 vector wave equation. In order to evaluate the eigenfunctions of this system, the dependent variables are assumed to be proportional to

$$e^{-i(\omega t - M\lambda)}F(y),$$

where λ is longitude, $y = \sin \varphi$, and M the zonal wave number. Substituting this into (8.93), the system becomes

$$(P - m)V = -2\alpha y D, \tag{8.95a}$$

$$(P + m)D = 2\alpha y V, \tag{8.95b}$$

Here $D = r/R = iv^n;_n / \omega$ and $V = aRz/c = -i\alpha v_\varphi \times \cos \varphi/cv$, while

$$P = a^2\Delta - \alpha^2 y^2 + \nu^2$$

is the prolate spheroidal wave operator with Lamb parameter $u = 2a\Omega/c$, $\nu = a\omega/c$, and $m = \alpha M/\nu$. In the form (8.95) the tidal problem emerges as a system of coupled spheroidal wave equations. In special cases, exact analytical solutions can be readily obtained without considering the complete fourth order system. Elimination of j_1 between the continuity equation and the one-component of the momentum budget yields

$$\alpha D = -(\mu + y^2)^{-1}\left[(1 - y^2)\partial_y - my\right]V,$$

with $\mu = (M^2 - \nu^2)/\nu^2$, while elimination of ji between the one-component and the two-component of the momentum budget leads to

$$\alpha V = -(n^2 - y^2)^{-1}\left[(1 - y^2)\partial_y + my\right]D,$$

with $n = \nu/\alpha$. Inserting these expressions into the right hand side (rhs) of (8.95) results in

$$(P + m)V = 2(\mu + y^2)^{-1} \left[y(1 - y^2)\partial_y + m\mu \right] V, \qquad (8.96a)$$

$$(P - m)D = -2(n^2 - y^2)^{-1} \left[y(1 - y^2)\partial_y + mn^2 \right] D. \qquad (8.96b)$$

Equation (8.96a) is the spherical generalization of the Matsuno equation, while (8.96b) is the form of the tidal equation studied by Longuet-Higgins. In view of the general solution, (8.96) may not be the most convenient form as it exhibits far less symmetry than the coupled system (8.95) suggests. It nevertheless lends itself readily to the evaluation of two special cases. For standing waves (M = 0) the spherical Matsuno equation (8.96a) is exactly solved by

$$V = (1 - y^2)^{1/2}(AS_L^1 + BS_L^{-1}),$$

with constants A and B and prolate spheroidal wave functions $S_L^{\pm 1}(y; \alpha^2)$ of order ± 1 and degree $L \geq 1$. The corwhere responding divergence becomes

$$D \sim \partial_y(1 - y^2)^{1/2}(AS_L^1 + BS_L^{-1}). \qquad (8.97)$$

A closed expression for the eigenvalues of the spheroidal wave equation does not exist and approximations depend strongly on the value of the Lamb parameter. On Earth, the value of the Lamb parameter ranges from $\alpha \approx 1$ for the atmospheric Lamb wave, over $\alpha \approx 5$ for barotropic gravity waves in the ocean to $\alpha \approx 300$ for the first baroclinic mode in the ocean. For $\alpha^2 \ll 1$, spheroidal wave functions are approximated by expansions in terms of associated Legendre polynominals, and the dispersion relation for standing waves becomes to $O(\alpha^0)$ [Flammer, 1957] [Abramowitz and Stegun, 1965].

$$\nu^2 = \Lambda^2 + \alpha^2(2\Lambda^2 - 3)/(4\Lambda^2 - 3) \tag{8.98a}$$

with $\Lambda^2 = L(L + 1)$. In the nonrotating limit, this reduces to the familiar $\nu^2 = L(L + 1)$. For $\alpha^2 \gg 1$, the prolate spheroidal wave function S_L^K is appropriately approximated by parabolic cylinder functions of nonnegative, integer order, and the corresponding eigenvalue to $O(\alpha^0)$ becomes [Flammer, 1957] [Abramowitz and Stegun, 1965]

$$\epsilon(L, K, \alpha) = \alpha q + K^2 - 1 + p,$$

with $q = 2N + 1$, $p = (3 - q^2)/8$, and $N = L - |K|$. In the present case of standing waves ($\epsilon = \nu^2$, $K^2 = 1$) this yields the dispersion relation to $O(\alpha^0)$

$$\nu^2 = \alpha(2N_2 + 1) + p, \tag{8.98b}$$

where the mode number $N_2 = L - 1$ measures the number of zeros of j_2 in the open interval $y \in (-1, 1)$. With the inclusion of higher orders in α and $1/\alpha$, respectively, the dispersion relation (8.98) and the corresponding expansion of the spheroidal wave function permit the construction of Fig. 1 and Fig. 7 of [Longuet-Higgins, 1968] to an arbitrary degree of accuracy from [Abramowitz and Stegun, 1965] . The asymptotic expansion of (8.96a) provides an estimate of the domain of validity of the β-plane approximation in physical and wave number space. For low frequencies $\nu^2 \ll M^2$, the first order Taylor expansion of the denominator on the rhs of (8.96a) is justified and yields

$$(\alpha^2 \Delta - \delta y^2 + \nu^2 - m)V = 0,$$

with $\delta = \alpha^2 - 2m/\mu$, . For large a in the low-frequency domain under consideration $\alpha^2 \gg 2m/\mu$, so that $\delta \approx \alpha^2$. Assuming $V = (1 - y^2)^{|M|/2}F$, where the absolute value of M ensures

the regularity of V at the poles for negative M, and transforming to $x = y\sqrt{\alpha}$

$$\left[(\alpha - x^2)\partial_x^2 - 2(|M| + 1)x\partial_x - \alpha x^2 + \alpha(2N_2 + 1)\right] F \approx 0$$

yields for $\alpha \gg x^2$, i.e., in the vicinity of the equator $y^2 \ll 1$

$$(\partial_x^2 - x^2 + 2N_2 + 1 - x + 2N2 + 1)F \approx O$$

the familiar β-plane version of the Matsuno equation with dispersion relation

$$\nu^3 - \left[\alpha(2N_2 + 1) + M^2\right]\nu - \alpha M = 0,$$

with $N_2 = L - |M|$. The β-plane approximation thus emerges asymptotically from the full tidal equation as an equatorial ($y^2 \ll 1$), baroclinic ($\alpha^2 \gg 1$) low-frequency approximation. While the original frequency restriction $\nu^2 \ll M^2$ admits Rossby waves only, the additional restriction to low latitudes also allows for low-frequency gravity waves in this approximation. The equatorial nature of these asymptotics is obviously not compatible with a "midlatitude β-plane," while the large u condition rules out a nonrotating limit. The β-plane approximation essentially neglects the coupling of Eqs. (8.95a) and (8.95b). In the full tidal vector equation it is this coupling that excludes the "wrong Kelvin wave" a priori. A second special, but exact solution can be obtained for inertial waves. At the inertial frequency $\nu = \alpha$, (8.96b) has the exact solution

$$D = (1 - y^2)^{1/2}(AS_L^{M-1} + BS_L^{1-M}), \tag{8.99}$$

with $L \geq 0$ and $1 - L \leq M \leq L + 1$. The dispertion relation in this case

$$\left[\epsilon = \alpha^2 - M, \ K^2 = (M - 1)^2\right]$$

can again be read from [Abramowitz and Stegun, 1965] . For small α one finds to $O(\alpha^2)$

$$\alpha^2 = \Lambda^2 + M + \alpha^2 \left[2\Lambda^2 - 2M(M-2) - 3\right] / (4\Lambda^2 - 3) \qquad (8.100a)$$

admitting only $L = 0$ and $M = 0$ in the nonrotationg case, while the dispersion relation for large α becomes $O(\alpha^2)$

$$\alpha^2 = \alpha(2N_0 + 1) + M^2 - M + p \qquad (8.100b)$$

where $N_0 = L - |M - 1|$ measures the number of seros of the mass perturbation r in the open interval $y \in (-1, 1)$. At $M = 0$, the relation (8.100) coincides obviously with (8.98) at $\nu = \alpha$, where $N_0(M = 0, \nu = \alpha = N_2(M = 0, \nu = \alpha)$, while in general the number of zeros of the mass perturbation r will differ from the number of zeros of j_2. The divergence (8.99) at M=0 agrees with (8.97) at $\nu = \alpha$, since for spheroidal wave functions [Flammer, 1957] :

$$M_{E/W} = \frac{1}{2} \pm \sqrt{\alpha^2 - \alpha q + p + \frac{1}{4}},$$

with

$$|M_W| = M_E - 1 < M_E. \qquad (8.101)$$

The same inequality is found from (8.100a). In contrast to the β-plane approximation, modes in this frequency domain are labeled by Np, and the phase speed of eastward propagating gravity waves is smaller than the westward speed at the same frequency and mode number. At high frequencies $\nu^2 \gg \alpha^2$, the approximation of (8.96b) by

$$(\alpha^2 \Delta - \delta y^2 + \nu^2 + m)D \approx 0,$$

with $\delta = \alpha^2 - 2M/n^3$, is uncritical. For large α and positive M not too large $\delta \approx \alpha^2$, and the

expansion of this equation similar to the Matsuno equation yields the

approximate dispersion relation

$$\nu^3 - \left[\alpha(2N_0 + 1) + M^2\right]\nu + \alpha M = 0,$$

with $N_0 = L - |M|$. The eastward propagating "Rossby" solutions of this dispersion have to be discarded, as they do not satisfy the defining inequality. On the other hand, the gravity solutions including the Kelvin wave satisfy this inequality. As a consequence of the positive sign of the last term, westward phase speeds are larger than eastward speeds. This is in agreement with the exact solution (8.100). The wave number space of the tidal equation is thus separated at the inertial frequency $\nu = \alpha$ into a lowfrequency domain, where modes are governed by N_2 and a high-frequency domain with gravity modes controlled by N_0. This is the mode number that survives the transition to the nonrotating case. For $\Omega = 0$, D is proportional to the associated Legendre polynominal P_L^M with $N_0 = L - |M|$ zeros in the open interval $y \in (-1, 1)$, and the dispersion relation $\nu^2 = L(L + 1)$ can alternatively be written

$$\nu^2(M) = N_0(N_0 + 1) + (2N_0 + 1)|M| + M^2 = \nu^2(-M)$$

. As indicated by the inequality (8.101), rotation leads primarily to the loss of this synunetry for gravity modes. At lower frequencies, additional Rossby modes emerge in rotating systems, which can no longer be accormnodated by Np. Hence, the mode number Np governs that domain of wave number space, where rotation merely modifies modes already existing in the nonrotating case. The transition between mode numbers is only possible due to the vector character of the tidal equa-tion. On Earth, this highfrequency domain of wave number space is occupied by Lamb waves and barotropic gravity waves, which are of minor significance on larger scales. The atmosphere of Venus, on the other hand, is characterized by much lower values of the Lamb parameter, and gravity waves with $\nu \geq \alpha$ gain greater relevance for large-scale aspects of the circulation. The concept of covariant differentiation renders the derivation of wave equations from the equations

of motion fairly straightforward. The prolate spheroidal wave operator assumes a central role in these vector wave equations. For the first time since Laplace's formulation of the problem, exact analytical solutions are presented in the special cases of standing waves and inertial waves. These solutions confirm corresponding numerical calculations, while the asymptotics of the spheroidal wave equation for equatorial, baroclinic, low-frequency waves yield indeed the β-plane approximation. Furthermore, the results demonstrate a fundamental separation of the wave number space of the tidal equation at the inertial frequency. At higher frequencies, the mode number N_0 controls the dispersion of gravity waves, which experience rotation effects merely as a loss of syrrunetry with respect to $M = 0$. For a corotating observer westward

propagating gravity waves at these frequencies are faster and longer than their eastward counterparts. At low frequencies the dispersion of additional Rossby modes is incorporated by transition to the mode number N_2. This transition is primarily possible due to the vector character of the tidal problem. It can be expected that in the framework of the complete analytical theory of the tidal equation the angular momentum of eigensolutions, measured by the degree L, will be essential for this separation of the wave number space.

Part III

Third part: Climate

Chapter 9

Paleoclimate

For present climate state we are able to directly measure all involved quantities. From measurements we can draw conclusions about physical, chemical and biological relationships between the variables. Our understanding about the involved processes is far from complete, but nevertheless we derive equations that describe and predict the observed phenomena.

One the other hand, the knowledge about the climate of the *past* relies on the use of indirect methods. Palaeotemperatures and -precipitation have to be derived from materials that had been exposed to the environment which in turn had an impact on this material. Then it had to be conserved and locked from any further alteration. Finally, the material has to be examined and the alterations have to be understood and interpreted as anomalies of temperature or precipitation. Therefore, assumptions have to be made that are independent of the type of the climate archive, the materials involved and the methods used to extract the information from the archive. For example, we assume that correlations between measured quantities and climate variables do not change in time. These measured quantities are called proxies if they convey indirect information about past climates.

Past climates left their imprints in a large variety of archives. Depending on the type of the archive, knowledge about the processes during the formation of the archive is useful and often essential for the interpretation. Some of the most influential climate archives are:

- Ice cores. They reflect local amount of precipitation plus information of the palaeo-atmosphere from trapped air bubbles that remain in the ice during the long tranformation process from snow to ice.

- Corals. They represent surface water isotopic content after modifications during shell formation. Depending on the coral type, the temporal resolution can range from several years to months.

- Speleothems. They consist of the material carried by drip water from the soil above the cave, in which the speleothem has grown.

- Tree rings. They convey information about the growth rate of the tree at annual resolution. Climate variables can be inferred from the knowledge of today's relationship between growth rate and climatic conditions.

- Sediments. They consist of material that precipitated through the water column and accumulated with or without chemical alterations, depending on the substance and environmental conditions, such as oxygen availability. For example, organic material is preserved under anoxic conditions at times of low mixing rates in the respective body of water.

Paleoclimate reconstructions, in particular from ice cores (e.g., Taylor et al., 1999) have also shown that climate can change over extremely short periods of time such as a few years to decades. Over the last century, humans have altered the Earth's surface and the composition of its atmosphere to the extent that these factors measurably affect current climate conditions (Hansen et al., 1988). There is concern that during one human generation we will gradually change climate conditions or even trigger a rapid and much more dramatic shift. We might be 'poking an angry beast' (Broecker, 1999).

The objective of the script is to examine fundamental concepts used to understand climate dynamics and the driving mechanisms for paleo and future climate change. Models of the Earth system, observational, and proxy data are used for the study of climate dynamics on decadal and

millennial time-scales (Fig. 9.1). Special focus is placed on the physical background and methodologies.

For present climate state we are able to directly measure all involved quantities. From measurements we can draw conclusions about physical, chemical and biological relationships between the variables. Our understanding about the involved processes is far from complete, but nevertheless we derive equations that describe and predict the observed phenomena.

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Past climates left their imprints in a large variety of archives. Depending on the type of the archive, knowledge about the processes during the formation of the archive is useful and often essential for the interpretation. Some of the most influential climate archives are:

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- Speleothems. They consist of the material carried by drip water from the soil above the cave,

Figure 9.1: Schematic view on the climate system. Global climate is a result of the complex interactions between the atmosphere, cryosphere (ice), hydrosphere (oceans), lithosphere (land), and biosphere (life), fueled by the non-uniform spatial distribution of incoming solar radiation (e.g. Peixoto and Oort, 1992). We know from climate reconstructions using recorders such as ice cores, ocean and lake sediment cores, tree rings, corals, cave deposits, and ground water that the Earth's climate has seen major changes over its history. An analysis of the temperature variations patched together from all these data reveals that climate change occurs in cycles with characteristic periods, for example 200 million, 100 thousand, or 4-7 years. For some of these cycles, particular mechanisms can be identified, for example forcing by changes in the Earth's orbital parameters or internal oscillations of the coupled ocean-atmosphere system. However, major uncertainties remain in our understanding of the interplay of the components of the climate system.

in which the speleothem has grown.

- Tree rings. They convey information about the growth rate of the tree at annual resolution. Climate variables can be inferred from the knowledge of today's relationship between growth rate and climatic conditions.

- Sediments. They consist of material that precipitated through the water column and accumulated with or without chemical alterations, depending on the substance and environmental conditions, such as oxygen availability. For example, organic material is preserved under anoxic conditions at times of low mixing rates in the respective body of water.

9.1 Temperature reconstructions

Alkenones are highly resistant organic compounds (ketones) produced by phytoplankton of the class Prymnesiophyceae. The exact function of the alkenones remains under debate. Coccolithophoroids, for instance Emiliania huxleyi, respond to changes in water temperature by altering the production of long-chain unsaturated alkenones in the structure of their cell. At higher temperatures, more of the di-unsaturated molecules are produced than tri-unsaturated (Prahl and Wakeham, 1987). The molecules are resistant to diagenesis, and can be recovered from sediments up to 110 million years old. The ambient water temperature in which the organisms dwelt can be estimated from ratio of their unsaturated alkenones (C37-C39) that are preserved in marine sediments. The Unsaturation Index of di- versus tri-unsaturated C37 alkenone is calculated according to the following relationship (Brassell et al., 1986):

$$UK'37 = C37:2/(C37:2 + C37:3) \qquad (9.1)$$

The Unsaturation Index can then be used to estimate the water temperature according to the following experimental relationship [Prahl and Wakeham]:

$$SST[^\circ C] = (UK'37 - 0.039)/0.034 \tag{9.2}$$

Magnesium (Mg) can be incorporated into the tests of bottom-dwelling foramanifera: higher temperatures make it easier to incorporate. Therefore a high Mg/Ca ratio implies a high temperature, although ecological factors may confound the signal. TEX86 is a paleothermometer based on the composition of membrane lipids of the marine picoplankton Crenarchaeota. Distributions of organic molecules in marine sediments reflect temperature. Certain plants prefer certain temperatures; if their pollen is found one can work out the approximate temperature.

Forcing and boundary conditions for the glacial runs are provided by data from two different reconstructions of the LGM's climate: *CLIMAP* and *GLAMAP 2000*

In 1971 a consortium of scientists from many institutions was formed in order to reconstruct the Earth's surface at particular times in the past, mainly using deep-sea sediments: the **C**limate: **L**ong-Range **I**nvestigation, **Ma**pping, and **P**rediction (CLIMAP) project **??**. The time period of maximum extent of glaciers and ice shields was chosen as time period representing the LGM and was dated to 18,000 B.P. The group responsible for the reconstruction of sea surface temperatures (SSTs) used a transfer function method after **?**. This method uses a factor analysis that provides a number of empirical orthogonal functions (EOF) to define biotic assemblages entombed on the present-day seabed. These assemblages, which reflect the distribution of surface water masses, are then related to seasonal temperatures by multivariate regression equations. Various species of plactonic foraminifera, radiolaria and coccolithophoridea from 247 deep-sea cores were used in this way to estimate past SSTs.

The results of the CLIMAP project show an ocean (Figure 9.2) which on a global average was not much colder than today (about 2.3°C, **?**). Although the higher latitude ocean was 2°C to 4°C colder at the LGM, the tropics were not. According to CLIMAP the tropical ocean had about the same SSTs as today and in particular much of the Pacific Ocean surface was even warmer than
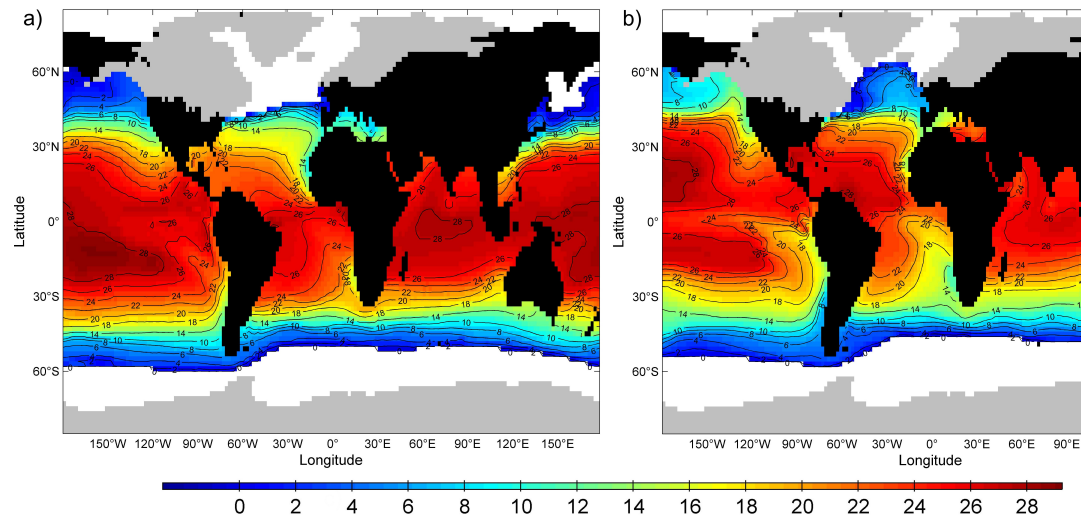
Figure 9.2: SSTs (°C) of February(a) and August(b) reconstructed from the CLIMAP project. White and grey areas represent sea ice and land ice cover, respectively, while land masses without ice cover are shown in black colour. Data from **?**

today. This result is in stark contrast to estimates obtained from various other proxies like corals [**???**], noble gas from groundwater [**??**], marine sediment pore fluid, tropical ice cores, snowline depression [**?**] and pollen studies [**??**]; and is physically not well understood. Therefore it led to a controversial debate throughout the 1980s and 1990s. Many suggestions for a solution of the discrepancy have been presented. Among others, C. **?** has shown, using a method of flux-weighted temperatures to reanalyse the original CLIMAP data set, that the tropical ocean SSTs have to be corrected by approximately 2°C while the values of higher latitudes are correct. **?** came to a similar conclusion with modelling studies on an AGCM: an additional cooling of 3°C between 30°N and 30°S yielded results which are in best agreement with reconstructed data of terrestrial and oceanic temperatures as well as of the hydrological cycle. The CLIMAP reconstructed data, with an additional cooling in the tropics following **?**, provides a consistent picture of the glacial climate and therefore is used as forcing for the glacial runs CB, CS, CBS3, CBS6 and CDEC in this thesis.

Despite the debate about the inconsistencies of the CLIMAP reconstruction [**???**] it took about 30 years before another large-scale reconstruction was undertaken [**?**]. The **Gl**acial **A**tlantic Ocean
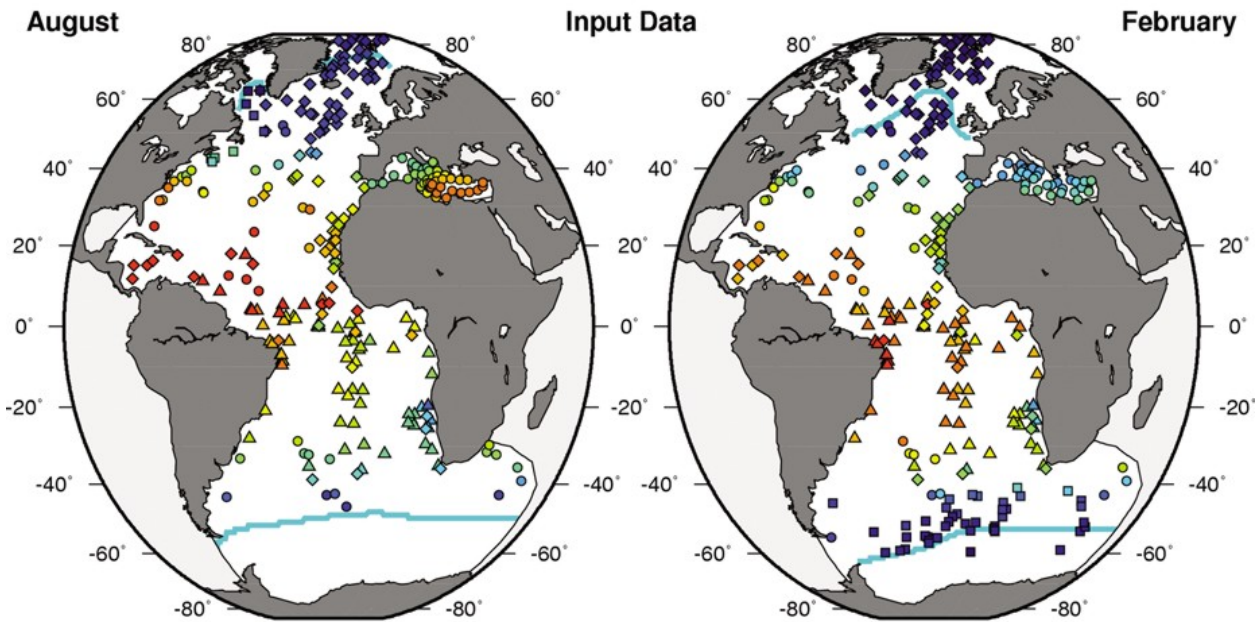
Figure 9.3: Input data set of the GLAMAP 2000 project. The symbols represent the positions where deep-sea core data was taken. Triangles: **?**. Diamonds: **?**. Boxes: **?**(Aug), **?**(Feb). Circles: **?**(Atlantic), **?**(Mediterranean). Blue lines: Ice edges after **?**, **?**, **?**. From **?**.

Mapping project *(GLAMAP 2000)* combined data from deep-sea sediment cores throughout the Atlantic Ocean to obtain a better and revised picture of the conditions during the Last Glacial Maximum. First of all the time period representing the LGM had to be chosen. Two major definitions for the LGM exist: The Last benthic oxygen Isotope Maximum (LIM), which assumably corresponds to the maximum volume of global ice sheets and the insolation minimum at 65° for summer. The former is dated to the time period between 21,500 and 18,000 years B.P. and the latter to 21,000 $\pm$ 2,000 years B.P. The project members agreed on an overlapping time period, covering both definitions: 22,000 - 19,000 years B.P. [**?**]. Furthermore compared to CLIMAP a larger number of sediment cores was available (Figure 9.3) which were calibrated in a standardised way using the World Ocean Atlas (WOA 94). The extent of sea-ice cover during summer and winter for the Southern Hemisphere was estimated by **?**, using diatom abundances. The same was performed for the Northern Atlantic through foraminiferic assemblages [**?**] and dinoflagellate cyst assemblages [**?**]. Moreover as compared to CLIMAP, the GLAMAP project used various new
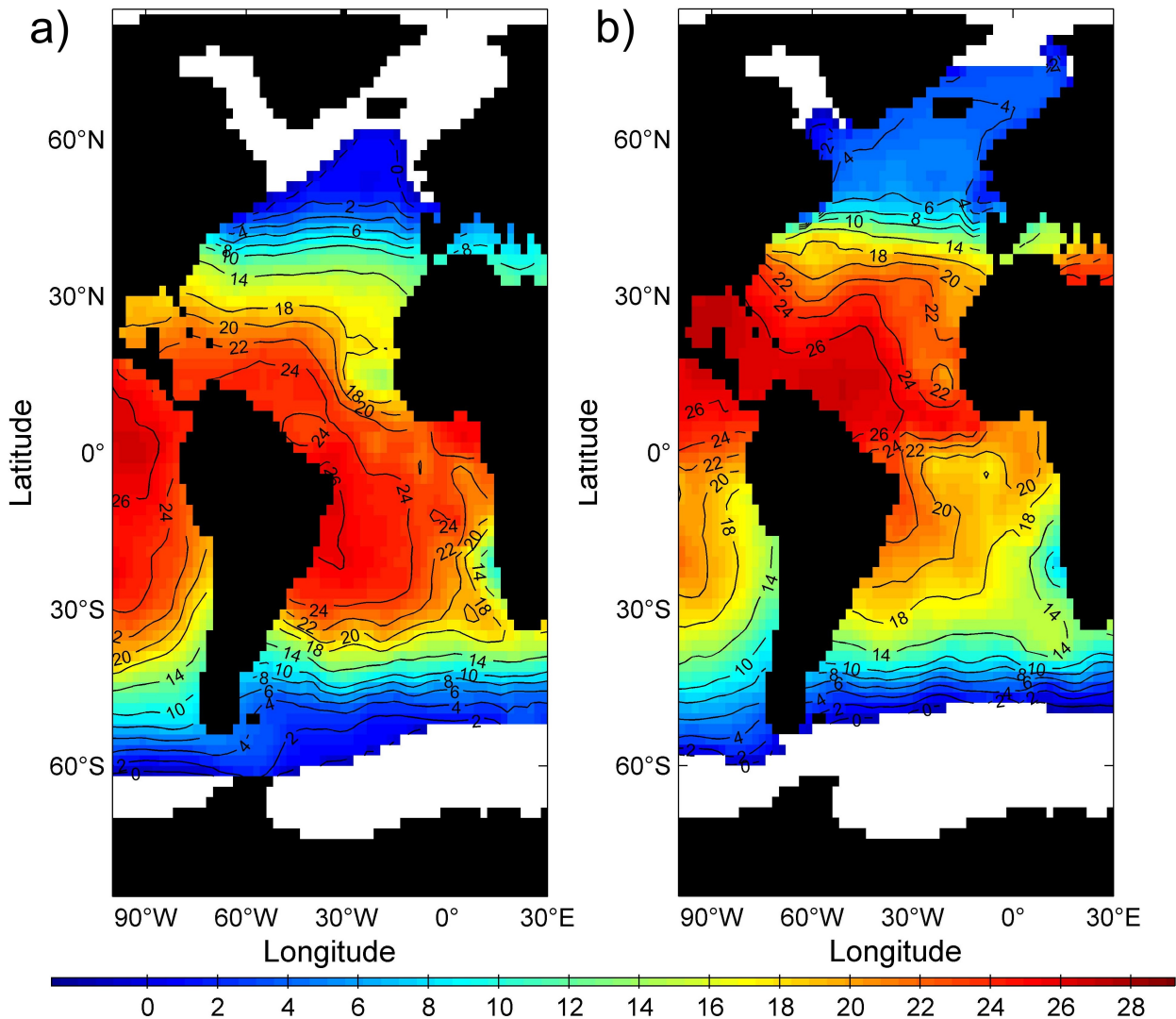
Figure 9.4: SSTs (°C) of February (a) and August (b) of the Atlantic Ocean reconstructed from the GLAMAP 2000 project. White areas represent sea-ice cover while black areas display land masses. Data from **?**.

transfer techniques to deduce SST from microfossil census counts, such as the Modern Analog Technique (MAT), similarity maximum-modern analog technique (SIMMAX-MAT), in addition to the Imbrie-Kipp technique using log-transfer techniques and improved statistical computer programs.

Unfortunately the reconstructed picture of the LGM, derived from these carefully analysed
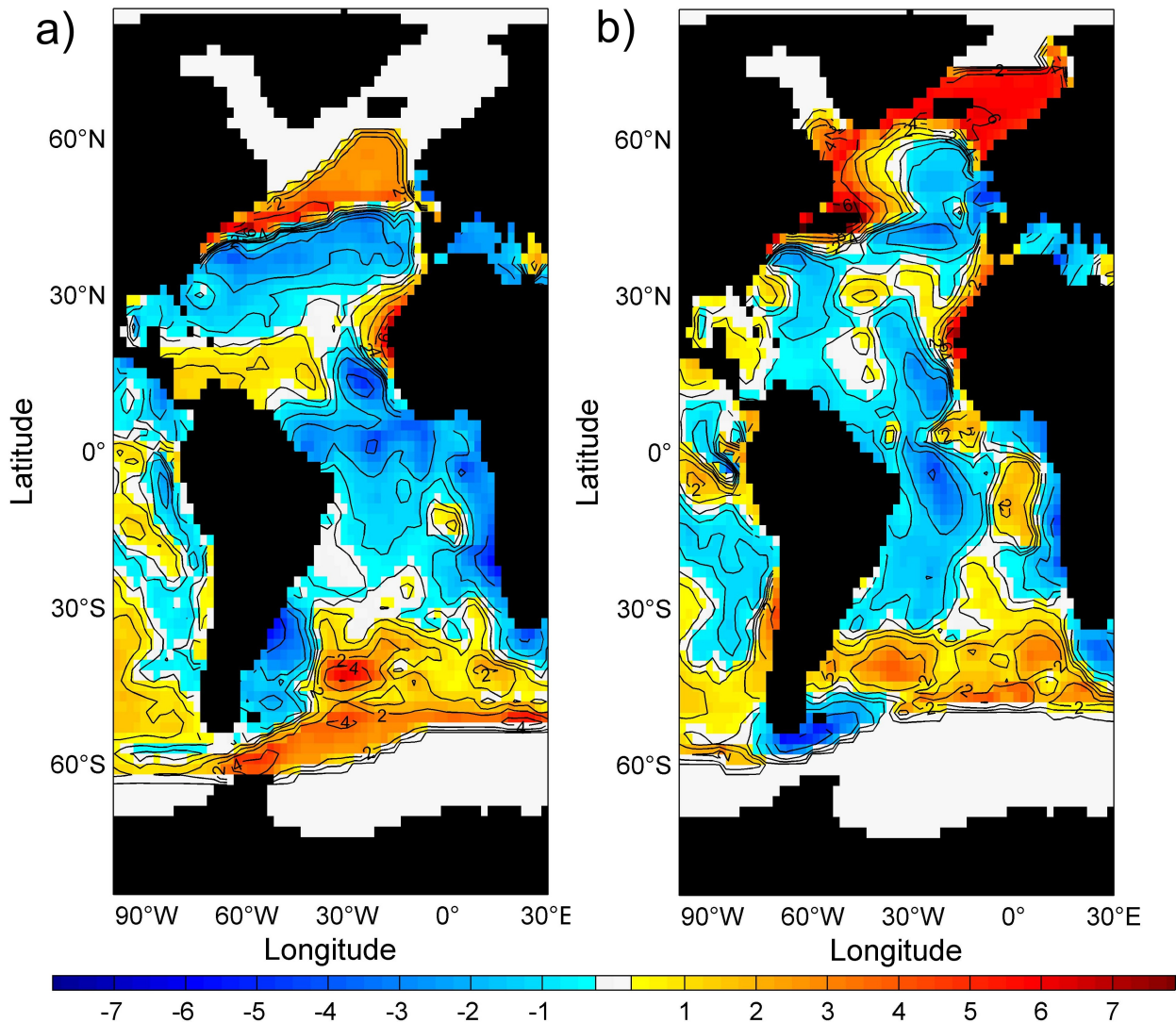
Figure 9.5: Comparison of the SSTs (February (a) and August (b)) in °C between GLAMAP 2000 and CLIMAP reconstructed data (GLAMAP minus CLIMAP). Black and white areas represent land masses and ice cover, respectively. Sea ice is shown where both data sets contain sea ice. The SST of water underneath the sea-ice cover is set to -1.8°C.

sediment cores, is confined to the Atlantic Ocean. Hence, **?** transformed the reconstructed SST, using variogram analysis and kriging for interpolation into a 1°x 1°grid and embedded it into the CLIMAP SST reconstruction of the other oceans in order to obtain a global gridded data set (Figure 9.4).

Upon comparing the newly gridded SST and the CLIMAP reconstruction (Figure 9.5) the most

prominent features are the different sea-ice margins, particularly in the northern Atlantic. The northernmost sea-ice margin reconstructed by the CLIMAP project is located at about 45°N in February and 60°N in August. On the other hand, GLAMAP 2000 locates the sea-ice margin to up to about 60°N in February and 80°N in August. Nevertheless the open ocean in front of the CLIMAP ice edge tends to be colder in the GLAMAP data set. Therefore the latter shows a weaker meridional temperature gradient in the mid latitudes of the northern Atlantic.

Except for the waters off the North African coast, the Atlantic Ocean of the GLAMAP reconstruction turns out to be generally colder in the low latitudes. This is most pronounced near the equator. These colder values are well supported by many studies focusing on the reconstruction of land temperatures in the tropics and subtropics.

The idea of ice-free Nordic Seas is not only proven by faunal assemblages, but as well by temperature reconstructions from alkenone concentrations [**?**]. While the band of higher SST in the northern mid-latitude Atlantic is well supported by deep-sea cores taken from different institutions, the high temperatures at approximately 40°- 50°S and 30°- 40°W are considered as an artefact, due to the lack of sediment cores from this area [**?**]. **?** also challenge the reliability of the very low temperatures off the Argentine coast, since a single sediment core is responsible for this local drop of SST.

9.2   Hydrological cycle and Oxygen isotope ratio cycle

The hydrological cycle describes the transport of water in all three phases on the global scale. Summing up all water volume contributions, about 14×10^{17} m^3 water are on the Earth and are stored in different components of the climate system. Most of the water (more than 97%) is in the salty oceans, and about 2% is stored in the glaciers. The remainder is unevenly divided between the reservoirs of groundwater, lakes and rivers, the atmosphere, and soil [*Baumgartner and Reichel,* 1975; *Peixoto and Oort,* 1992]. The atmosphere is a central subsystem in the global climate system, because it has the highest mobility and it connects the other subsystems.

Dansgaard et al. [1993] used stable isotope measurements from the full length of the 250 kyr (thousands of years) Greenland ice core record (Greenland Ice Sheet Project) to examine climate stability during glacial periods. They found that apart from the most recent 10,000 years, instability dominated the North Atlantic climate over the last 230 kyr. Other paleoclimate proxies point out an extreme rapidity in the changes of aridity, dust, and chemical composition, as recorded by Greenland ice cores [*Mayewski et al.,* 1993], or by marine sediment cores [*Lehman and Keigwin,* 1992], implying that events at the end of the last glaciation could have been responses to a threshold in the North Atlantic climate system.

One possible mechanism for such a threshold is associated with the hydrological cycle [*Birchfield,* 1989; *Broecker et al.,* 1990; *Zaucker et al.,* 1994; *Rahmstorf,* 1995]. Besides the exchange of energy and momentum between atmosphere and ocean, the freshwater flux is one of the main driving forces for the oceanic thermohaline circulation (THC). While air-sea heat fluxes damp sea surface temperature anomalies quite effectively, there is no direct feedback in the atmosphere for removing sea surface salinity anomalies, because precipitation and evaporation are not dependent on ocean salinity. Numerical experiments suggest that the THC is very sensitive to the atmospheric freshwater forcing [*Manabe and Stouffer,* 1995; *Rahmstorf,* 1996; *Weaver et al.,* 1998].

Warren [1983] suggested that the low surface salinities in the Pacific are due to an excess of precipitation over evaporation and that this can be the reason for the absence of deep water formation there. Because the salinity characteristic is affected by the interbasin transport of water vapor it is therefore conceivable that the hydrological cycle is responsible for the difference between the thermohaline circulation in the North Atlantic and Pacific Oceans. It has been argued [*Birchfield,* 1989; *Broecker et al.,* 1990; *Zaucker et al.,* 1994] that the interbasin water vapor transport provides for multiple equilibria of the climate system. A recent sensitivity study [*Weaver et al.,* 1998] confirms this idea: By changing the Atlantic catchment of a few rivers, the *Weaver et al.* [1998] model is able to enhance or to slow down the Atlantic thermohaline circulation.

The hydrological cycle and its associated distribution of moisture is furthermore of interest because of its important influence on the terrestrial climate, distribution of vegetation and deserts,

and the growth of ice sheets. These geographical features are documented by the proxy data available, as for example lake levels and vegetation types [*Crowley and North,* 1991; *Farrera et al.,* 1999]. Further links of the water vapor transport with climate are due to the ice sheets and the water's isotopic compositions [*Werner et al.,* 2000].

Oxygen isotope ratio cycles are cyclical variations in the ratio of the mass of oxygen with an atomic weight of 18 to the mass of oxygen with an atomic weight of 16 present in some substance, such as polar ice or calcite in ocean core samples. The ratio is linked to water temperature of ancient oceans, which in turn reflects ancient climates. Cycles in the ratio mirror climate changes in geologic history.

Oxygen (chemical symbol O) has three naturally occurring isotopes: ^{16}O, ^{17}O, and ^{18}O, where the 16, 17 and 18 refer to the atomic weights. The most abundant is ^{16}O, with a small percentage of ^{18}O and an even smaller percentage of ^{17}O. Oxygen isotope analysis considers only the ratio of ^{18}O to ^{16}O present in a sample. The calculated ratio of the masses of each present in the sample is then compared to a standard, which can yield information about the temperature at which the sample was formed.

^{18}O is two neutrons heavier than ^{16}O and causes the water molecule in which it occurs to be heavier by that amount. The addition of more energy is required to vaporize $H_2^{18}O$ than $H_2^{16}O$, and $H_2^{18}O$ liberates more energy when it condenses. In addition, $H_2^{16}O$ tends to diffuse more rapidly. Because $H_2^{16}O$ requires less energy to vaporize, and is more likely to diffuse to the liquid surface, the first water vapor formed during evaporation of liquid water is enriched in $H_2^{16}O$, and the residual liquid is enriched in $H_2^{18}O$. When water vapor condenses into liquid, $H_2^{18}O$ preferentially enters the liquid, while $H_2^{16}O$ is concentrated in the remaining vapor. As an air mass moves from a warm region to a cold region, water vapor condenses and is removed as precipitation. The precipitation removes $H_2^{18}O$, leaving progressively more $H_2^{16}O-$rich water vapor. This distillation process causes precipitation to have lower $^{18}O/^{16}O$ as the temperature decreases. Additional factors can affect the efficiency of the distillation, such as the direct precipitation of ice crystals, rather than liquid water, at low temperatures.

The $^{18}O/^{16}O$ ratio provides a record of ancient temperature. Precipitation and therefore glacial ice contain water with a low ^{18}O content. Since large amounts of ^{16}O water are being stored as glacial ice, the ^{18}O content of oceanic water is high. Water up to 5 degrees warmer than today represents an interglacial, when the ^{18}O content is lower. A plot of ancient water temperature over time indicates that climate has varied cyclically, with large cycles and harmonics, or smaller cycles, superimposed on the large ones. This technique has been especially valuable for identifying glacial maxima and minima in the Pleistocene.

Limestone is deposited from the calcite shells of microorganisms. Calcite, or calcium carbonate, chemical formula $CaCO_3$, is formed from water, H_2O, and carbon dioxide, CO_2, dissolved in the water. The carbon dioxide provides two of the oxygen atoms in the calcite. The calcium must rob the third from the water. The isotope ratio in the calcite is therefore the same, after compensation, as the ratio in the water from which the microorganisms of a given layer extracted the material of the shell. The microorganism most frequently referenced is foraminifera (Fig. 9.6).

There are two stable isotopes of hydrogen (^{1}H, protium, and ^{2}H, deuterium) and three stable isotopes of oxygen (^{16}O, ^{17}O, and ^{18}O). Further, there are three out of nine isotopically different water isotopes [1] that are available in measurable quantities: $H_2^{16}O$, $H_2^{18}O$, and $^1H^2H^{16}O$.

High values in the ratio of ^{18}O to ^{16}O reflect usually mean low temperatures in foram tests or ice cores. Confounded by ice volume - more ice means higher $\delta^{18}O$ values. Ocean water is mostly ^{18}O with small amounts of $HD^{16}O$ and ^{16}O.

Table 9.1 shows the relative quantities of the isotopes. ^{17}O appears equally in this table because $H_2^{17}O$ is involved in the same physical processes as $H_2^{18}O$. Only its abundance and fractionation rate is less useful for palaeoclimate research. However, it *is* used for research topics involving very long time scales. The ratio of $^{17}O/^{18}O$ in a given substance does change on geological time scales and can be used to distinguish the origins of meteorites and materials on different planets [e.g.][]Clayton:03.

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[1]The term *water isotope* is often used in the scientific community. It is not correct, since water is obviously not an element. In this study, this expression is used as an abbreviation for *water molecule containing a heavy stable isotope of oxygen or hydrogen.*
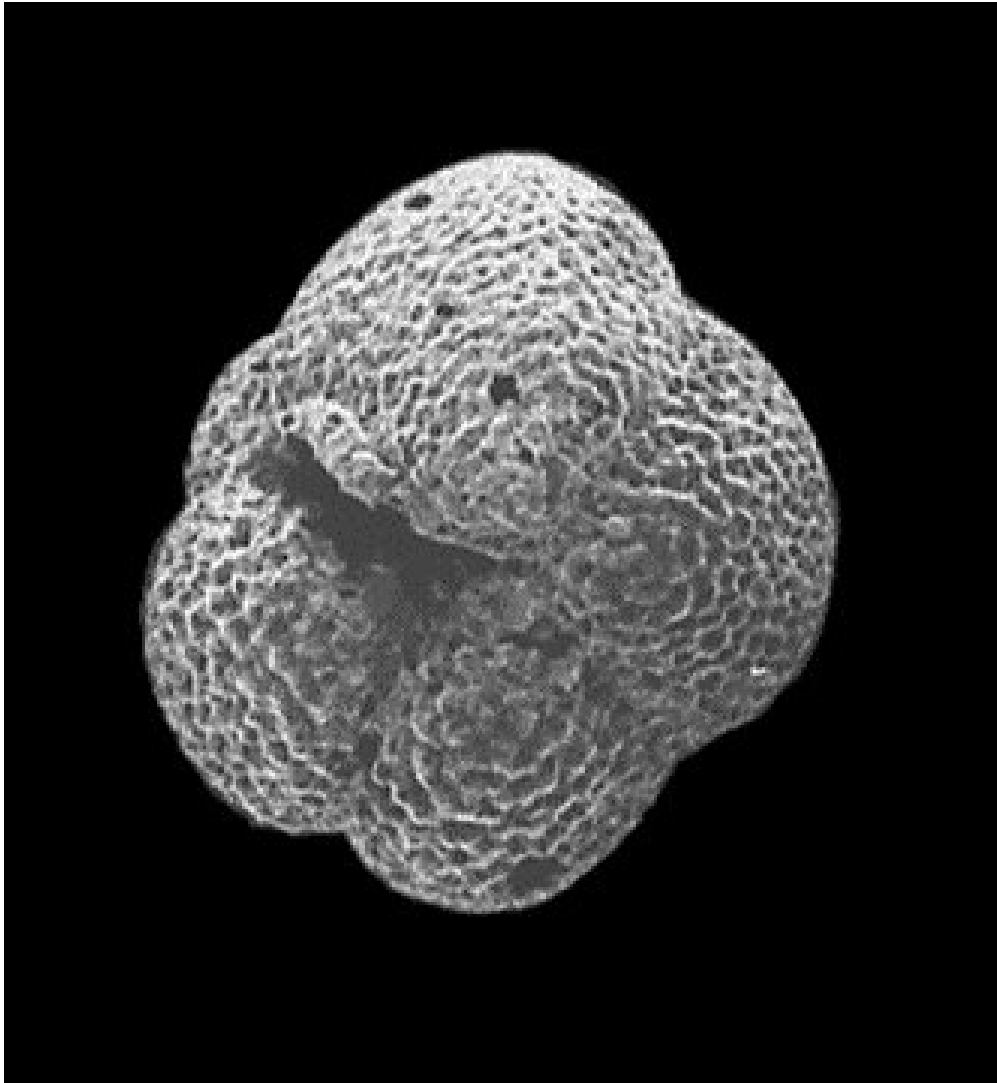
Figure 9.6: Foram neogloboquadrina pachyderma: Neogloboquadrina pachyderma dominates assemblages in transitional to polar water masses and occurs in low frequencies in warm subtropical and tropical environments. The species has broad tolerances for sea surface temperature and preferences for low sea surface salinities with little seasonal change. Its preferences for dense surface waters with low vertical temperature gradients and little stratification reflect its preferred high latitude habitats. Neogloboquadrina pachyderma has two coiling variants. The left-coiling (sinistral) variant is known to be frequent in cold water masses and has been found living in sea ice. Neogloboquadrina pachyderma (dextral) is a good indicator for surface water densities above about 25.5 kg/m^3.

Hydrogen		Oxygen		
^{1}H	^{2}H	^{16}O	^{17}O	^{18}O
99.985	0.015	99.759	0.037	0.204

Table 9.1: Natural abundance in ‰ of stable hydrogen and oxygen isotopes used in climate research.

In Standard Mean Ocean Water (SMOW) the ratio of D to H is $155.8 \cdot 10^{-6}$ and ^{18}O to ^{16}O is $2005 \cdot 10^{-6}$. The difference from VSMOW is expressed as

$$\delta^{18}O = \left(\frac{(^{18}O/^{16}O)}{(^{18}O/^{16}O)_{SMOW}} - 1 \right) \cdot 1000‰ \tag{9.3}$$

and a similar formula for δD. Fractionation occurs during changes between condensed and vapour phases: the vapour pressure of heavier isotopes is lower, so vapour contains relatively more of the lighter isotopes and when the vapour condenses the precipitation preferentially contains heavier isotopes. The major influence on $\delta^{18}O$ is the difference between ocean temperatures where the moisture evaporated and the place where the final precipitation occurred; since ocean temperatures are relatively stable the $\delta^{18}O$ value mostly reflects the temperature where precipitation occurs. Taking into account that the precipitation forms above the inversion layer, we are left with a linear relation:

$$\delta^{18}O = aT + b \tag{9.4}$$

which is empirically calibrated from measurements of temperature and $\delta^{18}O$ as $a = 0.67/ppm/°C$ for Greenland and $0.76/ppm/°C$ for East Antarctica. The calibration was initially done on the basis of spatial variations in temperature and it was assumed that this corresponded to temporal variations (Jouzel and Merlivat, 1984). More recently, borehole thermometry has shown that for glacial-interglacial variations, $a = 0.33/ppm/°C$ (Cuffey et al., 1995), implying that glacial-interglacial temperature changes were twice as large as previously believed.

The empirical relationships between the observed isotopic composition of meteoric water and environmental parameters were summarized by **?**. Relations between surface-air temperature, distance from the coast, height above sea level and amount of precipitation to the measured isotopic composition were termed 'effects' (temperature, continental, altitude and amount effect, respectively). These effects can be considered as a measure of rain-out of a given air mass on its way to the preipiation region. **?** used a Rayleigh model to explain the linear relationship between the isotopic composition of precipitation and the annual mean surface temperature. Under the simplifying assumption that this air mass is isolated and constantly cooled, the Rayleigh approach describes the observed gradient of 0.58 ‰/°C at higher latitudes, if the annual mean surface temperature $<15°$C. In the tropics, the annual temperature variation generally is low. On the other hand there are large variations in the rainfall intensity. Therefore, the isotopic composition is mainly influenced by the amount effect. Increased precipitation leads to isotopically more depleted rainfall and the δ^{18}O/temperature relationship is not valid. However, the effects mentioned above do not occur isolated or separated from each other. Any change in the environment in which water undergoes phase changes will have an effect on the fractionation.

The Rayleigh model approach has been refined to include kinetic fractionation during evaporation and formation of ice, the simultaneous existence of vapour, liquid and ice in clouds, and partial re-evaporation of precipitation from land surfaces [**?**]. Mixing of air masses from different origin cannot be described by Rayleigh models. The implementation of isotope physics in general circulation models has therefore been the next step in simulation of the global hydrological cycle.

The isotope signature of evaporating vapour depends on sea surface temperature, relative humidity and the delta-value of atmospheric vapour [**?**]. The latter is an independent quantity, allowing for example a degree of freedom in the interpretation of the deuterium excess.

The ratios of the heavy isotopic water $H_2^{18}O/H_2^{16}O$ and $HDO/H_2^{16}O$ usually are expressed in parts per thousand deviation relative to the standard V-SMOW (Vienna standard mean ocean wa-

ter). The delta notations $\delta^{18}O$ and δD are defined as

$$\delta_{sample} = \left(\frac{R_{sample}}{R_{standard}} - 1 \right) * 1000 \qquad (9.5)$$

where R_{sample} is the measured isotope ratio and $R_{standard}$ is the V-SMOW value for the respective isotope.

The water isotopes are transported in parallel to atmospheric water. The advection and diffusion routines therefore ensure constant ratios of $H_2^{18}O/H_2^{16}O$ and $HDO/H_2^{16}O$, respectively. The isotopic composition is then expressed in terms of VSMOW, the Vienna standard mean ocean water. For ^{18}O we use the formula

$$\delta^{18}O = [(^{18}O/^{16}O)_{sample}/(^{18}O/^{16}O)_{VSMOW} - 1]. \qquad (9.6)$$

δD is calculated correspondingly for $HD^{16}O$. The first fractionation process occurs during evaporation at the ocean surface. The evaporative flux of isotopes is calculated using

$$E_x = \rho C_v |\vec{v}_h|(1 - k)(x_{vap} - x_{sat}) \qquad (9.7)$$

where ρ is the density of air, C_v the drag coefficient depending on the stability of the atmospheric boundary layer, $|\vec{v}_h|$ the horizontal wind speed, x_{vap} the water isotope mixing ratio in the first model layer, and $x_{sat} = \alpha(T_{surf})^{-1}\beta R_{oc}q_{sat}$, with α the temperature-dependent equilibrium fractionation factor, β an enrichment factor for the oceanic surface due to evaporation, R_{oc} the oceanic mass relation corresponding to R_{SMOW}, and q_{sat} is the saturation mixing ratio. Non-equilibrium effects (kinetic diffusion from the thin sub-layer above the ocean surface to the atmosphere) are taken into account with the factor $1 - k$ [see ?]. Dividing E_x by the evaporative flux of $H_2^{16}O$ $E = \rho C_v |\vec{v}_h|(q_{vap} - q_{sat})$ results in the isotopic composition of the evaporative flux

$$\delta E_x + 1 = (1 - k)/(1 - h)[\alpha(T_{surf})^{-1}(\delta x_{oc} + 1) - (\delta x_{vap} + 1)h] \qquad (9.8)$$

with $\delta x_{oc} = \beta(R_{oc}/R_{SMOW}) - 1$, the relative humidity h, and the atmospheric delta-value δx_{vap}.

Condensation to liquid or ice phase is treated as an equilibrium process ($R_{l,i} = \alpha R_{vap}$).

A kinetic process becomes important at low temperatures, namely the diffusion of isotopes through the oversaturated zone around forming ice crystals. This is considered in the model with an effective fractionation coefficient

$$\alpha_{eff} = \alpha_{eq}\alpha_{kin} \tag{9.9}$$

with $\alpha_{kin} = S/(\alpha_{eq}R_D(S - 1) + 1)$. R_D is the ratio of the diffusivities of $H_2^{16}O$ and the isotopic water, and $S = 1 + 0.003T$ (T in °C) the oversaturation function [see **?**].

Re-evaporation of raindrops in the undersaturated air below cloud base occurs in an equilibrium and non-equilibrium process: The kinetic fractionation is formulated similar to equation (2), using undersaturation described by h_{eff}, the mean relative humidity of the air below cloud base in the grid box. The fraction of droplets that reach isotopic equilibrium depends on the droplet size. In the implementation of numerical models, the heavy isotopic water exactly follows the hydrological routines, except where fractionation during phase changes is altering the ratios $H_2^{18}O/H_2^{16}O$ in the vapour, liquid and solid phase, respectively. The fractionation coefficients α for ^{18}O are

$$\ln\alpha_l = 1137/T^2 - 0.4156/T + 0.00206, \text{ and} \tag{9.10}$$

$$\ln\alpha_i = 11.839/T - 0.28224, \tag{9.11}$$

where α_l and α_i are the fractionation factors for transitions from liquid to vapour phase and from solid to vapour phase, respectively. This modelling approach allows to examine climate variables, such as P and T independently from the isotopic water composition. The results presented here are therefore physically consistent and do not rely on any further concepts or models.

9.3   Role of the Ocean in Ice-Age Climate Fluctuations

What might happen when the production of deep water in the Atlantic is shut off? Information contained in the Greenland and Antarctic ice sheets and in north Atlantic sediments provide important clues. Several ice core through the Greenland ice sheet and three through the Antarctic sheet provide a continuous record of atmospheric conditions over Greenland and Antarctica extending back more than 400,000 years before the present in some cores. Annual layers in the core are counted to get age. Deeper in the core, where annual layers are hard to see, age is calculated from depth. Occasional world-wide dustings of volcanic ash provide common markers in cores. Oxygen-isotope ratios in the ice give temperatures over parts of the northern hemisphere; bubbles in the ice give atmospheric CO_2 and methane concentration; pollen, chemical composition, and particles give information about volcanic eruptions, wind speed, and direction; thickness of annual layers gives snow accumulation rates; and isotopes of some elements give solar and cosmic ray activity (Alley, 2000). Cores through deep-sea sediments in the north Atlantic made by the Ocean Drilling Program give information about sea-surface temperature and salinity above the core, the production of north Atlantic deep water, ice volume in glaciers, and production of icebergs.

- The oxygen-isotope record in the ice cores show abrupt temperature variability over the past 100,000 years. Many times during the last ice age, temperatures near Greenland warmed rapidly over periods of 1-100 years, followed by gradual cooling over longer periods (Dansgaard et al., 1993). For example, roughly 11,500 years ago, temperatures over Greenland warmed by roughly 8°C in 40 years in three steps, each spanning 5 years (Alley, 2000). Such abrupt warming is called a Dansgaard/Oeschger event. Other studies have shown that much of the northern hemisphere warmed and cooled in phase with temperatures calculated from the ice core.

- The climate of the past 8,000 years was constant with very little variability. Our perception of climate change is thus based on highly unusual circumstances. All of recorded history has been during a period of warm and stable climate.

- Hartmut Heinrich and colleagues (Bond et al., 1992), studying the sediments in the north Atlantic noticed periods when coarse material was deposited on the bottom in mid-ocean. Only icebergs can carry such material out to sea, and the find indicated times when large numbers of icebergs were released into the north Atlantic. These are now called Heinrich events.

- The correlation of Greenland temperature with iceberg production is related to the meridional overturning circulation. When icebergs melted, the surge of fresh water increased the stability of the water column shutting off the production of North Atlantic Deep Water. The shut-off of deep-water formation greatly reduced the transport of warm water in the north Atlantic, producing very cold northern hemisphere climate (Figure 13.2). The melting of the ice pushed the polar front, the boundary between cold and warm water in the north Atlantic further south than its present position. The location of the front, and the time it was at different positions can be determined from analysis of bottom sediments.

- When the meridional overturning circulation shuts down, heat normally carried from the south Atlantic to the north Atlantic becomes available to warm the southern hemisphere. This explains the Antarctic warming.

- The switching on and off of the meridional overturning circulation has large hysteresis (Figure 13.3). The circulation has two stable states. The first is the present circulation. In the second, deep water is produced mostly near Antarctica, and upwelling occurs in the far north Pacific (as it does today) and in the far north Atlantic. Once the circulation is shut off, the system switches to the second stable state. The return to normal salinity does not cause the circulation to turn on. Surface waters must become saltier than average for the first state to return (Rahmstorf, 1995)

- Heinrich events seem to precede the largest Dansgaard/Oeschger events (Stocker and Marchal, 2000). Here's what seems to happen. The Heinrich event shuts off the Atlantic deep circulation which leads to a very cold North Atlantic. This is followed about 1000 years later
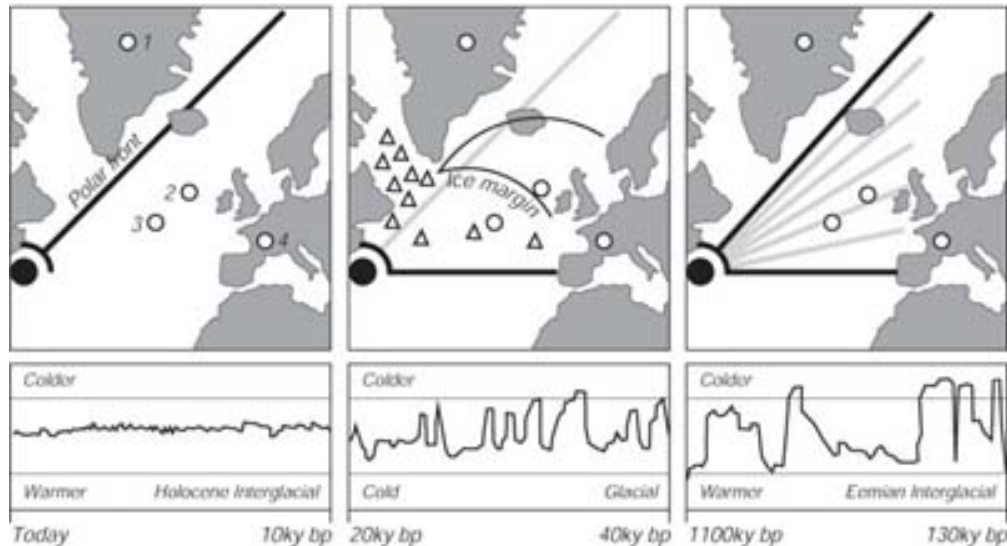
Figure 9.7: Periodic surges of icebergs during the last ice age appear to have modulated temperatures of the northern hemisphere by lowering the salinity of the far north Atlantic and reducing the meridional overturning circulation. Data from cores through the Greenland ice sheet (1), deep-sea sediments (2,3), and alpine-lake sediments (4) indicate that: Left: During recent times the circulation has been stable, and the polar front which separates warm and cold water masses has allowed warm water to penetrate beyond Norway. Center: During the last ice age, periodic surges of icebergs reduced salinity and reduced the meridional overturning circulation, causing the polar front to move southward and keeping warm water south of Spain. Right: Similar fluctuations during the last interglacial appear to have caused rapid, large changes in climate. The Bottom plot is a rough indication of temperature in the region, but the scales are not the same. From Zahn (1994).

by a Dansgaard/Oeschger event with rapid warming. Dansgaard/Oeschger-Heinrich tandem events have global influence, and they seem to be related to warming events seen in Antarctic ice cores. Temperatures changes in the two hemispheres are out of phase. When Greenland warms, Antarctica cools.

• A weakened version of this process with a period of about 1000 years may be modulating present-day climate in the north Atlantic, and it may have been responsible for the Little Ice Age from 1100 to 1800.

The relationship between variations in salinity, air temperature, and deep-water formation is not yet well understood. For example, we don't know what causes the ice sheets to surge. Surges may
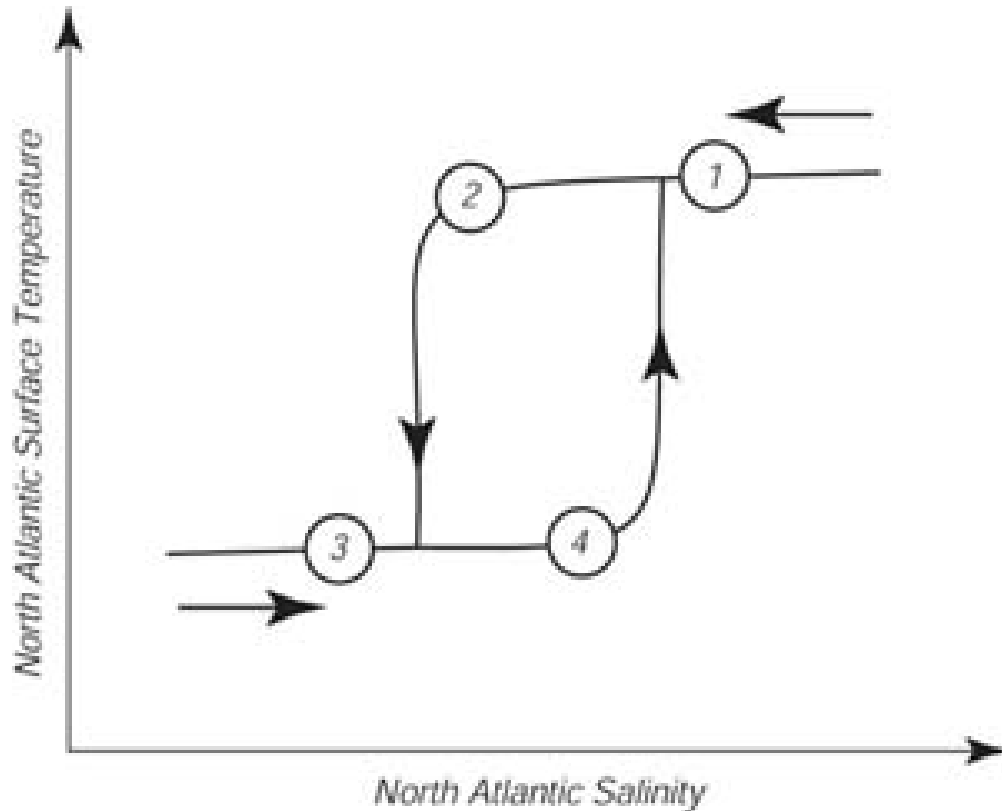
Figure 9.8: The meridional-overturning circulation is part of a non-linear system. The circulation has two stable states near 2 and 4. The switching of north Atlantic from a warm, salty regime to a cold, fresh regime and back has hysteresis. This means that as the warm salty ocean in an initial state 1 freshens, and becomes more fresh than 2 it quickly switches to a cold, fresh state 3. When the area again becomes salty, it must move past state 4 before it can switch back to 1.

result from warmer temperatures caused by increased water vapor from the tropics (a greenhouse gas) or from an internal instability of a large ice sheet. Nor do we know exactly how the oceanic circulation responds to changes in the deep circulation or surface moisture fluxes.

The oceans play a key role in the development of the ice ages. Every 100,000 years for the past million years, ice sheets have advanced over the continents. Shackleton (2000) finds that the 100,000-year period of Earth's orbital eccentricity, deep-sea temperature, and atmospheric carbon-dioxide concentration are well correlated over the 100,000-year cycle. He also finds that ice-sheet volume lagged behind CO_2 changes in the atmosphere, implying that ice sheets changed as a result of CO_2 changes, not the other way around. The deep water masses that participate in the MOC have chemical, temperature and isotopic ratio signatures and can be traced, their flow rate calculated, and their age determined. These include ^{231}Pa / ^{230}Th ratios.

Relatively well documented climate transitions are found at the end of the last ice-age (Lehman and Keigwin, 1992; Sarnthein et al., 1994). Paleoclimatic records show (Fig. 9.9) that deglacial meltwater has entered the Atlantic Ocean during the Bølling (ca. 14,000 years before present).

The question is what causes the abrupt warming at the onset of the Boelling as seen in the Greenland ice cores (Fig. 9.10). There is a clear antiphasing seen in the deglaciation interval between 20 and 10 ky ago: During the first half of this period, Antarctica steadily warmed, but little change occurred in Greenland. Then, at the time when Greenland's climate underwent an abrupt warming, the warming in Antarctica stopped.

Knorr and Lohmann **?**, also summarizing numerous modeling studies for deglaciation, describe how global warming (which may be induced by greenhouse gases and feedbacks) can induce a rapid intensification of the ocean circulation (Fig. 9.11). During the Boelling/Alleroed, a sudden increase of the northward heat transport draws more heat from the south, and leads to a strong warming in the north. This 'heat piracy' from the South Atlantic has been formulated by Crowley **?**. A logical consequence of this heat piracy is the Antarctic Cold Reversal (ACR) during the Northern Hemisphere warm Boelling/Alleroed. This particular example shows that an abrupt climate change of the ocean circulation (with large climate impacts in the North Atlantic) is related
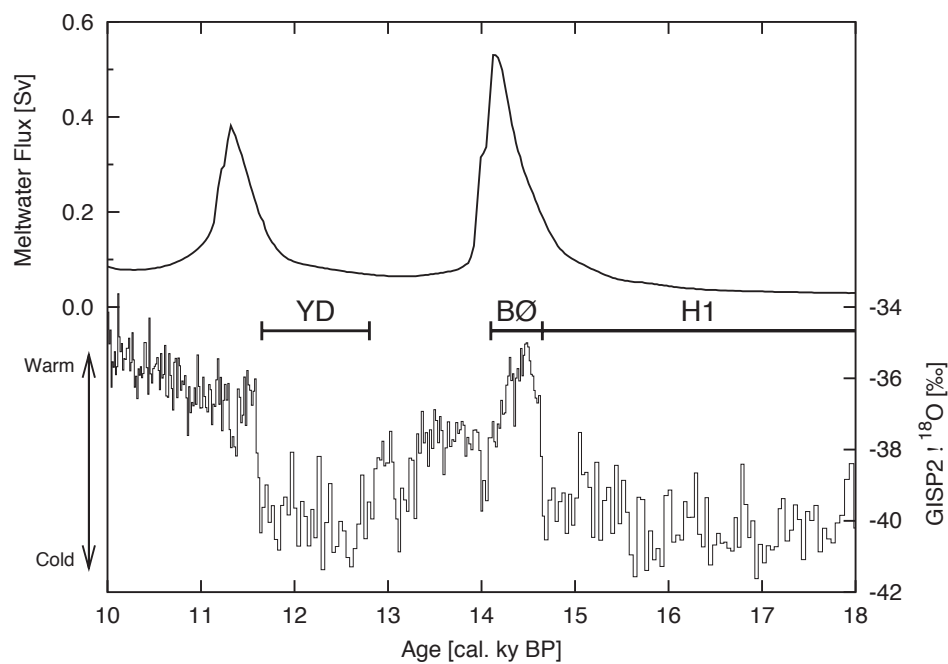
Figure 9.9: Top: Sea-level derived rate of deglacial meltwater discharge [*Fairbanks et al.,* 1992]. Bottom: Oxygen isotope record from Greenland GISP2 ice core [*Grootes and Stuiver,* 1997], reflecting air temperature. Meltwater influx is maximum during the Bølling warm period (BØ) and minimum during Heinrich event 1 (H1) and Younger Dryas (YD).
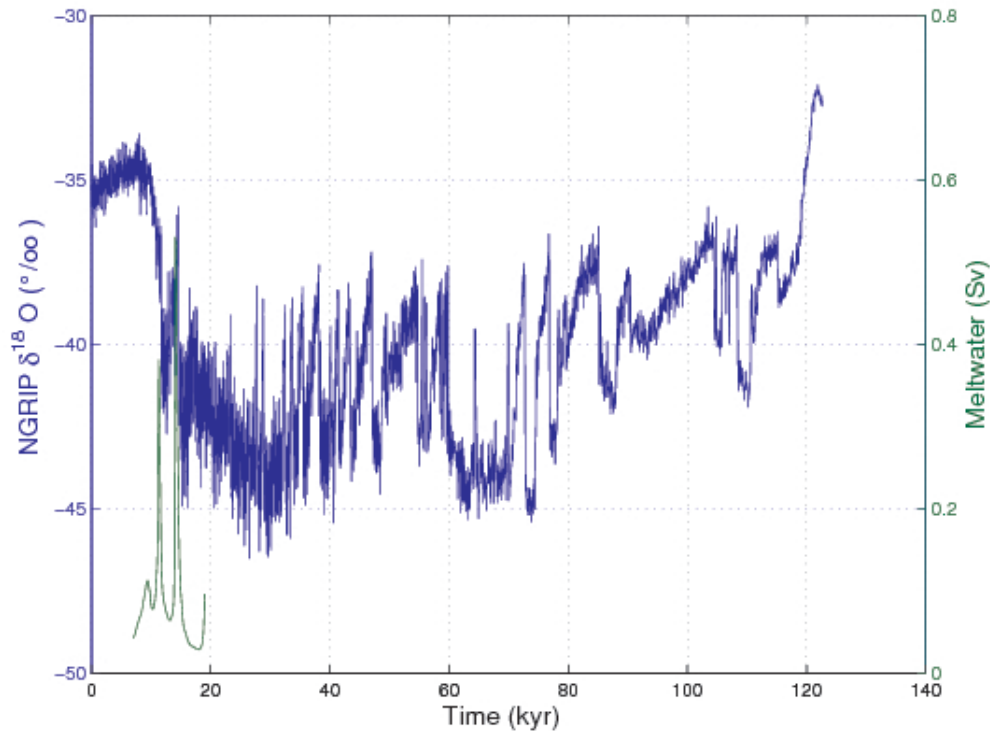
Figure 9.10: Oxygen isotope record from a Greenland ice core record **?** using an updated time scale for this record **?**. Green: Sea-level derived rate of deglacial meltwater discharge Fairbanks [1989] which is strong after deglacial warming.

to a smooth global warming. To understand the dynamical behavior of the system, the concept of hysteresis is applied, using the global warming after the last ice ages as the control parameter **?**. The system exhibits multiple steady states (Fig. 9.12): a weak glacial ocean circulation and a stronger circulation (which is comparable in strength to the modern mode of operation). Deglacial warming induces a transition from a weak glacial THC state to a stronger THC state, characterizing the abrupt warming during the deglaciation.
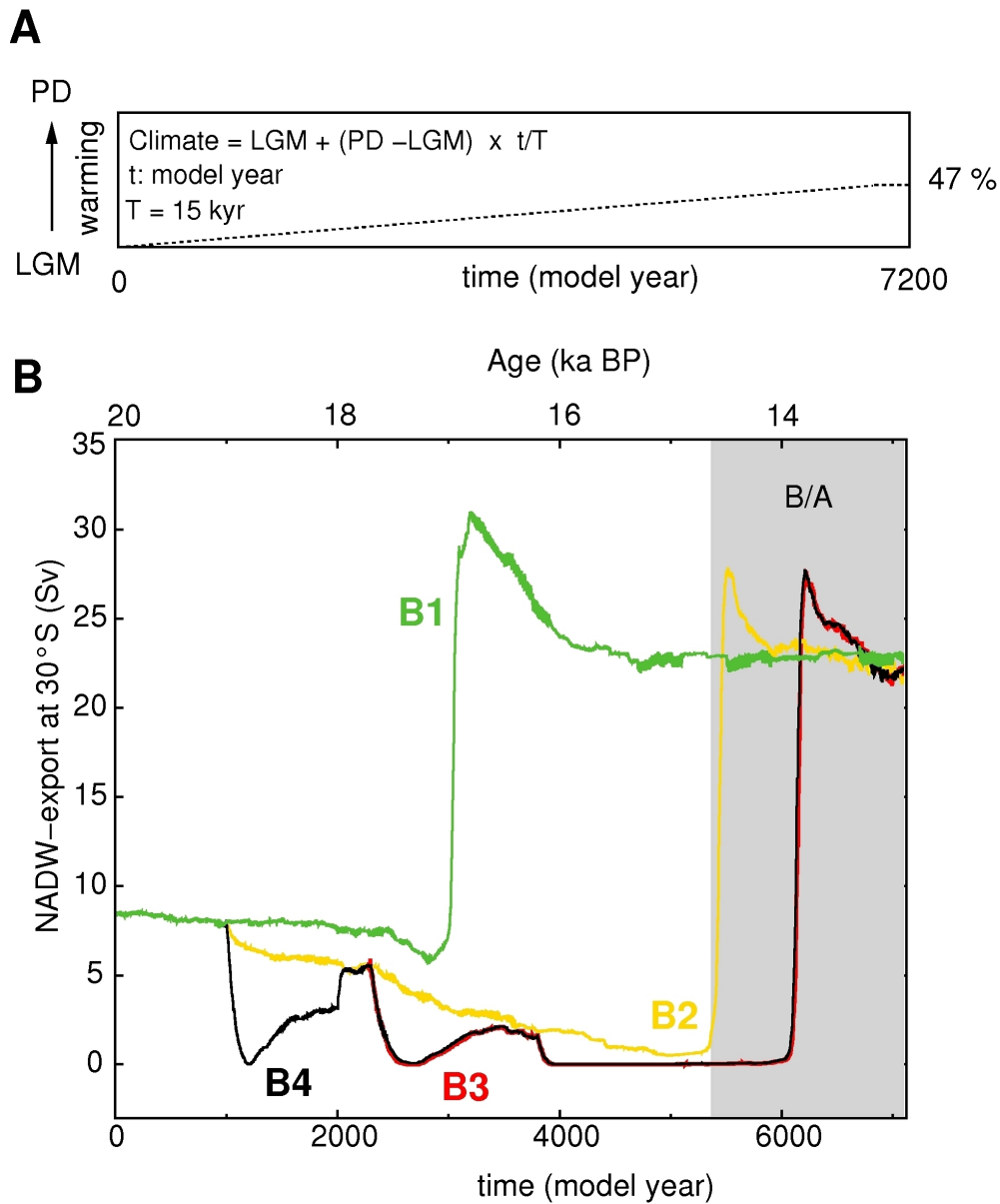
Figure 9.11: Forcing and model response of the ocean overturning rate. (a) The background climate conditions are linearly interpolated between glacial (LGM), and modern (PD), conditions. Gradual warming is stopped after 7000 model years, which is related to $\sim$ **47%** of the total warming. (b) circulation strength (export at **30°S**) versus time. The green curve (B1) represents the experiment without any deglacial freshwater release to the North Atlantic. Experiments B2 (yellow curve), B3 (red curve), and B4 (black curve), exhibit different successions of deglacial meltwater pulse scenarios to the North Atlantic **?**.
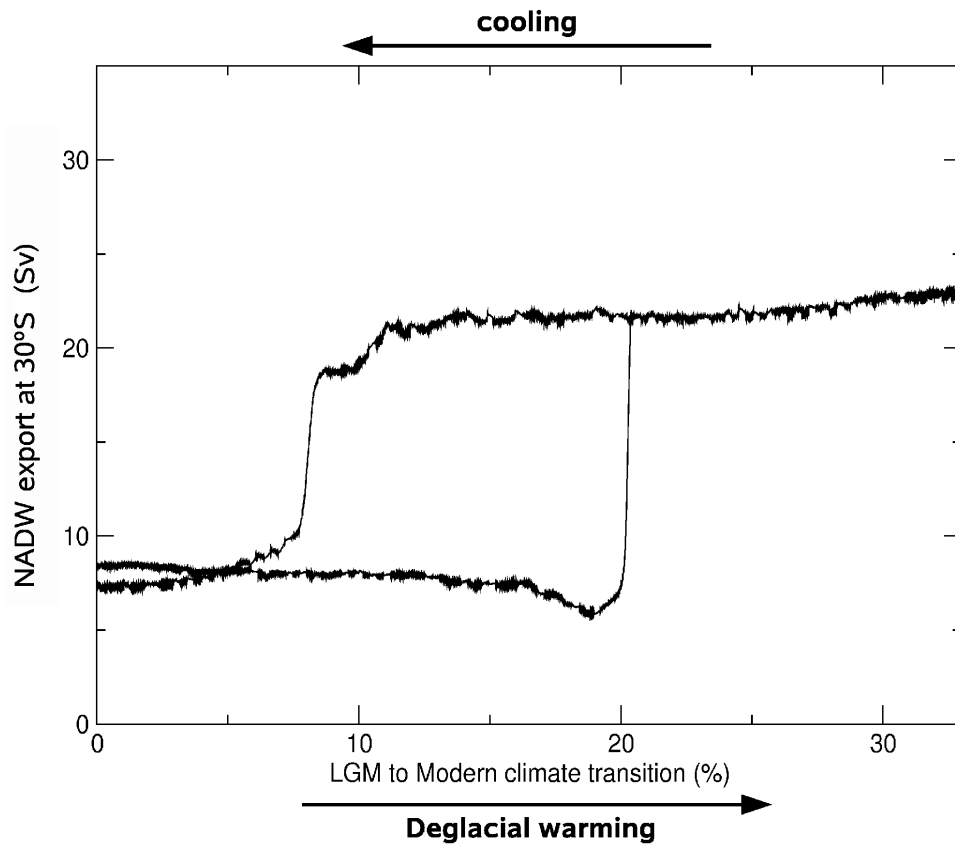
Figure 9.12: Hysteresis loop of the ocean overturning strength (black curve) with respect to slowly varying climate background conditions. The transition values are given in % of a full glacial-interglacial transition **?**.

9.4 Abrupt climate change

The occurrence of abrupt changes in the climate system at various time scales has attracted a great deal of interest for its theoretical and practical significance **?**; **?**; **?**. To some extent, a definition of what constitutes an abrupt climatic change depends on the sampling interval of the data being examined **?**. For the instrumental period covering approximately the last 100 years of annually or seasonally sampled data, an abrupt change in a particular climate variable will be taken to mean a statistically highly significant difference between adjacent 10-year sample means. In the paleoclimate context (i.e. on long time scales), an abrupt climate change can be in the order of decades to thousands of years. Since the climate dynamics can be often projected onto a limited number of modes or patterns of climate variability (e.g., **?**; **?**), the definition of abrupt climate change is also related to spatio-temporal patterns.

The concept of abrupt climate change is therefore applied for different time scales. For example, changes in climatic regimes were described associated with surface temperature, precipitation, atmospheric circulation in North America during the 1920s and 1960s **?**; **?**. Sometimes, the term 'climate jump' is used instead of 'abrupt climate change', e.g. **?**. Flohn **?** expanded the concept of abrupt climate change to include both singular events and catastrophes such as the extreme El Niño of 1982/1983, as well as discontinuities in paleoclimate indices taken from ice cores and other proxy data. In the instrumental record covering the last 150 years, there is a well-documented abrupt shift of sea surface temperature and atmospheric circulation features in the Northern Hemisphere in the mid-1970s, e.g. **?**; **?**; **?**. Some of the best-known and best-studied widespread abrupt climate changes started and ended during the last deglaciation, most pronounced at high latitudes.

In his classic studies of chaotic systems, Lorenz has proposed a deterministic theory of climate change with his concept of the 'almost-intransitivity' of the highly non-linear climate systems. In this set of equations, exists the possibility of multiple stable solutions to the governing equations, even in the absence of any variations in external forcing Lorenz [1976]. More complex models, e.g. Bryan [1986]; Dijkstra et al. [2004] also demonstrated this possibility. On the other hand, variations in external forcing, such as the changes of incoming solar radiation, volcanic activity,

deglacial meltwater, and increases of greenhouse gas concentration have also been proposed to account for abrupt changes in addition to climate intransitivity **?**; **?**; **?**; **?**, **?**. A particular climate change is linked to the widespread continental glaciation of Antarctica during the Cenozoic (65 Ma to present) at about 34 Ma, e.g. **?**. It should be noted that many facets of regional climate change are abrupt changes although the global means are rather smoothly changing.

Besides abrupt climate change as described in the time domain, we can find abrupt shifts in the frequency domain. A prominent example for an abrupt climate change in the frequency domain is the mid-Pleistocene transition or revolution (MPR), which is the last major 'event' in a secular trend towards more intensive global glaciation that characterizes the last few tens of millions of years. The MPR is the term used to describe the transition between 41 ky (ky=10^3 years) and 100 ky glacial-interglacial cycles which occurred about one million years ago (see a recent review in **?**). Evidence of this is provided by high-resolution oxygen isotope data from deep sea cores, e.g. **?**; **?**.

Another example is the possibility of greenhouse gas-driven warming leading to a change in El Niño events. Modeling studies indicate that a strong enhancement of El Niño conditions in the future is not inconceivable **?**. Such a shift would have enormous consequences for both the biosphere and humans. The apparent phase shift during the 1970s seems unique over this time period, and may thus represent a real climate shift although the available time series is probably too short to unequivocally prove that the shift is significant **?**. The inability to resolve questions of this kind from short instrumental time series provides one of the strongest arguments for extending the instrumental record of climate variability with well-dated, temporally finely resolved and rigorously calibrated proxy data.

The period of regular instrumental records of global climate is relatively short (100-200 years). Even so, this record shows many climatic fluctuations, some abrupt or sudden, as well as slow drifts in climate. Climatic changes become apparent on many temporal and spatial scales. Most abrupt climate changes are regional in their spatial extent. However, regional changes can have remote impacts due to atmospheric and oceanic teleconnections. Some of these shifts may be

termed abrupt or sudden in that they represent relatively rapid changes in otherwise comparatively stable conditions, but they can also be found superimposed on other much slower climatic changes.

The definition of 'abrupt' or 'rapid' climate changes is therefore necessarily subjective, since it depends to a large extent on the sample interval used in a particular study and on the pattern of longer-term variation within which the sudden shift is embedded. It is therefore useful to avoid a too general approach, but instead to focus on different types of rapid transitions as they are detected and modeled for different time periods. Although distinctions between types are somewhat arbitrary, together they cover a wide range of shifts in dominant climate modes on time scales ranging from the Cenozoic (the last 65 millions of years) to the recent and future climate.

In the frequency domain, abrupt climate shifts are due to changes in the dominant oscillations (as in the case of the MPR), or due to a shift in the phase between different climate signals. As an example, the phase between the Indian Monsoon and ENSO exhibits significant shifts for the past 100 years **?**.

9.4.1 Astronomical theory of ice ages

Over the past half million years, marine, polar ice core and terrestrial records all highlight the sudden and dramatic nature of glacial terminations, and the shifts in global climate that occurred as the world passed from dominantly glacial to interglacial conditions, e.g. **?**; **?**. These climate transitions, although probably of relatively minor relevance to the prediction of potential future rapid climate change, do provide the most compelling evidence available in the historical record for the role of greenhouse gas, oceanic and biospheric feedbacks as non-linear amplifiers in the climate system. It is such evidence for the dramatic effect of non-linear feedbacks that relatively minor changes in climatic forcing may lead to abrupt climate response.

A salient feature of glacial-interglacial climate change is furthermore its asymmetry (Fig. 9.13). Warmings are rapid, usually followed by slower descent into colder climate. Given the symmetry of orbital forcings $F(t)$, the cause of rapid warming at glacial 'terminations' must lie in a climate feedback **?**; **?**. Clearly, the asymmetric feedback is due to the albedo (reflectance) of ice and snow
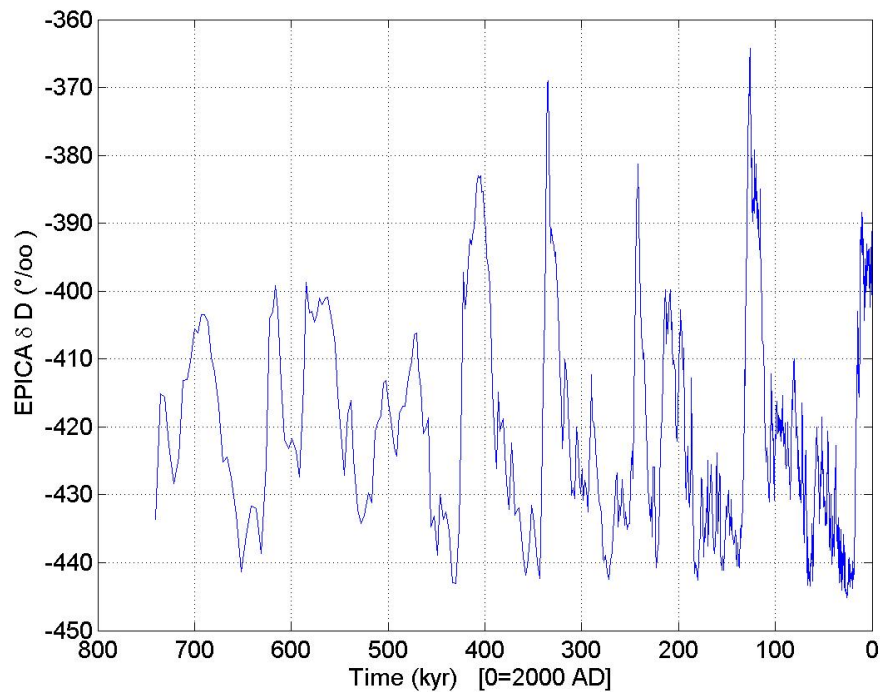
Figure 9.13: Hydrogen isotope record reflecting temperature from a Southern Hemisphere ice core **?** showing the glacial-interglacial changes. Note the asymmetry: the state is longer in the cold (glacials) phases than in the warm phases (interglacials).

changing from high values under glacial climates to low values under warm climates. The albedo feedback helps explain the rapidity of deglaciations and their beginnings in spring and summer. Increased absorption of sunlight caused by lower albedo provides the energy for rapid ice melt. The build-up of snow and ice takes much longer than melting.

Many simplified climate models consist of only a few coupled ordinary differential equations controled by carefully selected parameters. It is generally acknowledged that the 'best' models will be those that contain a minimum of adjustable parameters **?** and are robust with respect to changes in those parameters. Rial **?** formulated a logistic-delayed and energy balance model to understand

the saw-tooth shape in the paleoclimate record: A fast warming-slow cooling is described by

$$\frac{d}{dt}x(t) = R\left(1 - \frac{x(t-\tau)}{K(t)}\right)x(t-\tau) \tag{9.12}$$

$$C\frac{d}{dt}T(t) = Q\left(1 - \alpha(x)\right) - (A + BT) \tag{9.13}$$

with $x(t)$ for the normalized ice extent, τ time delay, $K(t) = 1 + e(t)T(t)$ carrying capacity, $1/R$ response time of the ice sheet, $T(t)$ global mean temperature, $\alpha(x)$ planetary albedo, external parameter $e(t)$, and $R\tau$ bifurcation parameter. A, B, C, Q are constants for the energy balance of the climate. The equation is calibrated so that for $x(t) = 1$ the albedo $\alpha(x) = 0.3$ and $T(t) = 15°C$. With (9.13), saw-toothed waveforms and frequency modulation can be understood ?. The delayed equation yields damped oscillations of $x(t)$ about the carrying capacity for small τ. If τ becomes long compared to the natural response time of the system, the oscillations will become strong, and will grow in amplitude, period and duration. As in the logistic equation for growth, here the product $R\tau$ is a bifurcation parameter, which when crossing the threshold value $\pi/2$ makes the solutions undergo a Hopf bifurcation and settle to a stable limit cycle with fundamental period $\sim 4\tau$?.

The astronomical theory of ice ages –also called Milankovitch ? theory– gained the status of a paradigm for explaining the multi-millennial variability. A key element of this theory is that summer insolation at high latitudes of the northern hemisphere determines glacial-interglacial transitions connected with the waxing and waning of large continental ice sheets, e.g. ?; ?, the dominant signal in the climate record for the last million years. Climate conditions of glacials and interglacials are very different. During the Last Glacial Maximum, about 20,000 years before present, surface temperature in the north Atlantic realm was 10-20°C lower than today ?. A recent study of Huybers and Wunsch ? has shown that the most simple system for the phase of ice volume $x(t)$ is given by

$$x(t+1) = x(t) + \sigma\xi \tag{9.14}$$

with ξ a Gaussian white noise process, but with mean $\mu = 1$, and $\sigma = 2$. ξ represents the unpredictable background weather and climate variability spanning all time scales out to the glacial/interglacial. This highly simplified model posits 1-ky steps in ice volume $x(t)$. The non-zero mean biases the Earth toward glaciation. Once $x(t)$ reaches a threshold, a termination is triggered, and ice-volume is linearly reset to zero over 10 ky. The following threshold condition for a termination makes it more likely for a termination of ice volume to occur when obliquity $\Theta(t)$ is large:

$$x(t) \geq T_0 - a\Theta(t) \quad . \tag{9.15}$$

$\Theta(t)$ has a frequency of about 41 ky, and is furthermore normalized to zero mean with unit variance. The other parameters are: amplitude $a = 15$, $T_0 = 105$. Furthermore, the initial ice volume at 700 ky before present is set to $x(t = -700) = 30$. Equation (9.14) resembles an order-one autoregressive process, similar to (7.57), plus the threshold condition (9.15). Models like (9.14, 9.15) are not theories of climate change, but rather attempts at efficient kinematic descriptions of the data, and that different mechanisms can be consistent with the limited observational records. In the next section, the process of deglaciation is modeled in a three-dimensional model including the spatial dimension.

9.4.2   Antarctic glaciation

During the Cenozoic (65 million years ago (Ma) to present), there was the widespread glaciation of the Antarctic continent at about 34 Ma, e.g. ?. Antarctic glaciation is the first part of a climate change from relatively warm and certainly ice-free conditions to massive ice sheets in both, the southern and northern hemispheres ?. Opening of circum-Antarctic seaways is one of the factors that have been ascribed as a cause for Antarctic climate change so far ?; ?. Besides gateway openings, the atmospheric carbon dioxide concentration is another important factor affecting the evolution of the Cenozoic climate ?; ?. As a third component in the long-term evolution of Antarctic glaciation, land topography is able to insert certain thresholds for abrupt ice sheet build-up.

Whereas tectonics, land topography, and long-term Cenozoic CO_2-decrease act as precondition-ing for Antarctic land ice formation, the cyclicities of the Earth's orbital configuration are super-imposed on shorter time scales and may have served as the ultimate trigger and pacemaker for ice-sheet growth at the Eocene-Oligocene boundary around 34 Ma **?**.

DeConto and Pollard **?** varied Southern Ocean heat transport to mimic gateway opening instead of an explicit simulation of ocean dynamics. They found a predominating role of pCO_2 in the onset of glaciation instead of a dominating tectonic role for 'thermal isolation'.

9.4.3 Mid-Pleistocene revolution

Glaciation in the Northern Hemisphere lagged behind, with the earliest recorded glaciation any-where in the Northern Hemisphere occurring between 10 and 6 Ma and continuing through to the major increases in global ice volume around 2-3 Ma **?**. A recent compilation of 57 globally dis-tributed records **?** is shown in Fig. 9.14. Let us focus now on the mid-Pleistocene transition or revolution (MPR), describing the transition from 41 ky to 100 ky glacial-interglacial cycles.

Milankovitch **?** initially suggested that the critical factor was total summer insolation at about 65°N, because for an ice sheet to grow some additional ice must survive each successive summer. In contrast, the Southern Hemisphere is limited in its response because the expansion of ice sheets is curtailed by the Southern Ocean around Antarctica. The conventional view of glaciation is thus that low summer insolation in the temperate North Hemisphere allows ice to survive summer and thus starts to build up on the northern continents. If so, how then do we account for the MPR? Despite the pronounced change in Earth system response evidenced in paleoclimatic records, the frequency and amplitude characteristics of the orbital parameters which force long-term global climate change, e.g., eccentricity ($\sim$100 ky), obliquity ($\sim$ **41** ky) and precession ($\sim$ **21** and $\sim$ **19** ky), do not vary during the MPR **?**. This suggests that the cause of change in response at the MPR is internal rather than external to the global climate system.

The result of a wavelet spectral analysis (Fig. 9.15) suggests several abrupt climate changes in the frequency domain (shown as schematic arrows in the figure). These abrupt climate shifts repre-
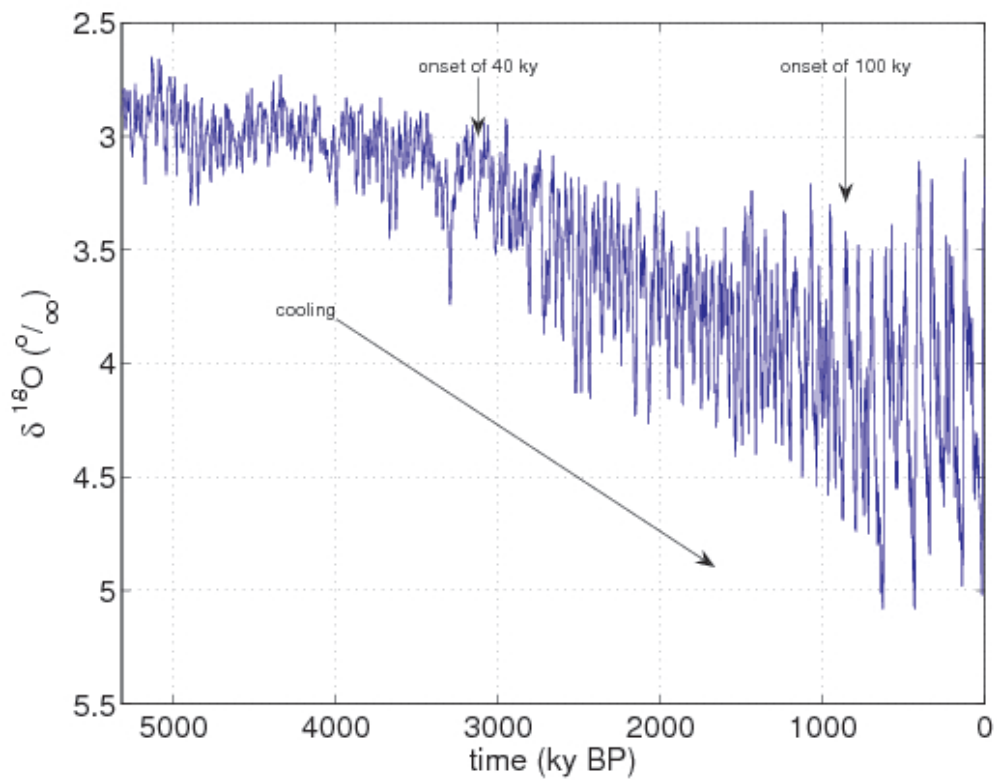
Figure 9.14: A compilation of 57 globally distributed records by Lisiecki and Raymo **?**: The $\delta^{18}O$ record reflects mainly the climate variables temperature and ice volume.
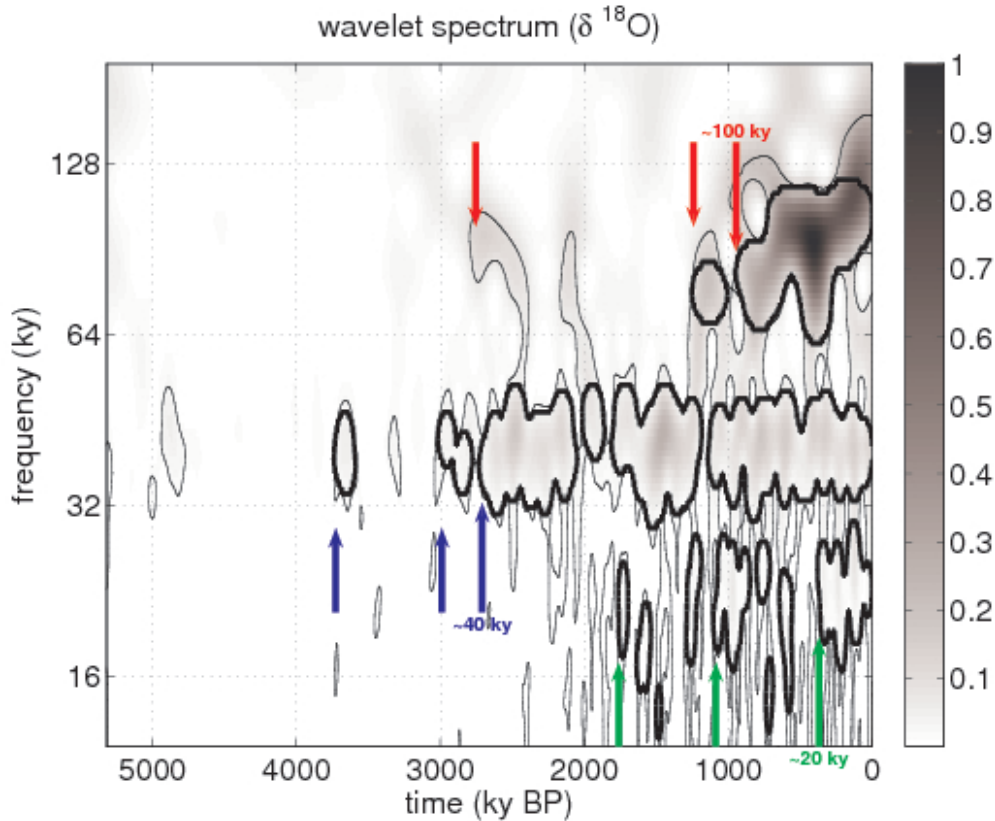
Figure 9.15: Lisiecki and Raymo **?**: The corresponding wavelet sample spectrum calculated using Morlet wavelet with $\omega_0 = 6$. Thin and thick lines surround pointwise and areawise significant patches, respectively.

sent major reorganizations in the climate system. Some of them are possibly linked to the development of Northern Hemisphere ice volume. The MPR marked a prolongation to and intensification of the $\sim$100 ky glacial-interglacial climate. Not only does the periodicity of glacial-interglacial cycles increase going through the MPR, but there is also an increase in the amplitude of global ice volume variations.

It is likely that the MPR is a transition to a more intense and prolonged glacial state, and associated subsequent rapid deglaciation becomes possible. The first occurrence of continental-scale ice sheets, especially on Greenland, is recorded as ice-rafted detritus released from drifting icebergs into sediments of the mid- and high-latitude ocean. After a transient precursor event at 3.2 Ma, signals of large-scale glaciations suddenly started in the subpolar North Atlantic in two

steps, at 2.9 and 2.7 Ma, e.g. **?**.

The ice volume increase may in part be attributed to the prolonging of glacial periods and thus of ice accumulation. The amplitude of ice volume variation is also accentuated by the extreme warmth of many interglacial periods. Thus, a colder climate with larger ice sheets should have the possibility of a greater sudden warming **?**. The MPR therefore marks a dramatic sharpening of the contrast between warm and cold periods. Note however, that the amount of energy at 40 ka period is hardly changed in the time after 1 Ma, and notably, one sees the addition of energy at longer periods, without any significant reduction in obliquity-band energy. After about 1 Ma, large glacial-interglacial changes begin to occur on an approximately 100 ka time scale (but not period-ically) superimposed upon the variability which continues largely unchanged **?**. Why did 100 ka glacial-interglacials also become possible in addition to the ice volume variability? Lowering of global CO_2 below some critical threshold, or changes in continental configuration, or atmospheric circulation patterns, or all together, are among the conceivable possibilities, e.g. **?**.

9.5   Carbon cycle and isotopes in the ocean

Two aspects of the deep circulation are especially important for understanding Earth's climate and its possible response to increased carbon dioxide CO_2 in the atmosphere:

- The ability of cold water to absorb CO_2 from the atmosphere, and

- the ability of deep currents to modulate the heat transported from the tropics to high latitudes.

The oceans are the primary reservoir of readily available CO_2 , an important greenhouse gas. The oceans contain 40,000 GtC of dissolved, particulate, and living forms of carbon. The land contains 2,200 GtC, and the atmosphere contains only 750 GtC. Thus the oceans hold 50 times more carbon than the air. Furthermore, the amount of new carbon put into the atmosphere since the industrial revolution, 150 GtC, is less than the amount of carbon cycled through the marine ecosystem in five years. (1 GtC = 1 gigaton of carbon = 10^{12} kilograms of carbon.) Carbonate

rocks such as limestone, the shells of marine animals, and coral are other, much larger, reservoirs. But this carbon is locked up. It cannot be easily exchanged with carbon in other reservoirs. More CO_2 dissolves in cold water than in warm water. Just imagine shaking and opening a hot can of tea. The CO_2 from a hot can will spew out far faster than from a cold can. Thus the cold deep water in the ocean is the major reservoir of dissolved CO_2 in the ocean.

New CO_2 is released into the atmosphere when fossil fuels and trees are burned. Very quickly, 48% of the CO_2 released into the atmosphere dissolves in the cold waters of the ocean, much of which ends up deep in the ocean. Forecasts of future climate change depend strongly on how much CO_2 is stored in the ocean and for how long. If little is stored, or if it is stored and later released into the atmosphere, the concentration in the atmosphere will change, modulating Earth's long-wave radiation balance. How much and how long CO_2 is stored in the ocean depends on the deep circulation and the net flux of carbon deposited on the seafloor. The amount that dissolves depends on the temperature of the deep water, the storage time in the deep ocean depends on the rate at which deep water is replenished, and the deposition depends on whether the dead plants and animals that drop to the sea floor are oxidized. Increased ventilation of deep layers, and warming of the deep layers could release large quantities of the gas to the atmosphere. The storage of carbon in the ocean also depends on the dynamics of marine ecosystems, upwelling, and the amount of dead plants and animals stored in sediments.

In contrast to investigations of present-day climate systems, an investigation of past climate lacks the possibility of measuring chemical, physical or biological properties directly. Therefore proxies are needed. In paleoceanography, elements and their isotopes incorporated in marine microorganisms (e.g. cadmium, strontium, oxygen) serve well for drawing a picture of the climate conditions of different time periods [?]. The ratio of the carbon isotope ^{13}C to ^{12}C in benthic foraminifers contains information about the amount of nutrients which were available to the phytoplankton upon uptake of the carbon. Furthermore the distribution of ^{13}C in an ocean reveals the basic ocean circulation and the net flow of water masses in that ocean. In this section I will introduce $\delta^{13}C$ as an important proxy for the paleo ocean and describe mechanisms for its distribution

pattern in the ocean.

Carbon consists of two stable isotopes: ^{12}C and ^{13}C. The former is used as the reference mass for the atomic weight scale and has a natural abundance of 98.89% while the latter has a natural abundance of 1.11%. Chemical behaviour of an element is determined by its electronic structure. Differences in chemical or physical properties of isotopes of the same element are smaller than between different elements and arise from variations in the atomic mass. During chemical processes one isotope is preferred to another. This is called fractionation. Such a fractionation occurs during equilibrium exchange reactions, for instance. For the air-sea exchange of CO_2 through the surface ocean interface an equilibrium is attained between atmospheric carbon dioxide and dissolved bicarbonate. Due to the more inert chemical behaviour of the "heavier" isotope the ocean is enriched in ^{13}C by about 8‰ compared to the atmosphere [**??**]. This effect is temperature dependent with -0.1‰ per K [**?**]. Thus, in equilibrium colder waters are enriched in ^{13}C compared to warmer waters. Though, the time required by the ocean to reach isotopical equilibrium with the atmosphere is much longer than an average residence time of carbon in the surface water of the ocean [**??**]. As a result, ^{13}C isotopes of dissolved inorganic carbon (DIC) in ocean surface waters generally never reach isotopic equilibrium with CO_2 of the atmosphere. Hence, different SSTs and residence times on the surface lead to regional variations of ^{13}C in surface DIC.

In geochemistry it is common to express the isotopic composition in terms of "delta" values.

$$\delta_A := \left(\frac{R_A}{R_{St}} - 1 \right) \cdot 10^3 \tag{9.16}$$

In this equation $\frac{R_A}{R_{St}}$ denotes the ratio of the number of isotopes of a substance to a standard value.

A second process alters the chemical and isotopical composition of the ocean surface water: the uptake of carbon by phytoplankton used for photosynthesis. This represents a kinetic fractionation process and is primarily caused by differences in reaction rates of isotopic molecules. Photosynthesis consists of two subprocesses: The uptake and intracellular diffusion of CO_2 as well as the biosynthesis of cellular components. The synthesised organic material has isotopic compositions of -20‰ to -30‰ [**?**]. In the same time as CO_2, nutrients are taken up as well. In cases where
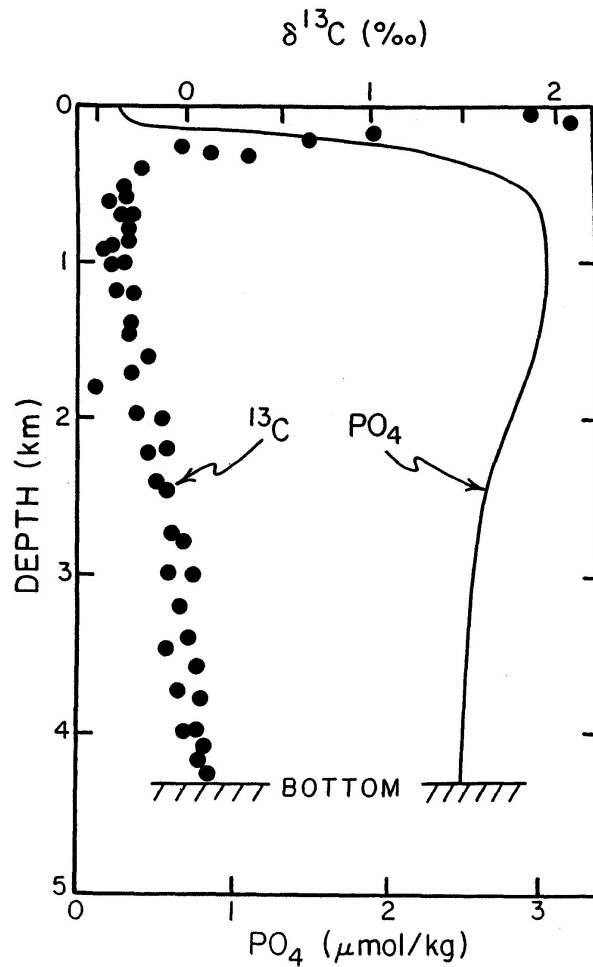
Figure 9.16: This figure displays $\delta^{13}C$ (‰) and PO_4 (μ mol kg^{-1}) versus depth (km) of the GEOSECS station 346 (28°N 121°W) From **?**.

the ocean waters are completely depleted in nutrients, concentration of DIC is reduced by about 10%. This results in an increase of $\delta^{13}C$ by about 2‰ to 3‰ in the surface waters compared to the deep ocean. Complete nutrient utilisation by phytoplankton generally occurs in the warm, well stratified regions of the oceans between 30°N and 30°S. **?** found $\delta^{13}C$-values of 2.2‰ in warm, tropical surface ocean water.

The simultaneous uptake of isotopically lighter carbon and nutrients, nitrogen and phosphorus, results in a near mirror image of the distribution of the nutrients to the carbon isotope ratios distribution (Figure 9.16) [**?**]. In fact, when $\delta^{13}C$ is plotted against PO_4, the values fall close to a
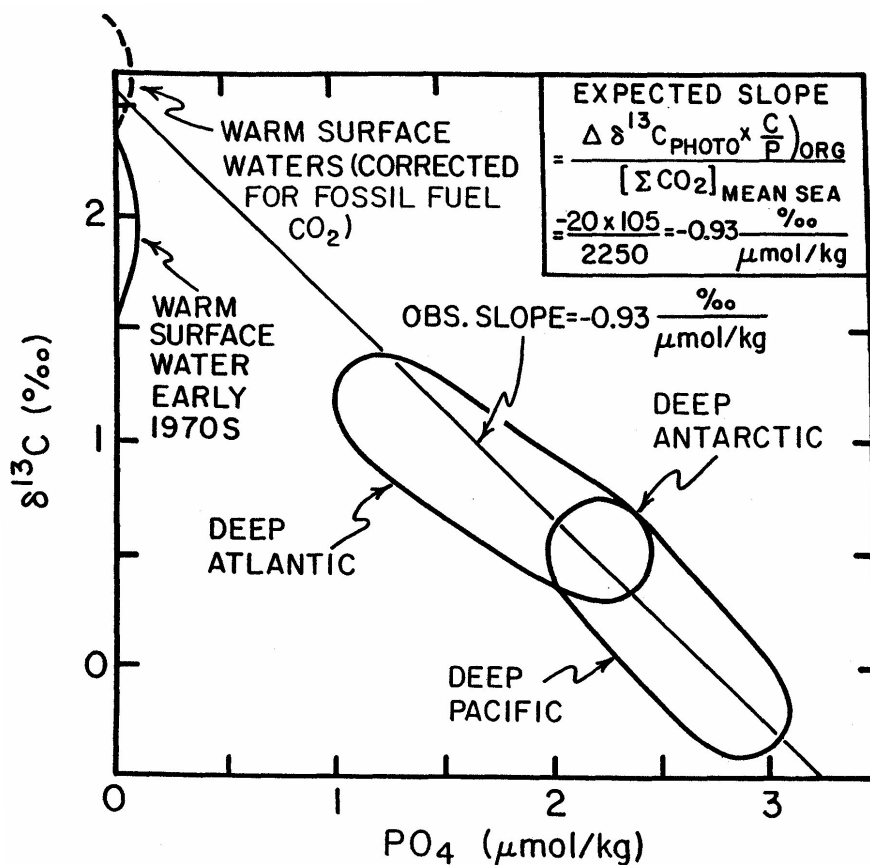
Figure 9.17: Relationship between $\delta^{13}C$ (‰) and PO_4 (μ mol kg^{-1}) in the world ocean (from **?**).

straight line with a slope of -0.93‰ $\delta^{13}C$ change per μ mol kg^{-1} PO_4 change (Figure 9.17). Thus, nutrient depleted waters are enriched in $\delta^{13}C$ while low values of $\delta^{13}C$ correlate with waters of high nutrient content. For this reason $\delta^{13}C$ is used as a tracer for nutrients in ocean water.

Besides photosynthesis a fractionation also occurs upon the formation of calcium carbonate. However, according to **?** this effect is negligible, so that temperature has almost no influence on $\delta^{13}C$ of marine calcite. Therefore phytoplankton reveals only changes in $\delta^{13}C$ of the waters in which they grew.

9.5.1 The water mass tracer δ^{13}C

After leaving the euphotic zone (e.g. formation of NADW in the North Atlantic) δ^{13}C of the ocean water is only influenced by 2 different processes: Remineralisation of particulate organic matter (POM) and mixing with other water masses. δ^{13}C of deep water generally decreases when it is moved from the position of its most recent ventilation, due to *in situ* oxidation of POM settling from the surface. POM, detrital remains of plankton of the euphotic zone, sinks down and, depending on its residence time in the water column, becomes degraded to a different degree. During the process, labile, ^{13}C-enriched amino acids and sugars are degraded first while the isotopically light lipids are left behind [**??**]. Thus, the "biological pump" creates a vertical gradient in the ocean with high δ^{13}C values close to the surface and nutrient-rich, ^{13}C-depleted deep waters. Generally, below the main thermocline of the oceans, the δ^{13}C distribution in deep waters is decoupled from that in surface waters. Only in regions of deep water formation the distribution is homogenous from the surface to the bottom of the ocean, due to convection [**?**]. In summary, the longer deep waters do not have contact to the surface of the ocean the more the isotopic signal δ^{13}C decreases.

The distribution of δ^{13}C in the present-day ocean displays the general distribution of water masses in the oceans. The net flow directions between ocean basins are represented by gradients of δ^{13}C. The first truly global measurements of δ^{13}C of dissolved inorganic carbon in the ocean were performed by the Geochemical Ocean Sections Study (GEOSECS) project [**???**]. Its results have drawn an interesting picture of the ^{13}C distribution in the ocean: Following the "conveyor belt" of the global ocean circulation and starting from the formation of NADW in the North Atlantic, δ^{13}C decreases continuously. The isotopic signal globally covers a range of 3‰. North Atlantic Deep Water is formed at about 1‰ - 1.5‰ and becomes gradually lower in δ^{13}C as it travels southward and mixes with water from the Southern Ocean, which has an average δ^{13}C value of 0.3‰ [**?**]. Mixing of NADW and AABW occurs at bathymetric gradients of δ^{13}C at clearly defined water mass boundaries [**?**]. Furthermore the confluence of ^{13}C-enriched NADW water with ^{13}C-depleted water of the deep Pacific in the Southern Ocean can clearly be identified by measurements, before forming AABW. There upwelling of waters from the deep circumpolar

current occurs which cools rapidly when it reaches the surface, due to heat exchange with sea ice and atmosphere. Formation of sea ice then leads to higher salinities which increases downwelling and thus, formation of AABW [?]. The residence time of these waters at the surface is short, so that neither CO_2 exchange with the atmosphere nor phytoplankton utilisation occurs [?]. However, mixing is strong and the whole southern ocean deep water mass below 1,500m has fairly homogenous $\delta^{13}C$ values south of the polar front [??]. Pacific bottom waters are farthest from the NADW source region on the "conveyor belt" of the global ocean circulation and therefore contain globally the lowest $\delta^{13}C$ values of 0.2‰ to -0.2‰ due to continuous oxidation of ^{13}C poor organic material.

Consequently, the global composition of $\delta^{13}C$ in ocean water or within an ocean basin, does not represent a conservative tracer for water masses if there is a high productivity rate of organic matter at the surface or long residence time of the deep water. The deep Pacific is an example of that. About 0.4‰ lower $\delta^{13}C$ values are measured than expected for a conservative tracer [?]. This discrepancy represents the so-called "age effect". The anomaly for $\delta^{13}C$ between North Atlantic Deep Water and Pacific Deep Water is about 1.2‰ where mixing accounts for a difference of about 0.8‰ and 0.4‰ which derives from the "age effect". A similar situation exists in the Eastern Atlantic. In the western basin of the South Atlantic the main body of the AABW flows northward at depth greater than 4,000m. It is driven eastward once it crosses the equator by Coriolis force, but the Mid-Atlantic Ridge presents a physical barrier to bottom water circulation at low and high latitudes in the Atlantic, almost isolating the deep East Atlantic from the thermohaline circulation. Here as well a longer residence time yields in an "age effect", resulting in a drop of $\delta^{13}C$ by about 0.2‰ [?]. On the other hand, production rates and degradation rates of POM in the present-day West Atlantic are small, so that $\delta^{13}C$ can be trusted as a conservative tracer [?].

Principally $\delta^{13}C$ does not serve as a tracer for present-day ocean circulation, but for the reconstruction of past condition in the ocean. Carbon isotopic composition of benthic foraminifers has proven to be a reliable source of information to measure changes in deep water circulation and intensity. In particular, the species *C. wuellerstorfi* became the scientific base of deep water carbon

chemistry [**?**]. Due to global changes in the distribution of carbon between the ocean and transient reservoirs in past climates, the $\delta^{13}C$ values incorporated in benthic foraminifers are significantly different to today. Because the mixing time for the ocean is of a timescale of 1,000 years, all foraminifers of present-day measurements are affected in the same way. So only the mean $\delta^{13}C$ is changed [**?**]. Furthermore reconstructions of the deep water and bottom water circulation using $\delta^{13}C$ generally are in good agreement with the results of other non-conservative tracer. Hence, nowadays $\delta^{13}C$ is widely used to describe paleo productivity and deep ocean circulation of the late Quaternary climate cycles [**?**].

Figure 9.18 displays the distribution of $\delta^{13}C$ in the Atlantic. The vertical cross-section follows the cruise track of the Geochemical Ocean Sections Study (GEOSECS) project.

North of 60°N, $\delta^{13}C$ shows a small vertical gradient with values of 1.5‰ at the surface to about 1‰ in the deep North Atlantic. Although $\delta^{13}C$ is not entirely homogeneous in this region, as found by **?**, it can be clearly identified as the source region of NADW.

At depths about 1,500 - 3,000m in the Atlantic between 60°N and 30°S, the meridional gradient of $\delta^{13}C$ is small compared to other depths, indicating a strong southward movement of NADW waters [**?**]. Nonetheless $\delta^{13}C$ of water below 1,500m depth decreases towards the south, reaching a minimum of -0.5‰ at depths between 1,000 - 2,500m at 60°S. This might be the inflow of deep Pacific water into the Southern Ocean [**?**], although its values are by 0.3‰ too low compared to measured values [**?**]. The vertical gradient between surface waters and waters at intermediate depth at 60°S exceeds 1.5‰ in the top 1,500m. This points to the fact that these waters are much less mixed than those in the North Atlantic at 60°N. Therefore AABW formation is weak compared to NADW formation [**?**]. $\delta^{13}C$ values of less than 0.5 ‰ are found below 3,000m depth in the area between 30°S and the equator and as far as 40°N below 3,500m. This points to an intrusion of Southern Ocean water into the Atlantic at depth below 3,000m.

Maximal values of $\delta^{13}C$ of about 2.1‰ are found in surface waters of the tropics and subtropics, although values at the equator are slightly lower. This is in good agreement with data obtained by the GEOSECS project [**??**].
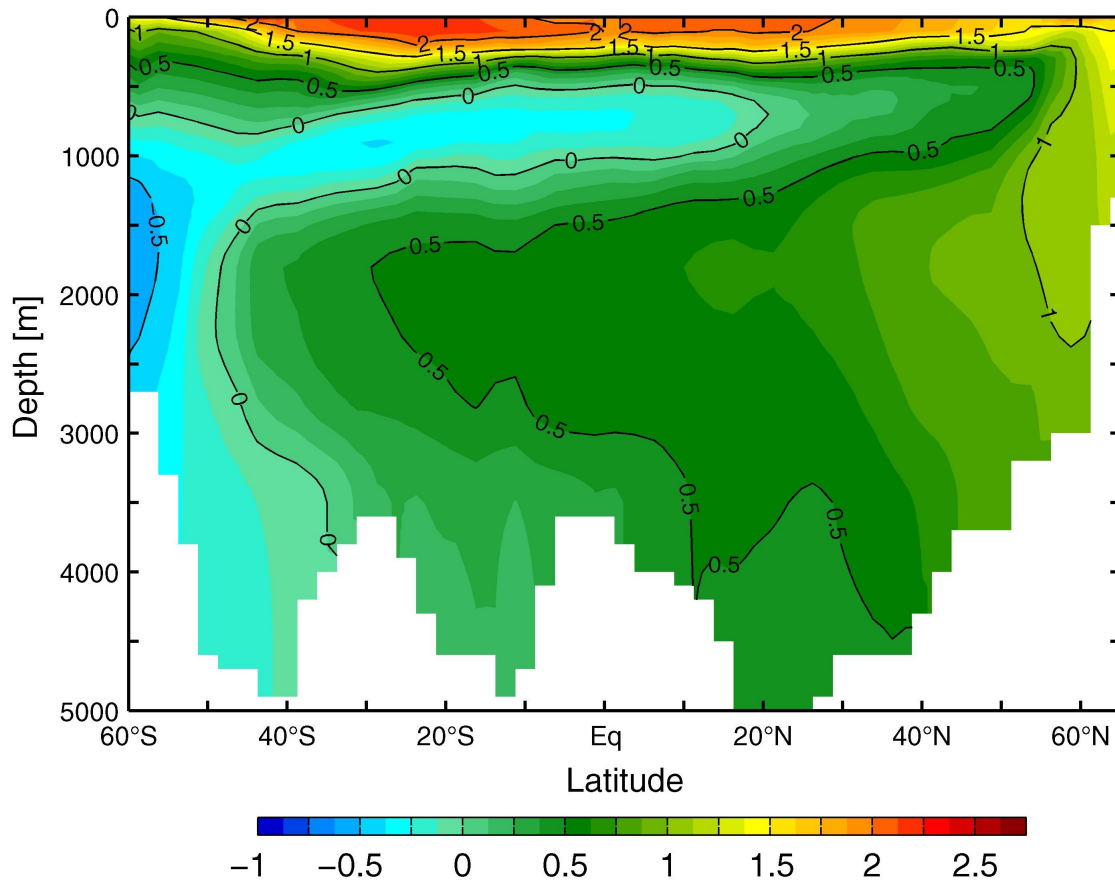
Figure 9.18: A cross-section along the GEOSECS cruise of the distribution of δ^{13}C is displayed in this figure. Values are in ‰.

The only unexpected feature of the δ^{13}C distribution of the Atlantic is a low-δ^{13}C water mass at intermediate depth, reaching from the Southern Ocean with δ^{13}C values of about -0.4‰ at 1,200m depth to the North Atlantic at about 50°N and δ^{13}C values of 0.5‰ at 500m depth. Antarctic Intermediate Water is the water mass dominating the South Atlantic at intermediate depths. It derives from the Southern Ocean and flow northward, reaching the equator. **?** measured δ^{13}C values of 1‰ to 1.8‰ for waters of the AAIW in the Brazil basin. Therefore the modelled δ^{13}C distribution at intermediate depths does not agree with measured values at all and thus, is considered an artefact.

9.5.2 Carbon Cycle Model

In order to model distributions of the carbon isotopes ^{12}C and ^{13}C in the ocean, a global marine carbon cycle model is needed. The Hamburg Ocean Carbon Cycle Model (HAMOCC, originally developed by **?**) in its refined version HAMOCC 2s [**?**] is employed for that purpose. The model simulates the inorganic carbon cycle, partly the organic carbon cycle as well as the silicate cycle.

It consists of three reservoirs: atmosphere, water column and lithosphere. The simple atmosphere model implemented in HAMOCC serves the purpose of exchanging carbon dioxide and oxygen with the surface ocean, and the meridional transport of CO_2. Inter-hemispheric mixing has a timescale of about one year while the atmosphere is considered to be zonally well mixed.

Prognostic tracer variables in the water column are alkalinity (ALK), dissolved inorganic carbon (DIC, equivalent to total CO_2), phosphate, oxygen, dissolved organic carbon (DOC) and silicic acid. Those of the simulated porewater in the bioturbated sediment are identical to those in the water column, except for DOC, while in the sediment the tracer variables are calcium carbonate, opal, organic carbon and clay.

Weathering fluxes are prescribed and matter enters the ocean globally. The cycle is closed by accumulation in the sediment. Therein the output rate from the water column asymptotically matches the input rate of terrigeneous matter. However, the spartial distribution of the sedimented matter can differ significantly from the input field. Therefore inventories of the model are variable, but stabilise in a steady-state.

The internal redistribution of tracers within the ocean is based on velocity fields and thermohaline fields of the Hamburg Large-Scale Geostrophic Ocean General Circulation Model (LSG OGCM, see above). Hence, they are not computed explicitly in HAMOCC. Convection adjustment and sea-ice cover are provided by LSG as well. For that reason the grid, the spartial and temporal resolution of the carbon cycle model as well as the advection scheme and topography are chosen identical to the OGCM.

The carbon cycle basic structure is according to **?** while the sediment simulation bases on **?**. The so-called "biological pump" is modelled in three components: The "soft tissue pump" or

organic carbon pump caused by formation of organic material and depletion of carbon and nutrients in surface waters, the calcium carbonate counterpump as well as biogenic silica (opal) pump. These three species of biogenic particles, particulate organic carbon (POC), calcium carbonate and biogenic silica, are produced in the euphotic zone. In the model the euphotic zone is represented by the uppermost ocean layer (0 - 50m). Silicic acid and phosphate are chosen as the biolimiting nutrients upon production. Biological export production of POC and opal is assumed to follow the Michaelis-Menten nutrient uptake kinetics [**?**]:

POC

$$P_{POC} = \frac{v_{max}^{POC} \cdot \left[PO_4^{3-}\right]^2 \cdot Red(C:P)}{K_s^{POC} + \left[PO_4^{3-}\right]} \tag{9.17}$$

where P_{POC} denotes the POC export production (mol$\,$l$^{-1}\,$a^{-1}), v_{max}^{POC} the maximum uptake rate of phosphate (a^{-1}), K_s^{POC} the half saturation constant for POC production (mol$\,$l^{-1}), $\left[PO_4^{3-}\right]$ the concentration of phosphate in the euphotic zone (mol$\,$l^{-1}) and $Red(C:P)$ the Redfield ratio C:P [**?**].

Opal

$$P_{opal} = \frac{v_{max}^{opal} \cdot [Si(OH)_4]^2}{K_s^{opal} + [Si(OH)_4]} \tag{9.18}$$

where P_{opal} describes the opal export production (mol$\,$l$^{-1}\,$a^{-1}), v_{max}^{opal} the maximum uptake rate of silicic acid (a^{-1}), K_s^{opal} the half saturation constant for opal production (mol$\,$l^{-1}) and $[Si(OH)_4]$ the concentration of silicic acid in the euphotic zone (mol$\,$l^{-1}). The uptake rate of silicic acid is assumed to be faster than that of phosphate.

The export production of CaCO$_3$ is linked to the local production ratio P_{opal}/P_{POC}. As long as the ratio does not exceed a threshold value S_{opal}, CaCO$_3$ is produced according to the following equation

$$P_{CaCO_3} = P_{POC} \cdot R \cdot \left(1 - \frac{P_{opal}/P_{POC}}{S_{opal}}\right) \tag{9.19}$$

where R marks the maximum possible rain ratio $C(CaCO_3):C(POC)$.

Fractionation of the carbon isotopes ^{12}C and ^{13}C upon organic matter production is also implemented in the model. A constant fractionation factor of -20‰ for present day and -18‰ for the LGM [?] is assumed.

When particulate organic matter (POM) leaves the euphotic zone, remineralisation takes place during the settling process. This is parameterised by an exponential redistribution profile in which both sinking velocity and redissolution rate are implicitly included:

$$F(z) = P(z_0) \cdot e^{(z-z_0)/d_p} \tag{9.20}$$

In this equation $F(z)$ denotes the particle flux at depth z, $P(z_0)$ the export production, $z_0 = 50m$ the depth of the euphotic zone and d_p marks the depth where flux is $1/e$ of export production. For the remineralisation, oxygen is utilised. If oxygen concentration drops below a certain threshold, no remineralisation takes place.

POM which leaves the deepest layer of the water column reaches the uppermost bioturbated sediment layer. These two layers also exchange concentration via pore water, which is modelled in HAMOCC. All matter deposited on the sediment, which is not redissolved, is accumulated.

Beside organic carbon cycle and silicate cycle, the inorganic carbon cycle is implemented in the model [?]. For the computation of the CO_2 partial pressure, which is equivalent to the free carbon dioxide in water H_2CO_3, the following reactions have to be considered:

$$H_2CO_3 \quad \rightleftharpoons \quad HCO_3^- + H^+ \tag{9.21}$$
$$HCO_3^- \quad \rightleftharpoons \quad CO_3^{2-} + H^+ \tag{9.22}$$
$$H_2O + B(OH)_3 \quad \rightleftharpoons \quad B(OH)_4^- + H^+ \tag{9.23}$$
$$H_2O \quad \rightleftharpoons \quad OH^- + H^+ \tag{9.24}$$
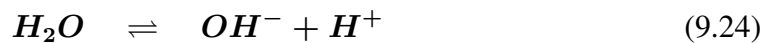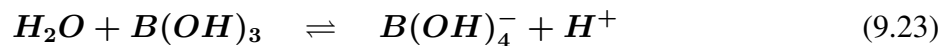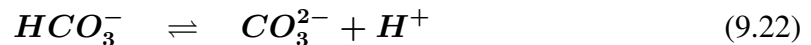
With the help of the total dissolved inorganic carbon (DIC), total boron (B_T) and total alkalinity (ALK)

$$[DIC] = [H_2CO_3] + [HCO_3^-] + [CO_3^{2-}] \tag{9.25}$$

$$[B_T] = [B(OH)_3] + [B(OH)_4^-] \tag{9.26}$$

$$[ALK] = [HCO_3^-] + 2 \cdot [CO_3^{2-}] + [B(OH)_4^-] + [OH^-] - [H^+] \tag{9.27}$$

all other reaction partners can be calculated.

All dissolved and suspended water column tracers are transported with the ocean velocity field using the tracer transport equation

$$\frac{\partial c}{\partial t} = -div(\vec{v} \cdot c) - q \tag{9.28}$$

where c is the tracer concentration, t time, $\vec{v}$ the velocity vector and q the sinks and sources. Local sinks are the marine biota as described above as well as the gas exchange with the atmosphere. The net flux of carbon isotopes ^{12}C and ^{13}C (F^{12} and F^{13}) through the air-sea interface is proportional to the difference between the atmosphere and ocean partial pressure of CO_2 [?]:

$$F^{12} = F_{as} - F_{sa} = \lambda(P_a - P_s) \tag{9.29}$$

$$F^{13} = \alpha_{as}F_{as}R_a^{13} - \alpha_{sa}F_{sa}R_s^{13} \tag{9.30}$$

where $\lambda = 0.06\, mol\, m^{-2}\, ppm^{-1}\, a^{-1}$ denotes the bulk coefficient, P_a and P_s the partial pressure of CO_2 in the atmosphere and the ocean and R_a^{13} and R_s^{13} the $^{13}C/^{12}C$ ratio of the atmosphere and the ocean, respectively. The isotopic fractionation through the air-sea interface is determined by the fractionation factors

$$\alpha_{as} = [-0.373/(T_s + 273.15) + 1.00019]\alpha_k \tag{9.31}$$

$$\alpha_{sa} = [1.02389 - 9.483/(T_s + 273.15)]\alpha_k \tag{9.32}$$

where T_s is the SST in °C and $\alpha_k = 0.9995$ is the kinetic fractionation factor [?].

9.5.3   Carbon isotope clock

Scientists have different tools to date samples, ranging from layer counting in trees and some ice cores, to the analysis of radiogenic isotopes the decay of which forms a clock marking long intervals of time. Denote $C(t)$ the concentration of the radiogenic isotopes ^{14}C at time t, the dynamics is as follows:

$$\frac{d}{dt}C(t) = -\lambda C + P \tag{9.33}$$

with P as the cosmic production rate and decay constant $\lambda = 1/(5730$ years$)$. Assume a constant production rate. Measuring the concentration at time t with respect to t_0, one can calculate the time difference

$$t - t_0 = -1/\lambda \, \ln\left(\frac{^{14}C(t)}{^{14}C(t_0)}\right) \tag{9.34}$$

Knowing the time between different samples (through (9.34)), and the magnitude of change observed in the proxy, the rate of change can be calculated. Uncertainties in the proxies, and uncertainties in the dating, are the main reasons that abrupt climate change is one of the more difficult topics in the field of paleoclimatology.

Exercise 74   –   Age of a water mass

Carbon-14 ages *(10 points)*

The number of decays per time is proportional to the current number of radioactive atoms. This is expressed by the following differential equation, where N is the number of radioactive atoms and

λ is a positive number called the decay constant:

$$\frac{dN}{dt} = -\lambda N \tag{9.35}$$

$\frac{1}{\lambda}$ is the radiocarbon mean- or average-life = 8033 years (Libby value)

a) Calculate the radiocarbon half-life based on Libby's value of $\frac{1}{\lambda}$! The correct half-life is $5,730 \pm 40$ years.

b) Plants take up atmospheric carbon dioxide by photosynthesis, and are ingested by animals, so every living thing is constantly exchanging carbon-14 with its environment as long as it lives. Once it dies, however, this exchange stops, and the amount of carbon-14 gradually decreases through radioactive beta decay. Calculate a raw radiocarbon date from N and an initial condition of N_0 for radioactive atoms at $t = 0$! Why is it difficult to report C-14 ages greater than 60,000 years?

c) Typical values for δ^{14}C are $-50\%_o$ in the mixed layer ocean with depth 50m and $-150\%_o$ in deep water (layer depth 4 km). What are the ^{14}C ages of surface and deep water?

d) Measurements of air-sea exchange fluxes suggest that the residence time of CO_2 in the mixed layer is about 5 years. Do you have an explanation for the difference?

9.6 Kepler orbit and the Earth-Sun geometry

In celestial mechanics, a Kepler orbit (or Keplerian orbit) describes the motion of an orbiting body as an ellipse, parabola, or hyperbola, which forms a two-dimensional orbital plane in three-dimensional space. The movement of the Earth and Moon around the Sun is fundamental for the annual cycle, tides as well on ice ages. n 1601, Johannes Kepler acquired the extensive, meticulous observations of the planets made by Tycho Brahe. Kepler would spend the next five years trying to fit the observations of the planet Mars to various curves. In 1609, Kepler published the first two of his three laws of planetary motion. The first law states: "The orbit of every planet is an ellipse with the sun at a focus."

More generally, the path of an object undergoing Keplerian motion may also follow a parabola or a hyperbola, which, along with ellipses, belong to a group of curves known as conic sections. Mathematically, the distance between a central body and an orbiting body can be expressed as:

$$r(\varphi) = \frac{a(1 - \varepsilon^2)}{1 + \varepsilon \cos(\varphi)} \tag{9.36}$$

where

- r is the distance

- a is the semi-major axis, which defines the size of the orbit

- ε is the eccentricity, which defines the shape of the orbit

- φ is the true anomaly, which is the angle between the current position of the orbiting object and the location in the orbit at which it is closest to the central body (called the periapsis)

Newton's law of gravitation states: Every point mass attracts every other point mass by a force pointing along the line intersecting both points. The force is proportional to the product of the two masses and inversely proportional to the square of the distance between the point masses:

$$F = G\frac{m_1 m_2}{r^2} \tag{9.37}$$

where:

- F is the magnitude of the gravitational force between the two point masses

- G is the gravitational constant

- m_1 is the mass of the first point mass

- m_2 is the mass of the second point mass

- r is the distance between the two point masses

From the laws of motion and the law of universal gravitation, Newton was able to derive Kepler's laws, demonstrating consistency between observation and theory. The laws of Kepler and Newton formed the basis of modern celestial mechanics until Albert Einstein introduced the concepts of special and general relativity in the early 20th century. For most applications, Keplerian motion approximates the motions of planets and satellites to relatively high degrees of accuracy and is used extensively in astronomy and astrodynamics.

To solve for the motion of an object in a two body system, two simplifying assumptions can be made:

1. The bodies are spherically symmetric and can be treated as point masses.

2. There are no external or internal forces acting upon the bodies other than their mutual gravitation.

The shapes of large celestial bodies are close to spheres. By symmetry, the net gravitational force attracting a mass point towards a homogeneous sphere must be directed towards its centre. The shell theorem (also proven by Isaac Newton) states that the magnitude of this force is the same as if all mass was concentrated in the middle of the sphere, even if the density of the sphere varies with depth (as it does for most celestial bodies). From this immediately follows that the attraction between two homogeneous spheres is as if both had its mass concentrated to its center.

Smaller objects, like asteroids or spacecraft often have a shape strongly deviating from a sphere. But the gravitational forces produced by these irregularities are generally small compared to the

gravity of the central body. The difference between an irregular shape and a perfect sphere also diminishes with distances, and most orbital distances are very large when compared with the diameter of a small orbiting body. Thus for some applications, shape irregularity can be neglected without significant impact on accuracy.

Planets rotate at varying rates and thus may take a slightly oblate shape because of the centrifugal force. With such an oblate shape, the gravitational attraction will deviate somewhat from that of a homogeneous sphere. This phenomenon is quite noticeable for artificial Earth satellites, especially those in low orbits. At larger distances the effect of this oblateness becomes negligible. Planetary motions in the Solar System can be computed with sufficient precision if they are treated as point masses.

Two point mass objects with masses m and M and position vectors $\mathbf{r_1}$ and $\mathbf{r_2}$ relative to some inertial reference frame experience gravitational forces:

$$m_1 \ddot{\mathbf{r}}_1 = \frac{-GmM}{r^2}\hat{\mathbf{r}} \tag{9.38}$$

$$m_2 \ddot{\mathbf{r}}_2 = \frac{GmM}{r^2}\hat{\mathbf{r}} \tag{9.39}$$

where $\mathbf{r}$ is the relative position vector of mass 1 with respect to mass 2, expressed as:

$$\mathbf{r} = \mathbf{r_1} - \mathbf{r_2} \tag{9.40}$$

and $\hat{\mathbf{r}}$ is the unit vector in that direction and r is the length of that vector. Dividing by their respective masses and subtracting (9.39) from (9.38) yields the equation of motion for the acceleration of the first object with respect to the second:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^2}\hat{\mathbf{r}} \tag{9.41}$$

where μ is the gravitational parameter and is equal to

$$\mu = G(m + M) \tag{9.42}$$

In many applications, a third simplifying assumption can be made:

3. When compared to the central body, the mass of the orbiting body is insignificant. Mathematically, $M >> m$, so $\mu = G(m + M) \approx GM$.

This assumption is not necessary to solve the simplified two body problem, but it simplifies calculations, particularly with Earth-orbiting satellites and planets orbiting the Sun. Even Jupiter's mass is less than the Sun's by a factor of 1047, which would constitute an error of 0.096% in the value of μ. Notable exceptions include the Earth-moon system (mass ratio of 81.3), the Pluto-Charon system (mass ratio of 8.9) and binary star systems.

The differential equation for the two body case can be completely solved mathematically and the resulting orbit which follows Kepler's laws of planetary motion is called a "Kepler orbit". Using the energy conservation per mass

$$E = \frac{\dot{x}^2}{2} + V(x) \tag{9.43}$$

one can derive an equation for the velocity $\dot{x}$:

$$\dot{x} = \sqrt{2(E - V(x))} \quad \rightarrow \quad t = \int^x dx' \frac{1}{\sqrt{2(E - V(x'))}} \tag{9.44}$$

For our case for the planetary motion, we have for the gravitational force per mass

$$\frac{F_G}{m} = -dV(r)/dr \quad \text{and} \quad V(r) = -GM/r \tag{9.45}$$

leading to

$$E = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + V(r) \quad . \tag{9.46}$$

Because the force has no $\varphi-$dependence, we have conservation of angular momentum l per mass:

$$\frac{d}{dt}\mathbf{l} = \frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = (\dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}) = \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times f(r)\mathbf{r} = 0 \tag{9.47}$$

with the gravitational function $f(r) = -\frac{GM}{r^3}$. The conservation of angular momentum is a special feature for all central body forces with $\mathbf{F} \sim \mathbf{r}$ and

$$l = r^2\dot{\varphi} \quad . \tag{9.48}$$

Since the cross product of the position vector and its velocity stays constant, they must lie in the same plane, orthogonal to $\mathbf{l}$. This implies the vector function is a plane curve.

The energy equation (9.46) can be rewritten as

$$E = \frac{1}{2}\left(\dot{r}^2 + \frac{l^2}{r^2}\right) + V(r) = \frac{1}{2}\dot{r}^2 + \left(\frac{l^2}{2r^2} - \frac{GM}{r}\right) = \frac{1}{2}\dot{r}^2 + V_{eff}(r) \tag{9.49}$$

with an effective potential $V_{eff}(r)$ depending only upon radial position, consisting of the attractive Newtonian gravitational potential energy and the repulsive "centrifugal" potential energy. The effective potential is shown in Fig. 9.19. For $E \geq 0$, one has unbounded solutions (parabola or hyperbola), for $E < 0$ bounded (ellipse or circle). One can show that the solutions of the problem are conic section that has one focus at the origin.

Formal solution

The solution of the problem uses the energy equation (9.49)

$$\dot{r} = \sqrt{2E + 2GM/r - l^2/r^2} \quad . \tag{9.50}$$

We calculate

$$\frac{d}{dt}(GM/r) = -GM\frac{\dot{r}}{r^2} = -\frac{\sqrt{2E + 2GM/r - l^2/r^2}}{r^2} \tag{9.51}$$
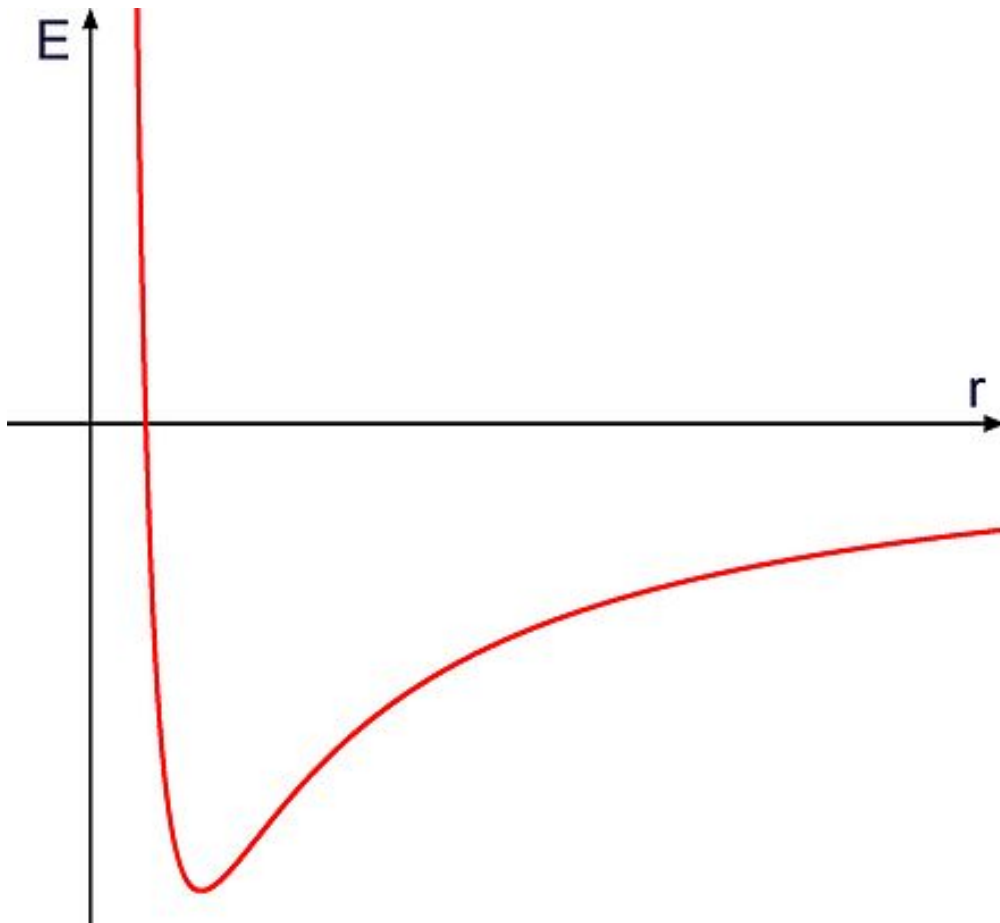
Figure 9.19: Effective potential in the gravitational force field.

and using the substitution $u = GM/r$ this yields

$$\dot{u} = -\frac{u^2}{GM}\sqrt{2E + 2u - l^2(GM)^{-2}u^2}\,. \tag{9.52}$$

Using (9.48,9.52)

$$\frac{d\varphi}{du} = \frac{\dot{\varphi}}{\dot{u}} = -\frac{lu^2(GM)^{-2}}{u^2(GM)^{-1}\sqrt{2E + 2u - l^2(GM)^{-2}u^2}} \tag{9.53}$$

$$= -\frac{l(GM)^{-1}}{\sqrt{2E + 2u - l^2(GM)^{-2}u^2}} \tag{9.54}$$

Separation of the variables in $\frac{d\varphi}{du}$ yields

$$-\varphi = \int du \frac{l(GM)^{-1}}{\sqrt{2E + 2u - l^2(GM)^{-2}u^2}} = \arccos\left(\frac{ul^2(GM)^{-2} - 1}{\sqrt{1 + 2El^2(GM)^{-2}}}\right) \quad .\tag{9.55}$$

Therefore,

$$\cos\varphi = \frac{ul^2(GM)^{-2} - 1}{\sqrt{1 + 2El^2(GM)^{-2}}} = \frac{ul^2(GM)^{-2} - 1}{\varepsilon} \tag{9.56}$$

$$1 + \varepsilon\,\cos\varphi = ul^2(GM)^{-2} \tag{9.57}$$

$$r = \frac{l^2(GM)^{-1}}{1 + \varepsilon\cos\varphi} \tag{9.58}$$

with $\varepsilon = \sqrt{1 + 2El^2(GM)^{-2}}$ as excentricity. (9.59) is the general formula for a conic section that has one focus at the origin:

$$r = \frac{p}{1 + \varepsilon\cos\varphi} = \frac{b^2}{a - \sqrt{a^2 - b^2}\cos\varphi} \tag{9.59}$$

$\varepsilon = 0$ corresponds to a circle, $0 < \varepsilon < 1$ corresponds to an ellipse, $\varepsilon = 1$ corresponds to a parabola, and $\varepsilon > 1$ corresponds to a hyperbola. The semi-major axes a and b are explained in the following and in Fig. 9.20.

The orbits of all planets are to high accuracy Kepler orbits around the Sun. The small devia-
tions are due to the much weaker gravitational attractions between the planets, and in the case of
Mercury, due to general relativity. The orbits of the artificial satellites around the Earth are, with a
fair approximation, Kepler orbits with small perturbations due to the gravitational attraction of the
sun, the moon and the oblateness of the Earth. In high accuracy applications for which the equa-
tion of motion must be integrated numerically with all gravitational and non-gravitational forces
(such as solar radiation pressure and atmospheric drag) being taken into account, the Kepler orbit
concepts are of paramount importance and heavily used.

Let us look at **Keppler's first law and the geometry of the ellipse.** At $\varphi = 0°$, perihelion,
the distance is minimum

$$r_{\min} = \frac{p}{1 + \varepsilon}. \tag{9.60}$$

At $\varphi = 90°$ and at $\varphi = 270°$, the distance is equal to the semi-latus rectum. At $\varphi = 180°$,
aphelion, the distance is maximum

$$r_{\max} = \frac{p}{1 - \varepsilon}. \tag{9.61}$$

The semi-major axis a is the arithmetic mean between $r_{\max}$ and $r_{\min}$:

$$r_{\max} - a = a - r_{\min} \tag{9.62}$$

$$a = \frac{p}{1 - \varepsilon^2}. \tag{9.63}$$

The semi-minor axis b is the geometric mean between $r_{\max}$ and $r_{\min}$:

$$\frac{r_{\max}}{b} = \frac{b}{r_{\min}} \tag{9.64}$$

$$b = \frac{p}{\sqrt{1 - \varepsilon^2}}. \tag{9.65}$$

The semi-latus rectum p is the harmonic mean between $r_{\max}$ and $r_{\min}$:

$$\frac{1}{r_{\min}} - \frac{1}{p} = \frac{1}{p} - \frac{1}{r_{\max}} \tag{9.66}$$

$$pa = r_{\max}r_{\min} = b^2 . \tag{9.67}$$

The eccentricity ε is the coefficient of variation between $r_{\max}$ and $r_{\min}$:

$$\varepsilon = \frac{r_{\max} - r_{\min}}{r_{\max} + r_{\min}}. \tag{9.68}$$

The area of the ellipse is

$$A = \pi ab . \tag{9.69}$$

The special case of a circle is $\varepsilon = 0$, resulting in $r = p = r_{\max} = r_{\min} = a = b$ and $A = \pi r^2.$

Let us look at **Keppler's second law and the mathematical derivation.** In a small time dt the planet sweeps out a small triangle (or, more precisely, a sector) having base line r and height $rd\varphi$ and area $dA = \frac{1}{2} \cdot r \cdot rd\varphi$ and so the constant areal velocity is $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\varphi}{dt}$. The planet moves faster when it is closer to the Sun. Since the area enclosed by the elliptical orbit is πab, the period P satisfies

$$P \cdot \tfrac{1}{2}r^2\frac{d\varphi}{dt} = \pi ab \tag{9.70}$$

and the mean motion of the planet around the Sun $n = 2\pi/P$ satisfies

$$r^2 d\varphi = abndt. \tag{9.71}$$
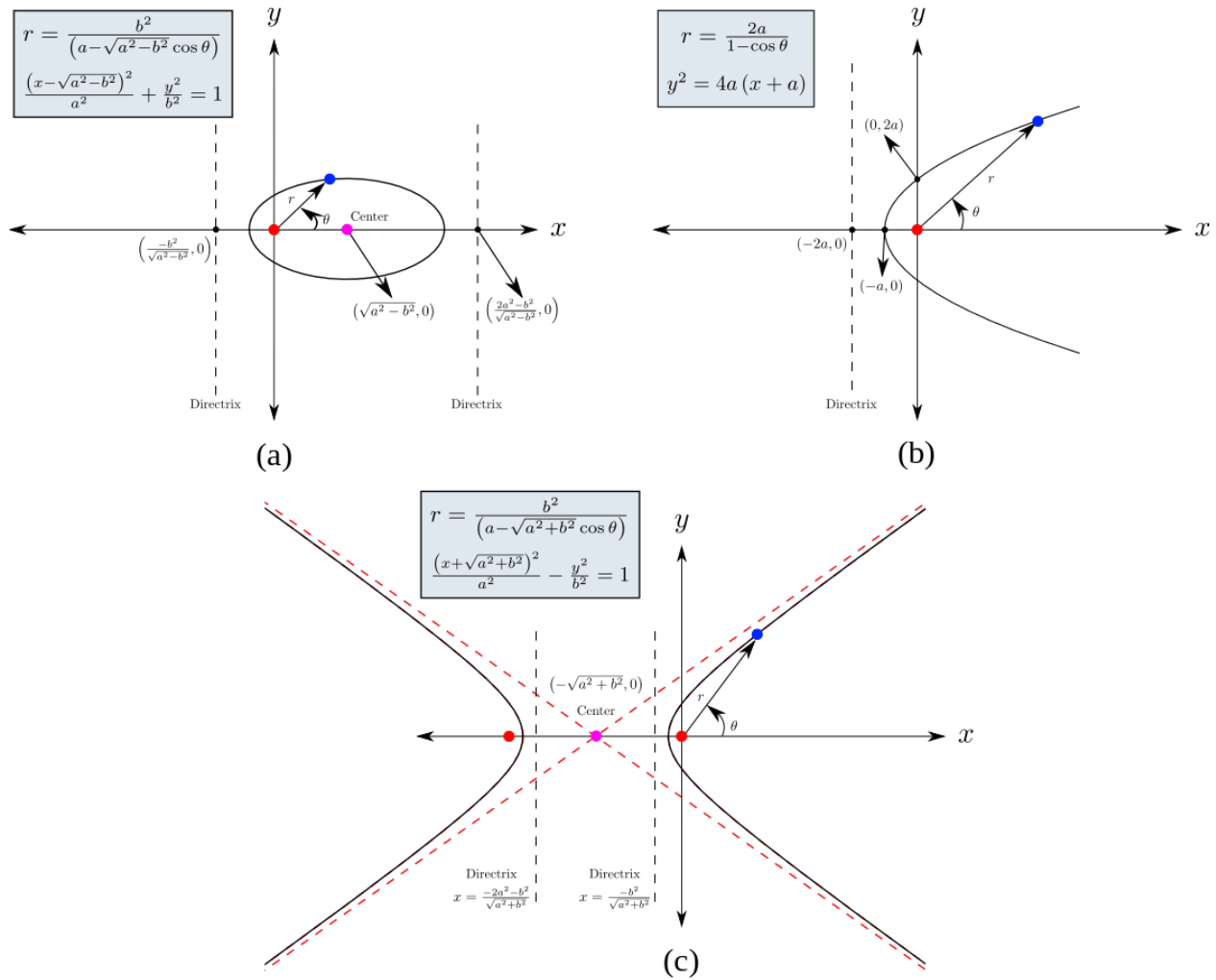
Figure 9.20: Three different types of conic sections. Focal-points corresponding to all conic sections are placed at the origin.

Keppler's third law reads as:

$$\frac{T^2}{r^3} = \frac{4\pi^2}{GM} \tag{9.72}$$

where T is the orbital period of the orbiting body. In the full formulation under Newton's laws of motion, M should be replaced by $M+m$, where m is the mass of the orbiting body. Consequently, the proportionality constant is not truly the same for each planet. Nevertheless, given that m is so small relative to M for planets in our solar system, the approximation is good in the original setting.

Kepler's laws refine the model of Copernicus, which assumed circular orbits. If the eccentricity of a planetary orbit is zero, then Kepler's laws state:

- The planetary orbit is a circle with the Sun at the center

- The speed of the planet in the orbit is constant

- The square of the sidereal period is proportionate to the cube of the distance from the Sun.

Actually, the eccentricities of the orbits of the six planets known to Copernicus and Kepler are quite small, so the rules above give excellent approximations of planetary motion, but Kepler's laws fit observations even better.

Kepler's corrections to the Copernican model are not at all obvious:

- The planetary orbit is not a circle, but an ellipse

- The Sun is not at the center but at a focal point

- Neither the linear speed nor the angular speed of the planet in the orbit is constant, but the area speed is constant.

- The square of the sidereal period is proportionate to the cube of the mean between the maximum and minimum distances from the Sun.

The nonzero eccentricity of the orbit of the earth makes the time from the March equinox to the September equinox, around 186 days, unequal to the time from the September equinox to the March equinox, around 179 days. A diameter would cut the orbit into equal parts, but the plane through the Sun parallel to the equator of the Earth cuts the orbit into two parts with areas in a 186 to 179 ratio, so the eccentricity of the orbit of the Earth is approximately

$$\varepsilon \approx \frac{\pi}{4} \frac{186 - 179}{186 + 179} \approx 0.015, \tag{9.73}$$

which is close to the correct value (0.016710219). The calculation is correct when the perihelion, the date that the Earth is closest to the Sun, is on a solstice. The current perihelion, near January 4, is fairly close to the solstice on December 21 or 22.

Earth's orbit about the sun is nearly circular at a mean distance of 1.5×10^8 km. The eccentricity of the orbit is small, 0.0168. Thus earth is 3.4% further from the Sun at aphelion than at perihelion, the time of closest approach to the sun. Perihelion occurs nowadays every year in January, and the exact time changes by about 20 minutes per year. In 1995, it occurred on 3 January. Earth's axis of rotation is inclined $23.45°$ to the plane of earth's orbit around the sun (Fig. 9.21). The orientation is such that the sun is directly overhead at the Equator on the vernal and autumnal equinoxes, which occur on or about 21 March and 21 September each year. The latitudes of $23.45°$ North and South are the Tropics of Cancer and Capricorn respectively. The tropics lie equatorward of these latitudes. As a result of the eccentricity of earth's orbit, maximum solar insolation averaged over the surface of the earth occurs in early January each year. As a result of the inclination of earth's axis of rotation, the maximum insolation at any location outside the tropics occurs around 21 June in the northern hemisphere, and around 21 December in the southern hemisphere.

Exercises: satellites as cellestrial objects
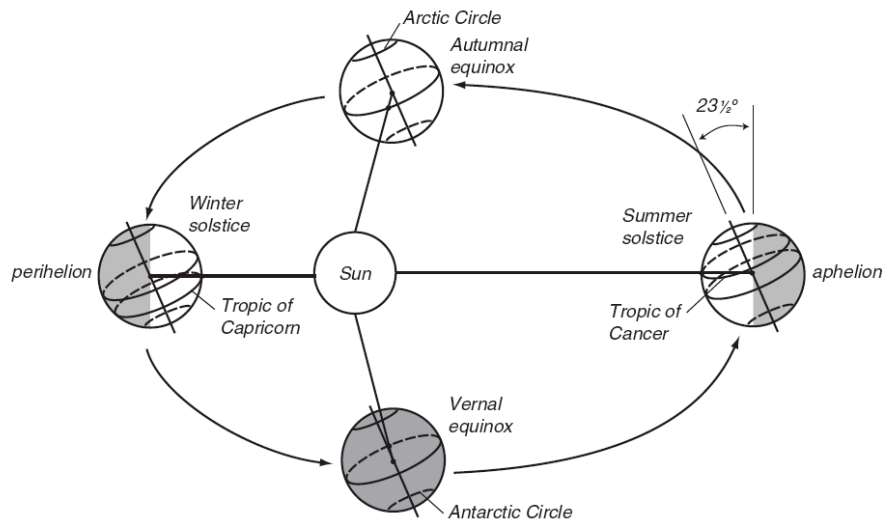
Figure 9.21: The earth in space. The ellipticity of earth's orbit around the sun and the tilt of earth's

9.7 Tides

High and low tides on Earth are caused mainly by gravitational forces exerted by the Moon (and the Sun). A tidal bulge is formed on the Earth's side facing the moon and the opposing side each (Fig. 9.22). Within 24 hours the Earth performs one rotational cycle underneath both of the tidal bulges. Thus, high tides occur every 12 hours. The question why two tidal bulges are formed will be answered here. The tidal bulge on the sublunar point is easily explained by the higher gravitational forces exerted at that point which pull the water into the direction of the moon. But what causes the tidal bulge at the antipodal point where the gravitational field is in fact weaker? As we will see, the crucial point is the heterogeneity of the gravitational field, i.e. the fact that the gravitation is different on either sides.

Figure 9.22: The tidal bulges on Earth are caused by the gravitational field of the Moon.

Tidal forces in the gravitational field of a celestial object. We having been arguing in a purely qualitative way so far and refrained from specifying in detail the form of the gravitational forces. To come to a quantitative description we take a closer look at a celestial object (mass M) of spherical symmetry. The gravitational forces it exerts on a mass m in the distance r are described in Newton's law of gravitation

$$F_G\left(r\right) = -G\frac{m\,M}{r^2} \tag{9.74}$$

where G is the universal gravitational constant and r is the radius, i.e. the semi-major axis of the ellipse.

The gravitational field is heterogeneous, its strength depending on the distance. Tidal forces will occur. The Earth is in the gravitational field of the Moon and Sun. The distance between the sea surface and the centre of gravity is Δr. The tidal force now is the difference of the gravitational forces between the sea surface and the centre of gravity:

$$\begin{aligned}
\boldsymbol{F_{tidal}}\,(r) &= \boldsymbol{F_G}(r_s + \Delta r) - \boldsymbol{F_G}(r_s) &\text{(9.75)}\\
&= -G\,m\,M\left(\frac{1}{(r_s + \Delta r)^2} - \frac{1}{r_s^2}\right) &\text{(9.76)}
\end{aligned}$$

We presume $\Delta r << r_s$ for a good approximation. Thus, having factored out r_s^2 in the denominator, we can use the approximation

$$\begin{aligned}
\boldsymbol{F_{tidal}}\,(r) &= -G\,\frac{m\,M}{r_s^2}\left(\frac{1}{(1 + \Delta r/r_s)^2} - 1\right) &\text{(9.77)}\\
&\approx -G\,\frac{m\,M}{r_s^2}\,(1 - 2\Delta r/r_s - 1) &\text{(9.78)}\\
&= 2G\,\frac{m\,M}{r_s^3}\Delta r &\text{(9.79)}
\end{aligned}$$

Its dependence on distance is typical for tidal force that diminishes by the third power of distance (and thus much stronger than the gravitational force). This explains why the Sun's tidal forces act more weakly on Earth than those of the moon despite the Sun's gravitational force being 178 times larger. The tidal forces that act on a physical object in the gravitation field of a celestial body diminish by $1/r^3$ and are proportional to the size Δr of the object.

9.8 The Earth-Sun geometry

Earth's orbit about the sun is nearly circular at a mean distance of 1.5×10^8 km. The eccentricity of the orbit is small, 0.0168. Thus earth is 3.4% further from the Sun at aphelion than at perihelion, the time of closest approach to the sun. Perihelion occurs every year in January, and the exact time changes by about 20 minutes per year. In 1995, it occurred on 3 January. Earth's axis of rotation is

inclined $23.45°$ to the plane of earth's orbit around the sun (Fig. 9.21). The orientation is such that the sun is directly overhead at the Equator on the vernal and autumnal equinoxes, which occur on or about 21 March and 21 September each year. The latitudes of $23.45°$ North and South are the Tropics of Cancer and Capricorn respectively. The tropics lie equatorward of these latitudes. As a result of the eccentricity of earth's orbit, maximum solar insolation averaged over the surface of the earth occurs in early January each year. As a result of the inclination of earth's axis of rotation, the maximum insolation at any location outside the tropics occurs around 21 June in the northern hemisphere, and around 21 December in the southern hemisphere.

On glacial-interglacial time scales, the geometry of the Earth-Sun system provides a pacing of climate change. Earth's climate is constantly undergoing changes due to temporal changes in the orbital parameters eccentricity e, precession $e \sin \tilde{\omega}$ and obliquity ϵ.

Eccentricity descibes the deviation of Earth's orbit from a circular shape (in this case, eccentricity equals zero). The precession of Earth's orbit around the Sun results in a variation of the distance between Earth and Sun at a given time in the year. The precession parameter is $e \sin \tilde{\omega}$ with $\tilde{\omega}$ measured from the vernal equinox. Currently, $\tilde{\omega}$ is $102°$ which means that the boreal winter solistice occurs close to perhelion. Therefore, the northern hemisphere receives more than average energy in winter and less in summer.

Obliquity is the tilt of Earth's rotational axis relative to a perpendicular drawn to the plane of the ecliptic. It varies with an average period of 41 ka. Currently, Earth's tilt is $23.44°$. This value lies in the middle of a $2.4°$ range between $22.1°$ and $24.5°$. At times of high ϵ-values the summer insolation is inceased and the winter insolation is decreased in both hemispheres. Therfore, the seasonality of incoming solar radiation is increased everywhere.

The eccentricity e is the only orbital parameter that changes the total average annual amount of incoming solar radiation. The other parameters modify the seasonal and hemispheric distribution of incoming solar radiation. Berger (1978(provided expressions for $e \sin \tilde{\omega}$, e and ϵ in a form

suitable for the astronomical theory of palaeoclimates:

$$e \sin \tilde{\omega} = \sum_i P_i \sin(\alpha_i t + \eta_i) \tag{9.80}$$

$$\epsilon = \epsilon^* + \sum_i A_i \cos(\gamma_i t + \zeta_i) \tag{9.81}$$

$$e = e^* + \sum_i E_i \cos(\lambda_i t + \Phi_i) \tag{9.82}$$

Milankovitch theory (1941) gained the status of a paradigm for explaining the Pleistocene ice-ages. A key element of this theory is that summer insolation at high latitudes of the northern hemisphere determines glacial-interglacial transitions connected with the waxing and waning of large continental ice sheets (e.g. Imbrie and Imbrie, 1980). In the last two million years, the glacial-interglacial cycles provide the dominant signal in the climate record. Climate conditions of glacials and interglacials are very different. During the Last Glacial Maximum, about 20,000 years before present, surface temperature in the north Atlantic realm was 10-20 degree lower than today (CLIMAP, 1976).

Although there are indications for Milankovitsch's astronomical theory (Hays et al., 1976), the driving mechanism of the northern hemisphere is under debate. Recent radiometric datings of past sea-level, which is intimately linked to continental ice mass, indicate that the penultimate deglaciation is in phase with an insolation maximum at high latitudes of the southern hemisphere (Henderson and Slowey, 2000). Moreover, paleoclimatic proxy data reveal that the Southern Ocean warmed before the northern hemisphere during deglaciations (e.g. Imbrie et al., 1992; 1993).

9.9 Template model

```
basedrive="/Users/tlaepple/data/"
path<-paste(basedrive,"paleoLibrary/src/",sep="")
source(paste(path,"header.R",sep=""))
setwd(paste(basedrive,"SmallHypothesis/",sep=""))

#Library for insolation->climate->proxy calculations
```

```r
#Read climatology of SST and SAT
sat.clim<-read_ncep.clim.day("sat.clim.day.nc",varname="air")


analyse.transferfunction<-function(lat1,lon1,dataset,bPlot=T,b3plot=T,main=NULL)
{
  a.matrix<-matrix(NA,365,365)

  #Insolation
  insol<-daily_insolation(0,lat1,1:365)$Fsw
  ins.1<-insol
  ins.2<-insol^2
  ins.3<-insol^3

  #daily temperature data
        a<-selspace(dataset,lat1=lat1,lon1=lon1)
  basis<-rep(a,2)

  for (iopt in 1:365) a.matrix[,iopt]<-basis[iopt:(iopt+364)]   #Jahre werden versch

  t<-lm(a.matrix~ins.1+ins.2+ins.3)                             #Das Polynom wird g
  rmse<-colMeans(t$residuals^2)

  #ul, il und index.neg wurde verwendet um eine positive Steigung zu erreichen,
         # dies brachte in Afrika auchdie richtige Monsoonantwort mit sich !

  ul<-max(insol)*t$coefficients[2,]+max(insol)^2*t$coefficients[3,]+max(insol)^3*t
  ll<-min(insol)*t$coefficients[2,]+min(insol)^2*t$coefficients[3,]+min(insol)^3*t

  index.neg<-((ul-ll)<=0)
  rmse[index.neg]<-1e6
  bestfit.index<-which.min(rmse)

  coeff<-t$coefficients[,bestfit.index]

  t.lin<-lm(a.matrix[,bestfit.index]~ins.1)
  coeff.lin<-t.lin$coeff


############# Plotting routine
if (is.null(main)) main=paste(lon1,"E / ",lat1,"N",sep="")
if (bPlot)
{
  at<-basis
  at<-filter(at,rep(1/20,20),circular=T)
  at.scale<-scale(at)
  insol.scale<-scale(rep(insol,2))

  at.sc<-attr(at.scale,"scaled:scale")
  at.offset<-attr(at.scale,"scaled:center")
```

```
  labels.at<-pretty(c(-3*at.sc+at.offset,3*at.sc+at.offset),5)
  at.at<-(labels.at-at.offset)/at.sc

  insol.sc<-attr(insol.scale,"scaled:scale")
  insol.offset<-attr(insol.scale,"scaled:center")

  labels.insol<-pretty(c(-3*insol.sc+insol.offset,3*insol.sc+insol.offset),5)
  labels.insol<-labels.insol[labels.insol>0]
  at.insol<-(labels.insol-insol.offset)/insol.sc

if (b3plot)
{

  par(mfrow=c(2,2))
  ylab="temperature (deg C)"
  plot(at.scale,ylim=c(-2,2),axes=F,main=main,ylab=ylab,xlab="day of year",type="l"
  lines(insol.scale,col="red",lwd=2,lty=2)


  axis(1,at=c(0,120,240,365,365+120,365+240,365*2),labels=c(0,120,240,0,120,240,365
  box()
  axis(2,at=at.at,labels=labels.at)
  axis(4,at=at.insol,labels=labels.insol)


}
  ############## Plot after cutting
  ylab="insolation (W/m2)"

  plot(at,ylim=c(-2*at.sc+at.offset,2*at.sc+at.offset),ylab=ylab,xlab="day of year"

  axis(1,at=c(0,120,240,365,365+120,365+240,365*2),labels=c(0,120,240,0,120,240,365
  box()
  axis(2)


  insol.scale<-rep(t.lin$fitted.values,2)
  iopt<-365-1*bestfit.index
  insol.scale<-rep(insol.scale[iopt:(iopt+364)],2)
  lines(insol.scale,col="red",lwd=2,lty=2)


  insol.scale<-rep(t$fitted.values[,bestfit.index],2)
  iopt<-365-1*bestfit.index
  insol.scale<-rep(insol.scale[iopt:(iopt+364)],2)
  lines(insol.scale,col="blue",lwd=2)



  add<-diff(range(a.matrix))/10
  main<-""
```

```
    ylab=""
    if (b3plot) {main<-"response function";ylab="temperature  (deg C)"}


    plot(ins.1,a.matrix[,bestfit.index],xlab="insolation (W/m2)",ylab=ylab,main=main,
    lines(ins.1,t$fitted.values[,bestfit.index],col="blue",lwd=2)
    lines(ins.1,t.lin$fitted.values,col="red",lwd=2)

}



    rsq1<-cor(a.matrix[,bestfit.index],ins.1)^2
    rsq2<-cor(a.matrix[,bestfit.index],t$fitted.values[,bestfit.index])^2
    coeff.lin<-lm(a.matrix[,bestfit.index]~ins.1)$coeff


    fitval<-rep(t$fitted.values[,bestfit.index],3)[(367-bestfit.index):(367-bestfit.

    #RSQ1 ist RSQ eines linearen Fits, RSQ2 der Fit des Polynoms
    return(list(coeff=coeff,coeff.lin=coeff.lin,rsq=c(rsq1,rsq2),lag=bestfit.index,f



}
#testcode
model<-analyse.transferfunction(40,100,sat.clim)



 #Apply Transferfunction

lat1=40
T.annual<-vector()
for (kyear in 1:400)
    {
        insol<-daily_insolation(kyear,lat1,1:365)$Fsw
T.annual[kyear]<-mean(model$coeff[1]+model$coeff[2]*insol+model$coeff[3]*insol^2+mo
}

plot(T.annual,type="l",xlab="kyr BP",ylab="T")
```

Exercise 75   –   Template model

Exercise 76   –   Insolation

Exercise 77   –   Some Questions

- What are the periodicities in the Earth orbital parameters?

  a) What is their physical meaning?

  b) Are these orbital parameters independent?

- What is the present day configuration?

- Please draw the situation for 6 kyBP!

- a) Let's once again let $\vec{x}$ a measurement vector, and $\vec{y} = M\vec{x}$ be a transformation to a new set of variables. Please write down C_y in terms of C_x and M !

  b) What happens with C_y if the columns of M^T are taken to be the eigenvectors of M ?

  c) Please describe in words and formula the singular value decomposition!

- While the preindustrial concentration of atmospheric $^{12}CO_2$ amounted to about 280 ppm, the $^{12}CO_2$ volume mixing ratio at the LGM was only about 200 ppm. Estimate the glacial value of atmospheric D14C, assuming that the cosmogenic production of ^{14}C has not changed (which in fact is an oversimplification) and that $\Delta^{14}C \approx \delta^{14}C$.

- Please solve the radioactive decay equation

$$\frac{d}{dt}x = -\lambda x + s$$

  with a constant source term s.

  b) Consider tow different consituents,

  $x(0) = 1000, s = 10, \lambda = 10$

  $y(0) = 100, s = 10, \lambda = 1$

  When is x = y ?

  c) How can the equation be used to date substances?

- Consider a reservoir with two separate sourses Q_1 and Q_2, and a single sink S. The magnitude of Q_1 and S and their uncertainties have been estimated to be 75 ± 20 and 100 ± 30 (arbitrary units). Assuming that there is no direct way of estimating Q_2, how would you derive its magnitude and uncertainty range from budget considerations? tipp: assume equilibrium.

b) Regarding the uncertainty, which assumption must be made for the estimation of uncertainty?

c) What is the turnover time in this system?

- Consider the non-linear case

$$\frac{d}{dt}M = Q - BM^2$$

with a constant source term Q, the removal rate is propotional to the square of the reservoir content. Assume $M(0) = 0$. Please solve and draw the solution!

- Consider the logistic map:

$$\frac{d}{dt}M = AM - BM^2$$

with a const. A, B. This is an example for the growth of many biological systems. Calculate the solution for

$M(0) = 0$

$M(0) = M_0 > 0$

- Classify the following models according to linear/non-linear, positive/negative feedback !

$$\frac{d}{dt}x = \lambda x + forcing$$

with $\lambda = const. > 0$

with $\lambda = \lambda_0 + \lambda_1 x, \lambda_0 < 0, \lambda_1 > 0$

- What is the main idea behind the Milankovitch Theory of ice ages? Describe!
- Describe the rectification process!

Chapter 10

Dynamics of spatio-temporal pattern

A major challenge is to understand the dynamics of past periods in which rapid climate changes have occurred and interactions between solid Earth and ocean that may drive climate change. The dependence on background climate, driving mechanisms and feedbacks will provide a suitable framework to study conditions which are expected to develop in the future. Since Earth System Models have to simplify the system and rely on parameterizations of unresolved processes using present data, paleoclimate records provide a unique tool to validate models for conditions which are different from our present one. Suitable data-model analyses provide therefore a proper basis to estimate and possibly reduce uncertainties of future climate change projections.

The current and future climate is subject to significant change due to the increasing human influence on the climate system. Polar regions are especially affected and contain key drivers for this change. The extent and the rate of this change are controversial, however. It is therefore necessary to improve the understanding of natural climate variability and trends by searching for their causes at all time scales. Furthermore, the model scenarios in conjunction with the long-term data can be used to examine mechanisms for the statistics of regional climate extremes under different boundary conditions.

10.1 Time domain

One view of climate change was that the Earth's climate system has changed gradually in response to both natural and human-induced processes. Researchers became intrigued by abrupt climate change when they discovered striking evidence of large, abrupt, and widespread changes preserved in paleoclimatic archives, the history of Earth's climate recorded in tree rings, ice cores, sediments, and other sources. For example, tree rings show the frequency of droughts, sediments reveal the number and type of organisms present, and gas bubbles trapped in ice cores indicate past atmospheric conditions.

The Earth's climate system is characterized by change on all time and space scales, and some of the changes are abrupt even relative to the short time scales of relevance to human societies. Paleoclimatic records show that abrupt climate changes have affected much or all of the Earth repeatedly over the last ice-age cycle as well as earlier - and these changes sometimes have occurred in periods as short as a few years, as documented in Greenland ice cores. Perturbations at northern high latitudes were spectacularly large: some had temperature increases of up to 10-20°C and a local doubling of precipitation within decades.

$$\frac{d}{dt}x_i(t) \text{ can be approximated by a function } \frac{B}{\pi}\frac{\epsilon}{x_i^2 + \epsilon^2} \tag{10.1}$$

for one $i \in \{1, \ldots, n\}$ in a time interval $[t_1, t_2]$. The case $\epsilon \to 0$ is called instantaneous climate shift, i.e. $x_i(t)$ can be approximated by the Heaviside step function. The degree of approximation can be specified by a proper norm.

An alternative way of defining an abrupt climate shift is through the identification of probable breaks in a time series (e.g., the surface temperature series). The formulation of a two phase

regression (TPR) test, e.g. **?**; **?**, describing a series $x(t)$ is given by

$$x(t) = \mu_1 + \alpha_1 t + \epsilon(t) \quad \text{for } t \le c \tag{10.2}$$

$$x(t) = \mu_2 + \alpha_2 t + \epsilon(t) \quad \text{for } t > c. \tag{10.3}$$

Under the null hypothesis of no changepoint, the two phases of the regression should be statistically equivalent and both the difference in means $\mu_{1,2}$, and the difference in slopes, $\alpha_{1,2}$, should be close to zero for each possible changepoint c.

```
# breakpoint1.R
# a piecewise regression
# Notice that the segments were not constrained to be touching or continuous.


x <- c(1:10, 13:22)
y <- numeric(20)
## Create first segment
y[1:10] <- 20:11 + rnorm(10, 0, 1.5)
## Create second segment
y[11:20] <- seq(11, 15, len=10) + rnorm(10, 0, 1.5)
## Plot it
par(mar=c(4,4,1,1)+0.2)
plot(x,y, ylim=c(5, 20), pch=16)


breaks <- x[which(x >= 9 & x <= 17)]


mse <- numeric(length(breaks))
for(i in 1:length(breaks)){
 piecewise1 <- lm(y ~ x*(x < breaks[i]) + x*(x>=breaks[i]))
 mse[i] <- summary(piecewise1)[6]
}
mse <- as.numeric(mse)

mse

br <- breaks[which(mse==min(mse))]
br

piecewise2 <- lm(y ~ x*(x < br) + x*(x > br))
summary(piecewise2)


plot(x,y, ylim=c(5, 20), pch=16)
curve((3.3133 + 16.6352) + (0.5843-1.3025)*x, add=T, from=1, to=br)
```

```r
curve((3.3133 - 0.9116) + 0.5843*x, add=T, from=br, to=max(x))
abline(v=br, lty=3)


##################


# http://www.r-bloggers.com/r-for-ecologists-putting-together-a-piecewise-regressi

x <- c(1:10, 13:22)
y <- numeric(20)
## Create first segment
y[1:10] <- 20:11 + rnorm(10, 0, 1.5)
## Create second segment
y[11:20] <- seq(11, 15, len=10) + rnorm(10, 0, 1.5)
## Plot it
par(mar=c(4,4,1,1)+0.2)
plot(x,y, ylim=c(5, 20), pch=16)


breaks <- x[which(x >= 9 & x <= 17)]


mse <- numeric(length(breaks))
for(i in 1:length(breaks)){
 piecewise1 <- lm(y ~ x*(x < breaks[i]) + x*(x>=breaks[i]))
 mse[i] <- summary(piecewise1)[6]
}
mse <- as.numeric(mse)

mse

br <- breaks[which(mse==min(mse))]
br

piecewise2 <- lm(y ~ x*(x < br) + x*(x > br))
summary(piecewise2)


plot(x,y, ylim=c(5, 20), pch=16)
curve((3.3133 + 16.6352) + (0.5843-1.3025)*x, add=T, from=1, to=br)
curve((3.3133 - 0.9116) + 0.5843*x, add=T, from=br, to=max(x))
abline(v=br, lty=3)


# Notice that the segments were not constrained to be touching or continuous. This


#Implementation of Rampfit
#Ramp (x1,y1 - x2,y2) + two horizontal pieces with a fix length
#Parameters: Search area for x1 and x2, length of horizontal piece
```

```r
#### Estimates for x1 and x2; Page 3, Mudelsee et al

#sigma2(t) contains the variances
#x(i) the values
#t(i) the time
#i0 = index of the start of flat part of the ramp (in Mudelsee = 1)
#i1 = index of the start of the ramp
#i2 = index of the end of the ramp
#i3 = index of the end of the second flat part (in Mudelsee = n)


ramp_xfit<-function(x,t,sigma2,i0,i1,i2,i3)
{

k1<-sum(1/sigma2[i0:i1])
k2<-sum(1/sigma2[i0:(i2-1)])
k3<-sum(1/sigma2[i2:i3])
k4<-sum(1/sigma2[(i1+1):(i2-1)])
k5<-sum(t[(i1+1):(i2-1)]/sigma2[(i1+1):(i2-1)])
k6<-sum(t[(i1+1):(i2-1)]^2/sigma2[(i1+1):(i2-1)])
k7<-sum(x[i0:(i2-1)]/sigma2[i0:(i2-1)])
k8<-sum(x[i0:i3]/sigma2[i0:i3])
k9<-sum(x[i2:i3]/sigma2[i2:i3])
k10<-sum(x[(i1+1):(i2-1)]/sigma2[(i1+1):(i2-1)])
k11<-sum(x[(i1+1):(i2-1)]*t[(i1+1):(i2-1)]/sigma2[(i1+1):(i2-1)])

t1<-t[i1]
t2<-t[i2]

K1<-k2+(t1*k4-k5)/(t2-t1) #Equation for the constants from Page 3
K2<-k3-(t1*k4-k5)/(t2-t1)
K3<-k8
K4<-k1+(t2*(t1+t2)*k4+2*k6-(t1+3*t2)*k5)/(t2-t1)^2
K5<-k3+(t1*(t1+t2)*k4+2*k6-(3*t1+t2)*k5)/(t2-t1)^2
K6<-k9-k7-2*(t1*k10-k11)/(t2-t1)

x2<-(K3*K4/K1+K6)/(K2*K4/K1+K5) #Eq (2)
x1<-(K3-x2*K2)/K1 #Eq (3)

return(list(x1=x1,x2=x2))
}


#Ramp function

#t(i) the time
#i0 = index of the start of flat part of the ramp (in Mudelsee = 1)
#i1 = index of the start of the ramp
#i2 = index of the end of the ramp
#i3 = index of the end of the second flat part (in Mudelsee = n)
#x1 = level of the start of the ramp
#x2 = level of the end of the ramp
```

```r
ramp<-function(t,i0,i1,i2,i3,x1,x2)
{
        result<-rep(NA,(i3-i0+1))
        result[(i0:(i1-1))-i0+1]<-x1

        result[(i1:i2)-i0+1]<-x1+((t[i1:i2])-t[i1])*(x2-x1)/(t[i2]-t[i1])


        result[((i2+1):i3)-i0+1]<-x2
        return(result)
}




#Rampfit besteht aus einer Brute Force Suche fÃŒr t und einer LSQ Fit fÃŒr x (x=We:
#x(i) time series
#t(i) the time

#i1_min, i1_max = index limits of the start of the ramp
#i2_min_i2_max = index limits of the end of the ramp
#tc1,tc2 = width of the flat part (in time units)
#sigma2 = uncertainty (variance) for every point in time

rampfit_xt<-function(x,t,i1_min,i1_max,i2_min,i2_max,tc1,tc2,sigma2)
{
        min_mse<-1e9
        for (i1 in i1_min:i1_max)
                for (i2 in i2_min:i2_max)
                {
                        i0<-which.min(abs(t-(t[i1]-tc1)))
                        i3<-which.min(abs(t-(t[i2]+tc2)))

                        if (i0 == 1) warning("Flat part at beginning of the time se
                        if (i3 == length(t)) warning("Flat part at end of the time

                        par<-ramp_xfit(x,t,sigma2,i0,i1,i2,i3)
                   rampfit<-ramp(t,i0,i1,i2,i3,par$x1,par$x2)
                        xpart<-x[i0:i3]
                        #plot(t[i0:i3],xpart)
                        #lines(t[i0:i3],rampfit)
                        mse<-mean((xpart-rampfit)^2)
                        if (mse<min_mse)
                        {
                                save<-par
                                save_i1<-i1
                                save_i2<-i2
                                min_mse<-mse
                                save_i0<-i0
                                save_i3<-i3
```

```
                                    }
                    }
            return(list(i1=save_i1,i2=save_i2,x1=save$x1,x2=save$x2,mse=min_mse,i0=save
}

x<-xClean+rnorm(length(x))*5
result<-rampfit_xt(x,t=seq(x),i1_min,i1_max,i2_min,i2_max,flatwidth=5,sigma2=rep(1,


rampfit.val<-function(x,t,param,bPlot=F)
{
        rampfit<-ramp(t,param$i0,param$i1,param$i2,param$i3,param$x1,param$x2)
        xpart<-x[param$i0:param$i3]

            if (bPlot) {
                    plot(t,x);
                    lines(t[param$i0:param$i3],rampfit,lwd=2,col="red")
            }
        return(list(residuals=xpart-rampfit,ramp=rampfit))

}




x<-data
t<-seq(data)
sigma2<-rep(1,length(data))

#Test ramp_xfit(x,t,sigma2,10,31,44,60) #Geschwindigkeit ca. 4000/Sekunde

plot(ramp(0,10,20,30,5,30))



###

rampfit<-function(x,t=seq(x),i1_min,i1_max,i2_min,i2_max,tc1,tc2,sigma2=rep(1,lengt
{
        bresult<-list()
        bresult$i1<-vector()
        bresult$i2<-vector()
        bresult$x1<-vector()
        bresult$x2<-vector()
        bresult$mse<-vector()


        bestfit<-rampfit_xt(x,t,i1_min,i1_max,i2_min,i2_max,tc1,tc2,sigma2)

        bestfit.val<-rampfit.val(x,t,bestfit,bPlot=T)
        a1<-acf(bestfit.val$residuals,plot=F)$acf[2]
        nblock<-lopt(a1,length(bestfit.val$residuals))
```

```r
        plot(x)
for (i in 1:N.R)
{
        x_sur<-x
        plot(x)

        x_sur[bestfit$i0:bestfit$i3]<-bestfit.val$ramp+blocksample(bestfit.val$res

        lines(x_sur)

        #bzw... fit auf allen durchfÃ©hren
        fit.sur<-rampfit_xt(x_sur,t,i1_min,i1_max,i2_min,i2_max,tc1,tc2,sigma2=sigm
        bresult$i0[i]<-fit.sur$i0
        bresult$i3[i]<-fit.sur$i3
        bresult$i1[i]<-fit.sur$i1
        bresult$i2[i]<-fit.sur$i2
        bresult$x1[i]<-fit.sur$x1
        bresult$x2[i]<-fit.sur$x2
        bresult$mse[i]<-fit.sur$mse

}
        return(list(bestfit=bestfit,bootstrap=bresult))
}

###
blocksample<-function(data,blocklength=10)
{
        nblock<-ceiling(length(data)/blocklength)
        starts<-floor(runif(nblock,min=1,max=length(data)-blocklength+1))
        index<-rep(1:blocklength,nblock) + rep(starts,each=blocklength)
        return(data[index[1:length(data)]])
}

lopt<-function(a1,n) {
                if (a1<0) a1=0
                return(max(1,ceiling((6^(0.5)*a1/(1-a1)^2)^(2/3)*n^(1/3))))
                }




## Test code for the ramp
t<-seq(from=1,to=100,by=1)
testramp1<-ramp(t,1,30,60,100,1,10)
sigma2<-rep(1,length(t))

ramp_xfit(testramp1,t,sigma2,1,30,60,100) #OK

#Testcase: non equidistant
index_ne<-cumsum(round(runif(50,min=1,max=5))) #some random sorted indices
testramp_ne<-testramp1[index_ne]
```

```
index<-!is.na(testramp_ne) #Remove the missing values in x and time
testramp_ne<-testramp_ne[index]
t_ne<-t[index_ne][index]

sigma_ne<-rep(1,length(t_ne)) #No weighting

i0<-1          #Position of the ramp
i1<-which(testramp_ne>1)[1]-1
i2<-which(testramp_ne==10)[1]
i3<-length(t_ne) #

ramp_xfit(testramp_ne,t_ne,sigma_ne,i0,i1,i2,i3) #OK


res<-rampfit_xt(testramp_ne,t_ne,i1_min=1,i1_max=12,i2_min=13,i2_max=19,tc1=10,tc2=


t1<-rampfit.val(testramp_ne,t_ne,res,bPlot=TRUE)

testramp_noise<-testramp_ne+rnorm(length(testramp_ne))
temp<-rampfit(testramp_noise,t_ne,i1_min,i1_max,i2_min,i2_max,tc1,tc2,sigma2=rep(1,

plot(t_ne,testramp_noise)
abline(v=t_ne[c(temp$bestfit$i1,temp$bestfit$i2)])

for (i in 1:10)
{
        sr<-ramp(t_ne,temp$bootstrap$i0[i],temp$bootstrap$i1[i],temp$bootstrap$i2[
        lines(t_ne[temp$bootstrap$i0[i]:temp$bootstrap$i3[i]],sr,col=i)
}


quantile(temp$bootstrap$x1,c(0.025,0.975))
quantile(temp$bootstrap$x2,c(0.025,0.975))

quantile(temp$bootstrap$i1,c(0.025,0.975))
quantile(temp$bootstrap$i2,c(0.025,0.975))



### Step1: Visual choice of start and endpoints
### Choice of width

## Result: Ramp + uncertainties



index<-!is.na(dO18b)

x<-dO18b[index]*-1
t<-time.huybers[index]
```

```
plot(t,x,type="l")
ival<-identify(t,x,n=4)

res<-rampfit(x,t,ival[1],ival[2],ival[3],ival[4],2,2,sigma2=rep(1,length(t)),N.R=10

abline(v=res$bootstrap$i1,col="red")
abline(v=res$bootstrap$i2,col="blue")

abline(h=res$bootstrap$x1,col="red")
abline(h=res$bootstrap$x2,col="blue")
```

10.1.1   Poisson process*

In probability theory, a Poisson process is a stochastic process that counts the number of events and the time points at which these events occur in a given time interval. The time between each pair of consecutive events has an exponential distribution with parameter Î» and each of these inter-arrival times is assumed to be independent of other inter-arrival times. The process is named after the Poisson distribution introduced by French mathematician SimÃ©on Denis Poisson. It describes the time of events in radioactive decay, telephone calls at a call center, document requests on a web server, and many other punctual phenomena where events occur independently from each other.

The Poisson process is a continuous-time stochastic process; the sum of a Bernoulli process can be thought of as its discrete-time counterpart. A Poisson process is a pure-birth process, the simplest example of a birth-death process. It is also a point process on the real half-line.

Link: https://en.wikipedia.org/?title=Poisson_process

10.2   Frequency domain

10.2.1 Discrete Fourier transform*

The discrete Fourier transform (DFT) converts a finite list of equally spaced samples of a function into the list of coefficients of a finite combination of complex sinusoids, ordered by their frequencies, that has those same sample values. It can be said to convert the sampled function from its original domain (often time or position along a line) to the frequency domain. The sequence of N complex numbers $x_0, x_1, \ldots, x_{N-1}$ is transformed into an N-periodic sequence of complex numbers:

$$X_k \overset{\text{def}}{=} \sum_{n=0}^{N-1} x_n \cdot e^{-2\pi i k n / N} \tag{10.4}$$

Each X_k is a complex number that encodes both amplitude and phase of a sinusoidal component of function x_n. It is the discrete analogy of the formula for the coefficients of a Fourier series:

$$x_n = \frac{1}{N} \sum_{k=0}^{N-1} X_k \cdot e^{i 2 \pi k n / N} \tag{10.5}$$

which is also N-periodic in the domain $n \in [0, N-1]$, this is the inverse transform of (10.4). The vectors

$$u_k = \left[e^{\frac{2\pi i}{N} k n} \mid n = 0, 1, \ldots, N-1 \right]^T \tag{10.6}$$

form an orthogonal basis over the set of N-dimensional complex vectors:

$$u_k^T u_{k'}^* = \sum_{n=0}^{N-1} \left(e^{\frac{2\pi i}{N} k n} \right) \left(e^{\frac{2\pi i}{N} (-k') n} \right) = \sum_{n=0}^{N-1} e^{\frac{2\pi i}{N} (k-k') n} = N \delta_{kk'} \tag{10.7}$$

where $\delta_{kk'}$ is the Kronecker delta. In the last step, the summation is trivial if $k = k'$, where it is $1 + 1 + \cdots = N$, and otherwise is a geometric series that can be explicitly summed to obtain zero.

The modern DFT algorithms are decribed elsewhere https://en.wikipedia.org/wiki/

Fast_Fourier_transform where the main idea goes back to Gauss (1866).

We previously saw a periodogram, a function/graph that displays information about the periodic components of a time series. Any time series can be expressed as a sum of cosine and sine waves oscillating at the fundamental (harmonic) frequencies $= j/n$, with $j = 1, 2, \ldots, n/2$. The periodogram gives information about the relative strengths of the various frequencies for explaining the variation in the time series. The periodogram is a sample estimate of a population function, i.e. the spectral density $S(\omega)$, which is the frequency domain characterization of a population stationary time series. The estimate through the periodogram is rough, in part, because we only use the discrete fundamental harmonic frequencies for the periodogram whereas the spectral density is defined over a continuum of frequencies.

The usual method for smoothing a periodogram, it is merely a centered moving average procedure with a few possible modifications. For a time series, the Daniell kernel with parameter m is a centered moving average which creates a smoothed value at time t by averaging all values between times $t - m$ and $t + m$(inclusive). For example, the smoothing formula for a Daniell kernel with $m = 2$ is

$$\hat{x}_t = \frac{x_{t-2} + x_{t-1} + x_t + x_{t+1} + x_{t+2}}{5} \tag{10.8}$$

The Fourier transform is used for the spectral analysis of time-series. The subject of statistical signal processing does not, however, usually apply the Fourier transformation to the signal itself. Even if a real signal is indeed transient, it has been found in practice advisable to model a signal by a function (or, alternatively, a stochastic process) which is stationary in the sense that its characteristic properties are constant over all time. The Fourier transform of such a function does not exist in the usual sense, and it has been found more useful for the analysis of signals to instead take the Fourier transform of its auto-correlation function.

In R, the weighting coefficients for a Daniell kernel with $m = 2$ can be generated with the command kernel("daniell", 2). The result is

```
coef[-2] = 0.2
```

```
coef[-1] = 0.2
coef[ 0] = 0.2
coef[ 1] = 0.2
coef[ 2] = 0.2
```

The subscripts for coef [ ] refer to the time difference from the center of the average at time t. Thus the smoothing formula in this instance is $\tilde{x}_t = 0.2x_{t-2} + 0.2x_{t-1} + 0.2x_t + 0.2x_{t+1} + 0.2x_{t+2}$, which is the same as the formula given above. The modified Daniell kernel is such that the two endpoints in the averaging receive half the weight that the interior points do. For a modified Daniell kernel with $m = 2$, the smoothing is

$$\tilde{x}_t = \frac{x_{t-2} + 2x_{t-1} + 2x_t + 2x_{t+1} + x_{t+2}}{8} \tag{10.9}$$

$$= 0.125x_{t-2} + 0.25x_{t-1} + 0.25x_t + 0.25x_{t+1} + 0.125x_{t+2} \tag{10.10}$$

In R, the command kernel("modified.daniell", 2) will list the weighting coefficients just used. Either the Daniell kernel or the modified Daniell kernel can be convoluted (repeated) so that the smoothing is applied again to the smoothed values. This produces a more extensive smoothing by averaging over a wider time interval. For instance, to repeat a Daniell kernel with $m = 2$ on the smoothed values that resulted from a Daniell kernel with $m = 2$, the formula would be

$$\tilde{\tilde{x}}_t = \frac{\hat{x}_{t-2} + \hat{x}_{t-1} + \hat{x}_t + \hat{x}_{t+1} + \hat{x}_{t+2}}{5}$$

This is the average of the smoothed values within two time periods of time t, in either direction.

In R, the command kernel("daniell",c(2,2)) will supply the coefficients that would be applied to as weights in averaging the original data values for a convoluted Daniell kernel with $m = 2$ in both smoothings. The result is

```
> kernel ("daniell",c(2,2))
coef[-4] = 0.04
coef[-3] = 0.08
coef[-2] = 0.12
coef[-1] = 0.16
coef[ 0] = 0.20
coef[ 1] = 0.16
coef[ 2] = 0.12
```

```
coef[ 3]  =  0.08
coef[ 4]  =  0.04
```

This generates the smoothing formula $\tilde{x}_t = 0.04x_{t-4} + 0.08x_{t-3} + 0.12x_{t-2} + 0.16x_{t-1} + 0.20x_t + 0.16x_{t+1} + 0.12x_{t+2} + 0.08x_{t+3} + 0.04x_{t+4}$. A convolution of the modified method in which the end points have less weight is also possible. The command

```
kernel("modified.daniell",c(2,2))
```

gives these coefficients:

```
coef[-4]  =  0.01563
coef[-3]  =  0.06250
coef[-2]  =  0.12500
coef[-1]  =  0.18750
coef[ 0]  =  0.21875
coef[ 1]  =  0.18750
coef[ 2]  =  0.12500
coef[ 3]  =  0.06250
coef[ 4]  =  0.01563
```

Thus the center values are weighted slightly more heavily than in the unmodified Daniell kernel.

When we smooth a periodogram, we are smoothing across a frequency interval rather than a time interval. Remember that the periodogram is determined at the fundamental frequencies $\omega_j = j/n$ for $j = 1, 2, \ldots, n/2$. Let $I(\omega_j)$ denote the periodogram value at frequency $\omega_j = j/n$. When we use a Daniell kernel with parameter m to smooth a periodogram, the smoothed value $\hat{I}(\omega_j)$ is a weighted average of periodogram values for frequencies in the range $(j - m)/n$ to $(j + m)/n$.

Bandwidth: There are $L = 2m + 1$ fundamental frequency values in the range $(j - m)/n$ to $(j+m)/n$, the range of values used for smoothing. The bandwidth for the smoothed periodogram is defined as

$$B_\omega = \frac{L}{n}$$

The bandwidth is a measure of the width of the frequency interval(s) used for smoothing the periodogram. When unequal weights are used in the smoothing, the bandwidth definition is modified.

Denote the smoothed periodogram value at $\omega_j = j/n$ as

$$\hat{I}(\omega_j) = \sum_{k=-m}^{+m} h_k I\left(\omega_j + \frac{k}{n}\right).$$

The h_k are the possibly unequal weights used in the smoothing. The bandwidth formula is then modified to

$$B_\omega = \frac{L_h}{n} = \frac{1/\sum h_k^2}{n}$$

Actually, this formula works for equal weights too. The bandwidth should be sufficient to smooth our estimate, but if we use a bandwidth that is too great, we will smooth out the periodogram too much and miss seeing important peaks. In practice, it usually takes some experimentation to find the bandwidth that gives a suitable smoothing.

The bandwidth is predominately controlled by the number of values that are averaged in the smoothing. In other words, the m parameter for the Daniell kernel and whether the kernel is convoluted (repeated) affect the bandwidth.

Averaging/smoothing the periodogram with a Daniell kernel can be accomplished in R using a sequence of two commands. The first defines a Daniell kernel and the second creates the smoothed periodogram. As an example, suppose that the observed series is named x and we wish to smooth the periodogram using a Daniell kernel with m = 4. The commands are

```
k = kernel("daniell", 4)
spec.pgram(x, k, taper=0, log = "no")
```

The first command creates the weighting coefficients needed for the smoothing and stores them in a vector named k. (It is arbitrary to call it k. It could be called anything.) The second command asks for a spectral density estimate based on the periodogram for the series x, using the weighting coefficients stored in k, with no taper, and the plot will be on an ordinary scale, not a log scale. If a convolution is desired, the kernel command could be modified to something like k = kernel("daniell", c(4,4)).

There are two possible ways to achieve a modified Daniell kernel. You can either change the

kernel command to refer to the "modified.daniell" rather than "daniell" or you can skip using the kernel command and use a spans parameter in the spec.pgram command.[1]

The spans parameter gives the length $(= 2m + 1)$ of the desired modified Daniell kernel. For instance, a modified Daniell kernel with $m = 4$ has length $L = 2m + 1 = 9$ so the we could use the command

```
spec.pgram(x, spans=9, taper = 0, log="no")
```

Two passes of a modified Daniell kernel with m = 4 on each pass can be done using

```
spec.pgram(x, spans=c(9,9), taper = 0, log="no")
```

To learn where the two dominant peaks are located, assign a name to the spec.pgram output and then you can list it. For instance,

```
specvalues = spec.pgram(x, k, taper=0, log="no")
specvalues
```

One can put vertical dotted lines onto the (estimated) spectral density plot at the approximate locations of the peak densities.

```
abline(v=1/44, lty="dotted")
```

A series should be de-trended prior to a spectral analysis. A trend will cause such a dominant spectral density at a low frequency that other peaks would not be seen. By default, the R command spec.pgram performs a de-trending using a linear trend model. That is, the spectral density is estimated using the residuals from a regression done where the y-variable = observed data and the x-variable = t. If a different type of trend is present, a quadratic for instance, then a polynomial regression could be used to de-trend the data before the estimated spectral density is explored. Note, however, that the R command spec.ar, however does not perform a de-trending by default.

Exercise 78   –   Calculate a spectrum

---

[1]The smoothing method of spectral density estimation is called a nonparametric method because it does not use any parametric model for the underlying time series process. An alternative method is a parametric method which entails finding the best fitting AR model for the series and then plotting the spectral density of that model. This method is supported by a theorem which says that the spectral density of any time series process can be approximated by the spectral density of an AR model (of some order, possibly a high one). In R, parametric estimation of the spectral density is easily done with the command/function spec.ar

Tasks: Using R https://stat.ethz.ch/R-manual/R-devel/library/stats/html/spectrum.html:

1. Create a vector containing a sinus signal and plot the spectrum on the right top corner you see a cross which shows the bandwith (horizontal bar) and 95% confidence interval (vertical bar)

2. Vary the averaging and look how the shape (width) of the peak varies

3. Now add noise to the data and repeat 1 and 2. You will see that when you increase the smoothing you get a smaller variance and a larger bandwith. The information content is limited and you have to make your choice.

Solution

```
#Spectrum, bandwith and variance


a<-sin(2*pi*(1:5000)*1/10)+0.5*rnorm(5000)/10
#a<-sin((1:1000)/10)
plot(a,type="l")

spectrum(a)
spectrum(a, method = "ar")
spectrum(a,spans=10,main="spans=10")
spectrum(a,spans=30,main="spans=30")
spectrum(a,spans=100,main="spans=100")
spectrum(a,spans = c(5,7), log = "dB", ci = 0.8)
dev.print(postscript, file="spectrumtest.ps")
```

With the R-command

```
??spectrum
```

you get help a description of the function. It says that the as arguments is has a univariate or multivariate time series x. The spectrum here is defined with scaling 1/frequency(x). A confidence interval will be plotted by plot.spec: this is asymmetric, and the width of the centre mark indicates the equivalent bandwidth.

Another important method is the cross-spectrum analysis where you can have two processes x and y.

```
#Coherence spectrum - cross-spectrum analysis
library(clim.pact)
coherence(x,y,dt=1,M=NULL,plot=TRUE)
# or for the spectrum if both processes are the same
coherence(a,a,dt=1,M=NULL,plot=TRUE)
```

10.2.2   Wavelet spectrum*

Many geophysical processes are furthermore often non-stationary. In this regard, the optimal method is continuous wavelet analysis as it intrinsically adjusts the time resolution to the analyzed scale, e.g. **?**; **?**. Wavelet analysis is based on the convolution of $f(t)$ with a set of functions $g_{ab}(t)$ derived from the translations and dilations (and rotations in higher dimensions) of a mother

wavelet $g(t)$, where

$$g_{ab} = \frac{1}{\sqrt{a}} g \left(\frac{t-b}{a} \right) \tag{10.11}$$

$a > 0$ and b are real. Any set of functions $g_{ab}(t)$ constructed from (10.11) and meeting the conditions outlined below are called wavelets. The convolution of $f(t)$ with the set of wavelets is the wavelet transform (WT)

$$T_g(b, a) = \frac{1}{\sqrt{a}} \int g \left(\frac{t-b}{a} \right) \ f(t) \ dt \tag{10.12}$$

This is known as the continuous wavelet transform since "a" and "b" may be varied continuously. Translation parameter "b" corresponds to position or time if the data is spatial or temporal, especially. Dilation parameter "a" then corresponds to scale length or temporal period. Equation (10.12) expands a one-dimensional time series into the two-dimensional parameter space (b, a) and yields a local measure of the relative amplitude of activity at scale a at time b. This is in contrast to the Fourier transform that yields an average amplitude over the entire dataset. Note, we have avoided the use of the words "wavelength" or "frequency" in our description of the WT. Though wavelets have a definite scale, they need not to bear any resemblance to Fourier modes(sines and cosines). However, a correspondence between wavelength and scale a sometimes can be achieved. In contrast to the standard Fourier analysis and the incentive for the development of wavelet analysis, the WT produces "instantaneous" coefficients and therefore can yield information on the evolution of nonstationary processes.

A major question concerns the significance testing of wavelet spectra. Torrence and Compo **?** formulated pointwise significance tests against reasonable background spectra. However, Maraun and Kurths **?** pointed out a serious deficiency of pointwise significance testing: Given a realization of white noise, large patches of spurious significance are detected, making it - without further insight - impossible to judge which features of an estimated wavelet spectrum differ from background noise and which are just artefacts of multiple testing. Under these conditions, a reliable

corroboration of a given hypothesis is impossible. This demonstrates the necessity to study the significance testing of continuous wavelet spectra in terms of sensitivity and specificity. Given the set of all patches with pointwise significant values, areawise significant patches are defined as the subset of additionally areawise significant wavelet spectral coefficients given as the union of all critical areas that completely lie inside the patches of pointwise significant values. Whereas the specificity of the areawise test appears to be - almost independently of the signal to noise ratio - close to one, that of the pointwise test decreases for high background noise, as more and more spurious patches appear **?**.

10.2.3   Pseudospectrum*

Another spectral method characterizing the abruptness and resonance of the linear system (7.57) is the pseudospectrum. As we will see later in the context of atmosphere and ocean instabilities, an eigenvalue analysis is inappropriate in describing the dynamics of the system (**??**). One of the most prominent of these problems is equations of the form

$$\frac{d}{dt}x(t) = Ax + forcing \quad , \tag{10.13}$$

A perturbation in a system with a negative feedback mechanism will be reduced whereas in a system with positive feedback mechanisms, the perturbation will grow. In the one dimensional case, A can be rewritten as $-\lambda$. The real part of λ determines then the stability of the system and is called feedback factor. We have discussed this dynamics in the context of the mixed layer model (**??**). Let us start with a simple model: We have a two-dimensional system with matrix A, eigenvectors and eigenvalues are simply calculated:

```
a = [-1 20; 0 -5 ]
[v,l]=eig(a)
v
l
v1=v(:,1)
v2=v(:,2)
v1'*v2
```

It is easy to recognize that the eigenvectors are not orthogonal sind the vector product is non-zero. Let us now calculate the solution with initial conditions $x(0) = -4$ and $y(0) = 6$, then using the Laplace backtransformation the the solution is simply:

```
% y dyn
figure(1)
[t1,ft1]=INVLAP('6/(s + 5)',0.001,4,100);
plot(t1,ft1), grid on
xlabel('time '), ylabel('y(t)','Fontsize',16)
title('Laplace backtransformation from 6/(s+5) ','Fontsize',16)
%set(gca,'XLim',xlim(:))
%   line([0 0],ylim,'Color',[0 0 0],'LineWidth',4)
ca=gca
set(ca,'Fontsize',16)
set(gca,'XLim',[0 4],'YLim',[0 7])

%  x dyn
figure(2)
[t2,ft2]=INVLAP('1/(s+1)   * (-4  + 20* (6/(s + 5)))',0.001,4,100);
plot(t2,ft2), grid on
xlabel('time '), ylabel('x(t)','Fontsize',16)
title('Laplace back from 1/(s+1) * (-4  + 20* (6/(s + 5)) ','Fontsize',16)
ca=gca
set(ca,'Fontsize',16)
%print -dpdf x_t_nonnorm.pdf
```

The graphs are shown in Fig. 10.3. The Laplace transform of x(t) and y(t) are denoted as

$$Y(s) = \frac{y(0)}{s+5} \tag{10.14}$$

$$X(s) = \frac{x(0) + 20Y(s)}{s+1} = \frac{x(0)}{s+1} + \frac{20y(0)}{(s+1)(s+5)} \tag{10.15}$$

Either solve x(t) and y(t) numerically or use the method of partial fraction expansion (Exercise 7). With $a = 1, b = 5$, and $N = 20$, we have

$$y(t) = y(0) * e^{-bt} = 6\, e^{-5t} \tag{10.16}$$

$$x(t) = x(0) * e^{-at} + Ny(0) * \frac{e^{-at} - e^{-bt}}{b-a} = -4e^{-t} + 6 * 20 * \frac{(e^{-t} - e^{-5t})}{5-1} \tag{10.17}$$

It is interesting to write an approximation of x(t) for small t:

$$x(t) \;\approx\; x(0) * (1 - at) + Ny(0) * \frac{(1 - at) - (1 - bt)}{b - a} \tag{10.18}$$

$$=\; x(0) + t(-ax(0) + Ny(0)) \tag{10.19}$$

A linear increase in t with $-ax(0) + Ny(0) = (-1) * (-4) + 20 * 6 = 124$ is nicely seen in Fig. 10.3. It strongly depends on N, the higher N (the more non-normal the matrix A is), the stronger can be the transient growth. The example shows furthermore, that the transfer function with a multiplication in the denominator $\frac{Ny(0)}{(s-a)(s-b)}$ provides the source of the transient growth.

In the one-dimensional case for $x(t) = \exp(at)$ we have the the inverse Laplace transform

$$\exp(at) = \mathcal{L}^{-1}\{F(s)\}(t) = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} \frac{1}{s - a}\, ds, \tag{10.20}$$

and and the entire range of t is controlled t by the resolvent $|\frac{1}{s-a}|$. Using the Fourier transformation, (10.13) with forcing F(t) is tranformed to

$$(i\omega I - A)\hat{x} \;=\; \hat{F} \tag{10.21}$$

$$\hat{x} \;=\; (i\omega I - A)^{-1}\hat{F} \tag{10.22}$$

where I is the identity. The so-called resolvent operator of matrix A is $R(\omega) = (i\omega I - A)^{-1}$ The behavior of the norms $\|\exp(At)\|$ over the entire range of t is controlled t by the resolvent norm $\|R(\omega)\|$. If A is a normal operator

$$A\, A^{+} \;=\; A^{+}\, A \;, \tag{10.23}$$

where $^{+}$ denotes the adjoint-complex operator, then

$$\|R(\omega)\| = 1/dist(i\omega, \sigma(A)) \tag{10.24}$$

is completely determined by the spectrum $\sigma(A)$ alone. The operator dist denotes the shortest distance of ω to the eigenvalues, the spectrum $\sigma(A)$. This explains the success of eigenvalue analysis. In contrast to this, for non-normal operators the behavior of $||R(\omega)||$ may deviate from that dramatically and hence in this context pseudospectral analysis is just the right tool **?**; **?**. More about the dynamics can be learned by examining the pseudospectrum of A in the complex plane. Inspection of many geophysical systems shows that most of the systems fail the normality condition (10.23). The $\epsilon-$pseudospectrum of operator A is defined by two equivalent formulations:

$$
\begin{aligned}
\Lambda_\epsilon(A) \quad &= \quad \{z \in \mathsf{C} : ||(zI - A)^{-1}|| \geq \epsilon^{-1}\} \\
&= \quad \{z \in \mathsf{C} : [\text{ smallest singular value of } (zI - A)] \leq \epsilon\} \quad . \qquad (10.25)
\end{aligned}
$$

This set of values z in the complex plane are defined by contourlines of the resolvent $(zI - A)^{-1}$. The resolvent determines the system's response to a forcing as supplied by external forcing $F(x, t)$, stochastic forcing $g(x)\xi$, or initial/boundary conditions. The pseudospectrum reflects the robustness of the spectrum and provides information about instability and resonance. One theorem is derived from Laplace transformation stating that transient growth is related to how far the $\epsilon-$pseudospectrum extends into the right half plane:

$$
|| \exp(A\,t) || \quad \geq \quad \frac{1}{\epsilon} \sup_{z \in \Lambda_\epsilon(A)} Real(z) \quad . \qquad (10.26)
$$

In terms of climate theory, the pseudospectrum indicates resonant amplification. Maximal amplification is at the poles of $(zI - A)^{-1}$, characterized by the eigenfrequencies. In a system satisfying (10.23), the system's response is characterized solely by the proximity to the eigenfrequencies. In the non-normal case, the pseudospectrum shows large resonant amplification for frequencies which are not eigenfrequencies. This transient growth mechanism is important for both initial value and forced problems.

10.2.4   Resonance in an atmospheric circulation model*

An atmospheric general circulation model PUMA **?** is applied to the problem. The model is based on the multi-level spectral model described by Hoskins and Simmons **?**. For our experiments we chose five vertical levels and a T21 horizontal resolution. PUMA belongs to the class of models of intermediate complexity **?**; it has been used to understand principle feedbacks **?**, and dynamics on long time scales **?**. For simplicity, the equations are scaled here such that they are dimensionless. The model is linearized about a zonally symmetric mean state providing for a realistic storm track at mid-latitudes **?**. In a simplified version of the model and calculating the linear model A with $n = 214$, one can derive the pseudospectrum. Fig. 10.4 indicates resonances besides the poles (the eigenvalues) indicated by crosses. The $Im(z)-$axis shows the frequencies, the $Re(z)-$axis the damping/amplification of the modes. Important modes for the climate system are those with $-0.5 < Im(z) < 0.5$ representing planetary Rossby waves. The basic feature is that transient growth of initially small perturbations can occur even if all the eigenmodes decay exponentially. Mathematically, an arbitrary matrix A can be decomposed as a sum

$$A = D + N \tag{10.27}$$

where A is diagonalizable, and N is nilpotent (there exists an integer $q \in \mathbf{N}$ with $N^q = 0$), and D commutes with N (i.e. $DN = NA$). This fact follows from the Jordan-Chevalley decomposition theorem. This means that we can compute the exponential of (A t) by reducing to the cases:

$$\exp(At) = \exp((D + N)\, t) = \exp(Dt) \quad \exp(Nt) \tag{10.28}$$

where the exponential of Nt can be computed directly from the series expansion, as the series terminates after a finite number of terms. Basically, the number $q \in \mathbf{N}$ is related to the transient growth of the system ($q = 1$ means no transient growth).

The resonant structures are due to the mode interaction: It is not possible to change one variable

without the others, because they are not orthogonal. Interestingly, one can also compute the A^+ model, showing the optimal perturbation of a mode e_i through its biorthogonal vector (2.60).

The analysis indicates that non-normality of the system is a fundamental feature of the atmospheric dynamics. This has consequences for the error growth dynamics, and instability of the system, e.g. **?**; **?**. Similar features are obtained in shear flow systems **?**; **?** and other hydrodynamic applications. This transient growth mechanism is important for both initial value and forced problems of the climate system **?**.
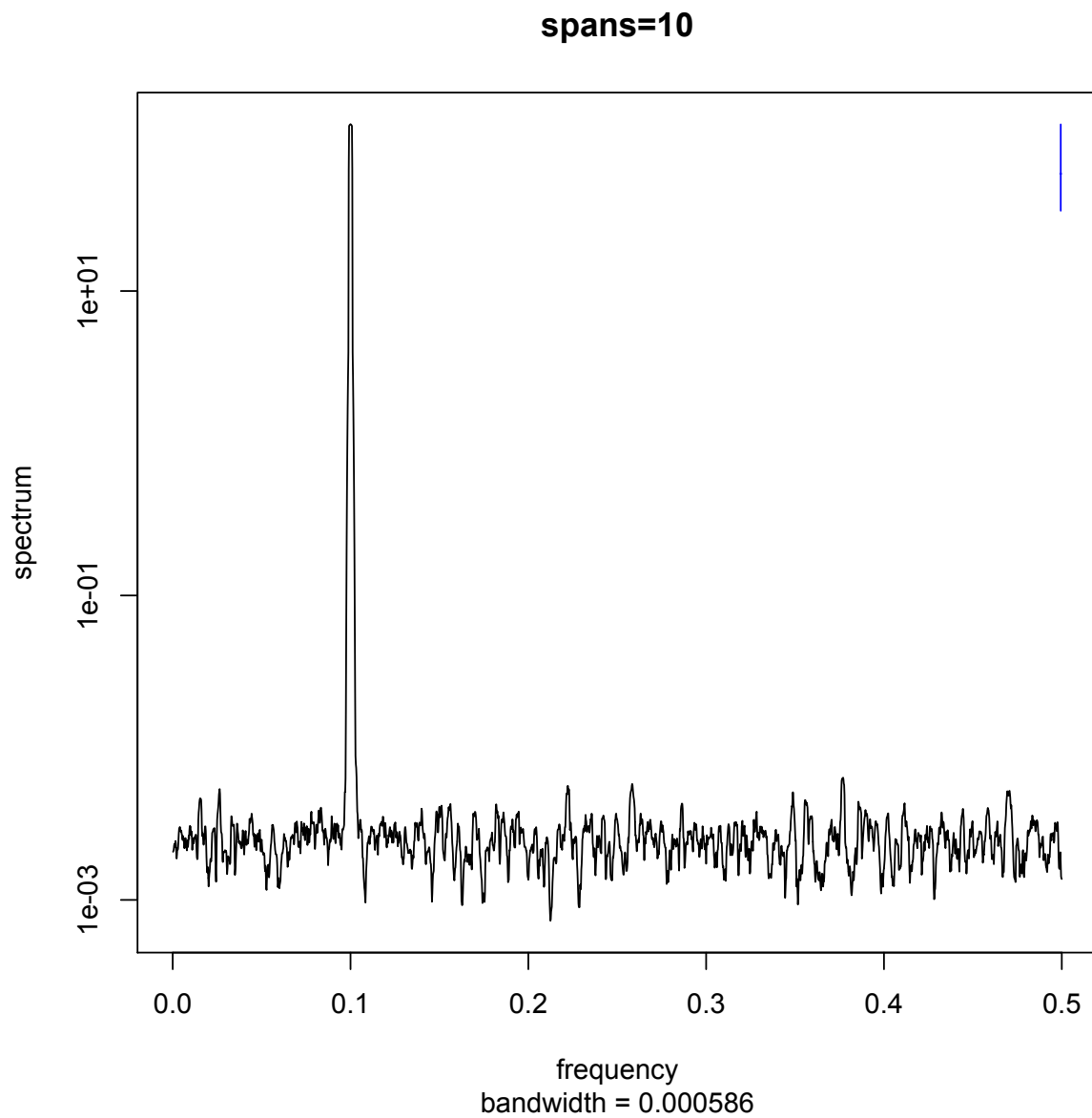
Figure 10.1: Numerical solution of the spectrum for $\sin(2\pi 0.1 t) + \xi$. On the right top corner the cross which shows the bandwith (horizontal bar) and 95% confidence interval (vertical bar). Spans $= 10$.
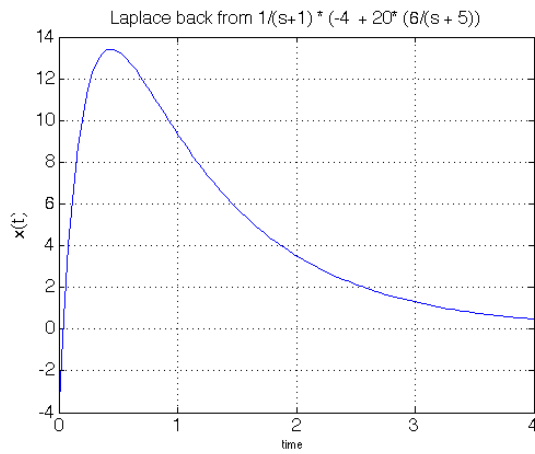
(a) Solution of x(t)
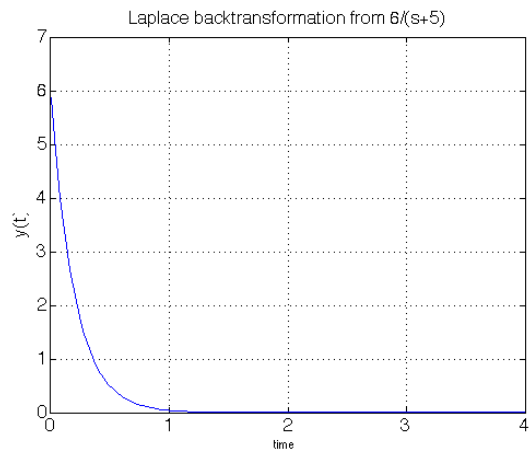
(b) Solution of y(t)

Figure 10.2: Non-normal 2d dynamics with transient growth in x(t).
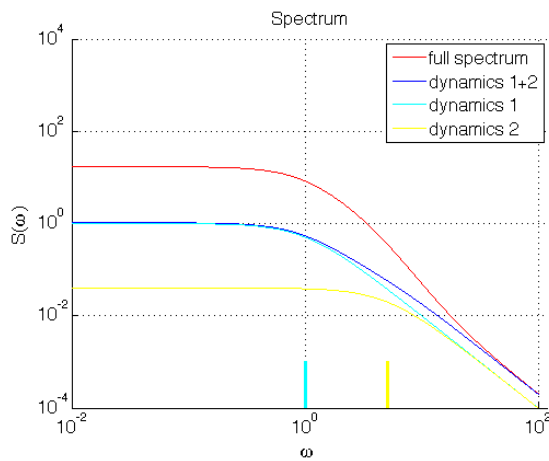
(a) Spectrum
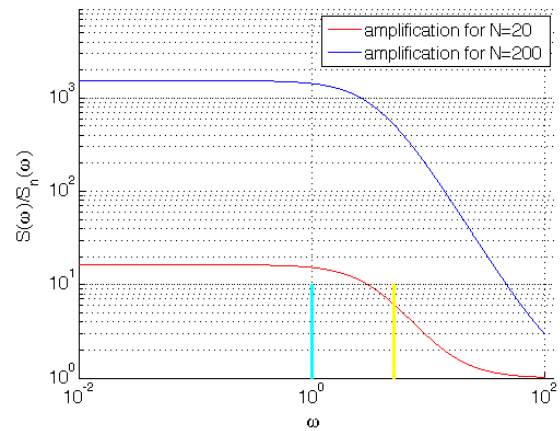
(b) Amplification factor

Figure 10.3: Spectrum of the non-normal 2d dynamics. Dynamics 1 and 2 denote the individual spectra of the modes, the blue curve is the sum. The full spectrum shows more variance in the order of magnitudes depending on the degree of non-normality (b).
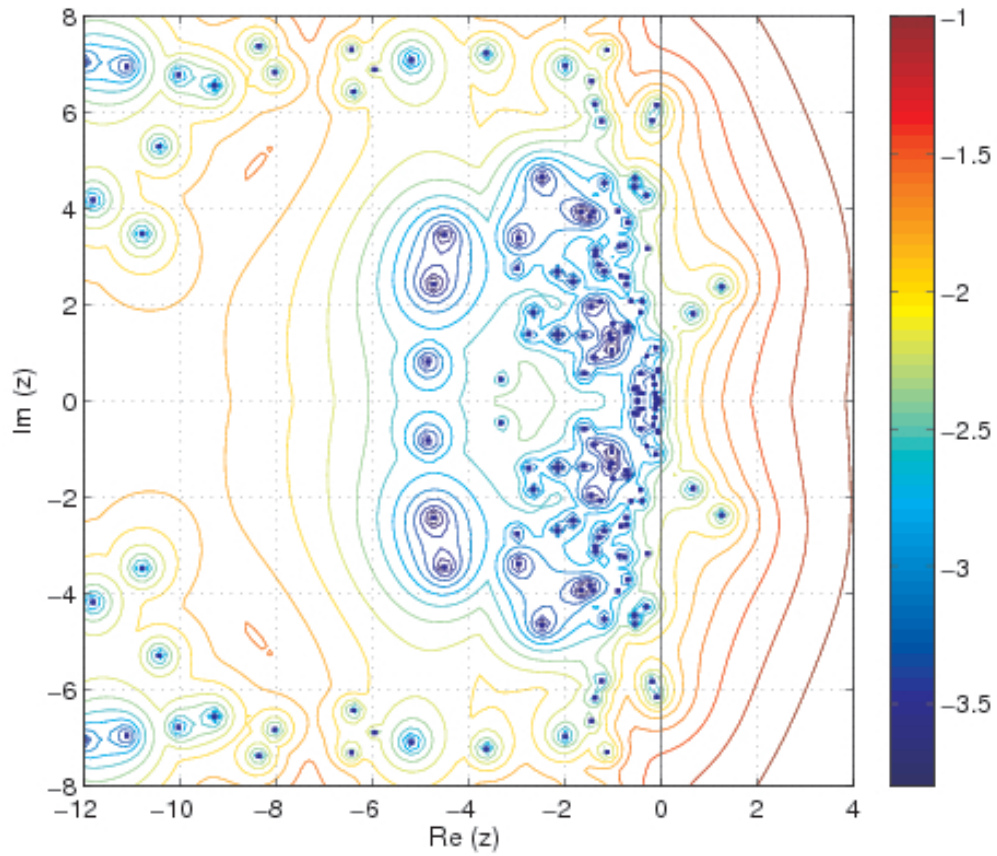
Figure 10.4: Contours of $\log_{10}(1/\epsilon)$. The figure displays resonant structures of the linearized atmospheric circulation model. The modes extend to the right half plane and are connected through resonant structures, indicating for transient growth mechanism inherent in atmospheric dynamics.
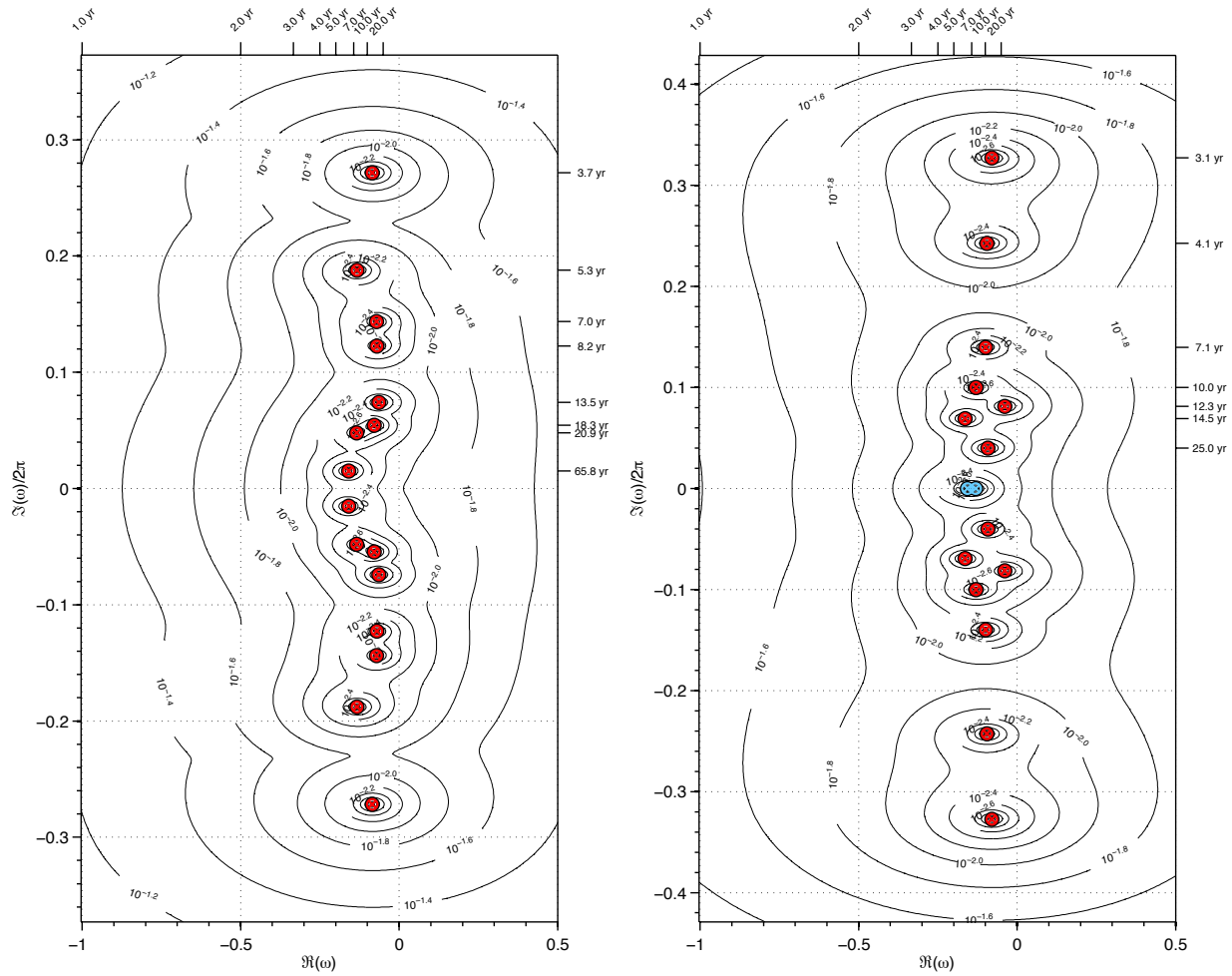
(a) Pseudospectrum North Atlantic Modes

(b) Pseudospectrum Pacific Modes

Figure 10.5: Pseudospectrum with FESOM for the period 1958-1995. The figure displays resonant structures of the linearized ocean circulation model.

10.3   Principal Component Analysis

Principal component analysis (PCA) is a statistical procedure that uses an orthogonal transformation to convert a set of observations of possibly correlated variables into a set of values of linearly uncorrelated variables called principal components. The number of principal components is less than or equal to the number of original variables. This transformation is defined in such a way that the first principal component has the largest possible variance.

One of the most ubiquitous uses of eigenanalysis in data analysis is the construction of EOFs, the topic of this section. EOFs are a transform of the data; the original set of numbers is transformed into a different set with some desirable properties. In this sense the EOF transform is similar to other transforms such as the Fourier or Laplace transforms. In all these cases, we project the original data onto a set of orthogonal functions, thus replacing the original data with the set of projection coefficients on the basis vectors. However, the choice of the specific basis set varies from case to case.

In the Fourier case, for example, the choice is a set of sines and cosines of various frequencies. This is motivated by the desire to identify the principal modes of oscillation of the system. Thus if the signal projects strongly on sine waves of 2 frequencies, we will say that the signal is approximately the linear combination of these 2 frequencies. We will then attribute the remainder to other processes that are more weakly represented in the signal (the signal has low projection on them), and are thus assumed unimportant for the signal. Another important property for a basis is orthogonality (like sines or various frequencies); we would like to account for a certain component of the signal only once. An alternative to the sine/cosine set is a set of orthogonal polynomials, such as those named after Legendre.

The representation of the signal in terms of the projection coefficients on a basis set is often very useful at separating cleanly various scales. For example, if our data is the sea surface temperature of a given ocean basin, we can think of the projection on the lowest frequency wave (the one which has one crest and one trough within the spatial extent of the domain) as representing the ocean's 'large-scale', while that on wavelengths of order 10-100 km as 'eddies'. In EOF analysis we also

project the original data on a set of orthogonal basis vectors. However, the choice of the basis is different. Here, the first EOF is chosen to be the pattern, without the constraint of a particular analytic form, on which the data project most strongly. In other words, the leading EOF (sometime called the 'gravest', or 'leading', mode) is the pattern most frequently realized. The second mode is the one most commonly realized under the constraint of orthogonality to the first one, the third is the most frequently realized pattern that is orthogonal to both higher modes, and so on. Hence the term 'empirical'; we still have an orthogonal basis, like the Fourier or Legendre bases, but whose members are not chosen based on analytic considerations, but based on maximization of the projection of the data on them.

10.3.1 Singular Value Decomposition

In linear algebra, the singular value decomposition (SVD) is a factorization of a real or complex matrix. It is the generalization of the eigendecomposition of a positive semidefinite normal matrix (for example, a symmetric matrix with positive eigenvalues) to any $m \times n$ matrix. It has many useful applications in signal processing and statistics. Suppose M is a $m \times n$ matrix whose entries are real or complex numbers. Then there exists a factorization, called a singular value decomposition of M, of the form

$$\mathrm{M} = \mathrm{U}\Sigma\mathrm{V}^*$$ (10.29)

where U is a $m \times m$ unitary matrix, Σ is a $m \times n$ diagonal matrix with non-negative real numbers on the diagonal, and V^* is a $n \times n$ unitary matrix. The diagonal entries σ_i of Σ are known as the singular values of M. A common convention is to list the singular values in descending order. In this case, the diagonal matrix Σ is uniquely determined by M (though not the matrices U and V).

The columns of U and V are orthonormal bases. Since U and V* are unitary, the columns of each of them form a set of orthonormal vectors, which can be regarded as basis vectors. In short, the columns of $U, U^*, V,$ and V^* are orthonormal bases.

Example: Consider the 4×5 **matrix**

$$
\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{bmatrix}
\tag{10.30}
$$

A singular value decomposition of this matrix is given by $\mathbf{U\Sigma V^*}$

$$
\mathbf{U} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{bmatrix}
\tag{10.31}
$$

$$
\mathbf{\Sigma} = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\tag{10.32}
$$

$$
\mathbf{V^*} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\ 0 & 0 & 0 & 1 & 0 \\ -\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2} \end{bmatrix}
\tag{10.33}
$$

Notice Σ is zero outside of the diagonal and one diagonal element is zero. Furthermore, because the matrices U and V^* are unitary, multiplying by their respective conjugate transposes yields identity matrices. In this case, because U and V^* are real valued, they each are an orthogonal

matrix.

$$
\mathbf{U}\mathbf{U}^{\mathbf{T}} =
\begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix}
=
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
= \mathbf{I_4}
\tag{10.34}
$$

$$
\mathbf{V}\mathbf{V}^{\mathbf{T}} =
\begin{bmatrix}
0 & 0 & \sqrt{0.2} & 0 & -\sqrt{0.8} \\
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & \sqrt{0.8} & 0 & \sqrt{0.2}
\end{bmatrix}
\cdot
\begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
\sqrt{0.2} & 0 & 0 & 0 & \sqrt{0.8} \\
0 & 0 & 0 & 1 & 0 \\
-\sqrt{0.8} & 0 & 0 & 0 & \sqrt{0.2}
\end{bmatrix}
$$

$$
=
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
= \mathbf{I_5}
\tag{10.35}
$$

10.3.2 Empirical orthogonal functions

The spatial inter-dependence implies that there is a lot of redundant information stored in a $n_x \times n_y$ map, and that the information can be compressed to just a few numbers describing the state of that field. In geophysics, geographically weighted PCAs are often used, which normally are referred to as empirical orthogonal functions (EOFs). The EOFs can be regarded as a kind of eigenvectors, which are aligned so that the leading EOFs describe the spatially coherent pattern that maximises its variance. The EOFs are often used as basis functions (a new set of axes or reference frame).

Often data matrices have 2 distinct dimensions that correspond to different physical units. For example, suppose we have $\mathbf{A}_{M \times N}$ representing monthly surface air temperatures along the 35°N parallel at fixed spatial intervals over 1 month=N/12 years. The column M-vector $\mathbf{a}_j$ comprising

all space points at time j is **A**'s *jth* column, $j = 1, 2, \ldots, N$ for the N times. Using the SVD representation $\mathbf{A} = \mathbf{U\Sigma V}^T$ we get the modes of **A**; **U**'s columns are **A**'s EOFs, while **V**'s columns are the corresponding 'principal components'.[2]

Given the above arrangement of **A**, with time running along the rows and space running along the columns (which is a very common convention), **U**'s columns span **A**'s column space, which corresponds to the spatial dimension. They are **A**'s EOFs. Similarly, **V**'s columns span **A**'s row space, which corresponds to the timeseries at a given spatial location. Because the modes are arranged in descending order ($\sigma_i > \sigma_{i+1}$), $\mathbf{u}_1$, **U**'s 1st column, is the spatial pattern most frequently realized, the 2nd is the spatial pattern orthogonal to **U**'s first column that is most frequently realized, and so on.

Example 1 of an EOF analysis   Consider the signal

$$f(x, y) = A \, \cos\left(\frac{2\pi x}{196}\right) \, \cos\left(\frac{2\pi y}{100}\right) + B \, \cos\left(\frac{2\pi x}{98}\right) \, \cos\left(\frac{2\pi y}{50}\right) + \xi$$

with

$$A \sim N(0, 0.7) \quad x = 0, 1, \ldots, 98$$
$$B \sim N(0, 0.1) \quad y = 0, 1, \ldots, 50$$
$$\xi \sim N(0, 0.1)$$

The signal is thus a linear combination of (primarily) the first rhs term (because A's variance is 7 times larger than other additive terms), (some of) the 2nd rhs term, and unstructured noise $\xi(x, y)$. The 2 deterministic patterns are shown in Fig. 10.6, panels b and d. Note that they are mutually orthogonal (the cosines in both x and y are Fourier frequencies).

Now imagine 50 such $f(x, y)$ fields (x-y maps representing random combinations of the 2 patterns plus noise as given above), or a series of 51×99 matrices $\mathbf{F}_i, \quad i = 1, 2, \ldots, 50$. This is meant to simulate a geophysical situation in which a certain time-dependent field, say sea-level pressure, is generated by some known, deterministic, physics, plus other, low-amplitude,

---

[2]For this reason, in some fields the exact same analysis is called 'principal component analysis'.
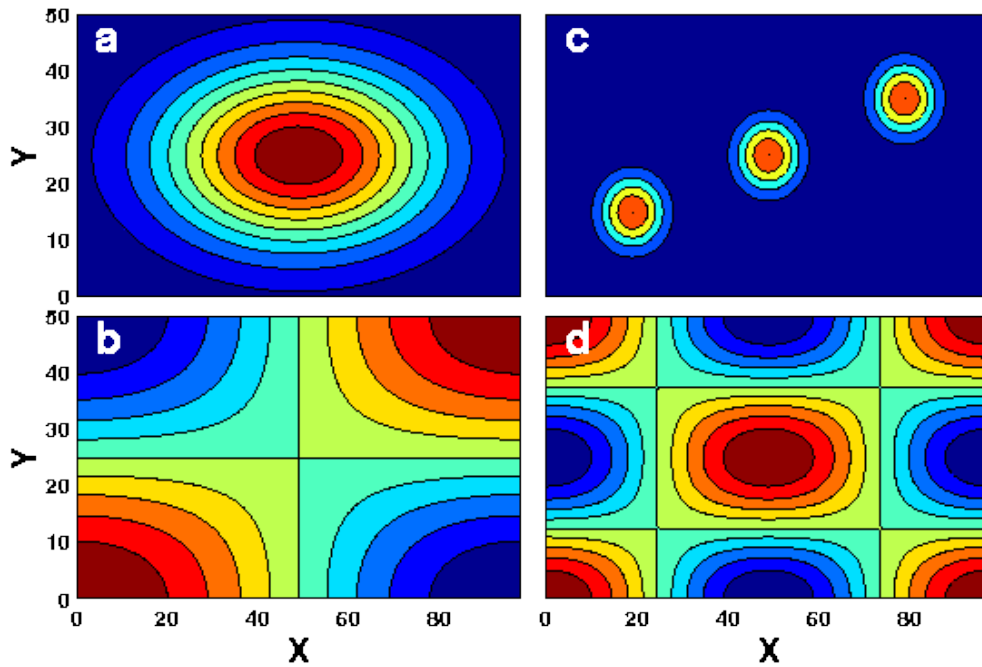
Figure 10.6: Four spatial patterns used to generate the combined synthetic signal.

processes, collectively represented here as $\xi(x, y, t)$. Given 50 realizations of this process, we want to identify the dominant spatial patterns of **F**, or, put differently, the spatial structures of **F** that are most frequently realized.

To identify these structures, we first make the 3-dimensional array $\mathbf{F}(t) = \left\{ F_{ij}^{k} \right\}$ (where i is the latitude index, j is the longitude index, and k is the time index) into a 2-dimensional array (matrix), by storing an entire field in one column vector. That is

$$
\mathbf{A} = \begin{pmatrix} \vdots & \vdots & & \vdots \\ \mathbf{a_1} & \mathbf{a_2} & \cdots & \mathbf{a_{50}} \\ \vdots & \vdots & & \vdots \end{pmatrix},
$$

where each of **A**'s columns, $\mathbf{a_k}$, comprises the F_{ij}^{k} for all i and j of a given k. The order of the reshaping of each of the $\mathbf{F}^{k}$ matrices into a single column vector is not important. Now all the information we have about **F** is condensed into a single matrix, **A**. If we next use **A**'s SVD

representation, $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$, and reshape $\mathbf{U}$'s columns in a manner exactly opposite to the one we employed while forming $\mathbf{A}$ from $\mathbf{F}$, we get $\mathbf{F}$'s EOFs.

To demonstrate the method in action, let's use the 50 fields of the above $\boldsymbol{f(x,y,t)}$, generated from the patterns shown in Fig. 10.6 b and d. Fig. 10.7 a, b and c show the 3 leading EOFs of the cosine signal. Note how both generating patterns are well reproduced by the method (the leading 2 patterns), despite the noise and the random blending of the the signals by the amplitudes A and B. Note also the sign reversal, which is completely immaterial - the singular values (and hence the EOFs and Principal Components) are all known to within a sign, as they are the square root of the eigenvalues of $\mathbf{A}\mathbf{A}^T$ and $\mathbf{A}^T\mathbf{A}$. Clearly, the 3rd pattern is structureless noise.

A possible critique of the previous example is that we made the method's job particularly (and artificially) easy by using 2 mutually orthogonal generating patterns. This can be fair - if the method is designed to turn arbitrary signals into an orthogonal decomposition of those signals, the real test of the method is with non-orthogonal signals.

Fig. 10.7 d-f show that the method functions well even when the input signal is not artificially orthogonal. The 2 leading modes are nicely reproduced with good fidelity (compared with the generating patterns). The 3rd, while clearly structured (unlike the case with the 2 orthogonal cosines), is a combination of the generating patterns, not an individual pattern. This is the consequence of the non-orthogonality of the exponential generating patterns. Since they are not orthogonal either to each other or to the cosines, they project on them, resulting in the blend shown in Fig. 10.7 f.

It is always extremely important to examine the fraction of the total variance the various modes account for. For the 2 synthetic cases above, this is shown in Fig. 10.8. (Note that only the leading 9 are shown, of of the 50 total. The rest are very near zero in both cases.) The cosine signals are very similar in both (modes 1 and 2). Higher modes differ. In the cosine only, where the only reminder is noise, it is roughly equally distributed over the entire spectrum. Conversely, in the case of the added exponentials, the remainder has 2 structured modes (the 2 exponential terms, and indeed the singular values 3 and 4 are distinct from zero. The rest, just like in the pure cosine case,
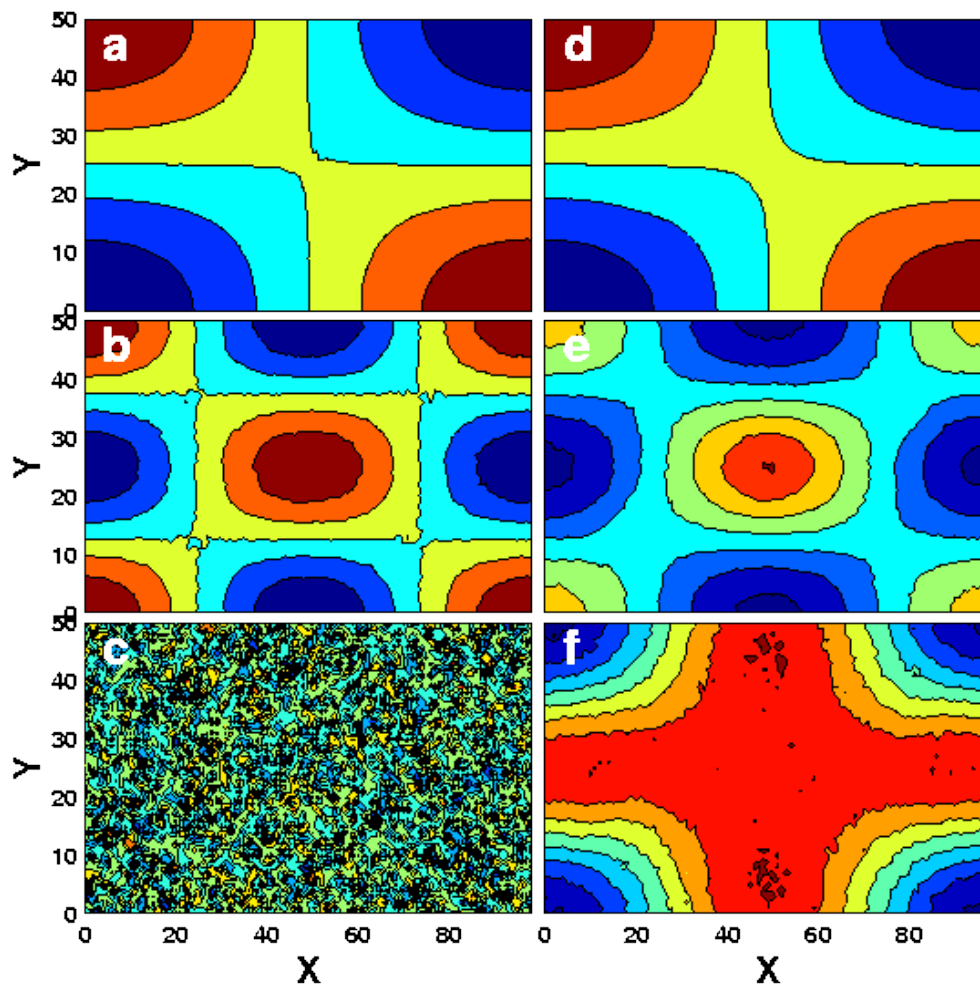
Figure 10.7: The 3 leading spatial modes (EOFs) of 2 signals. The left panels are for the signal comprising cosines only. The right panels show the EOFs of the signal with both cosine and exponential terms, as described in the text.
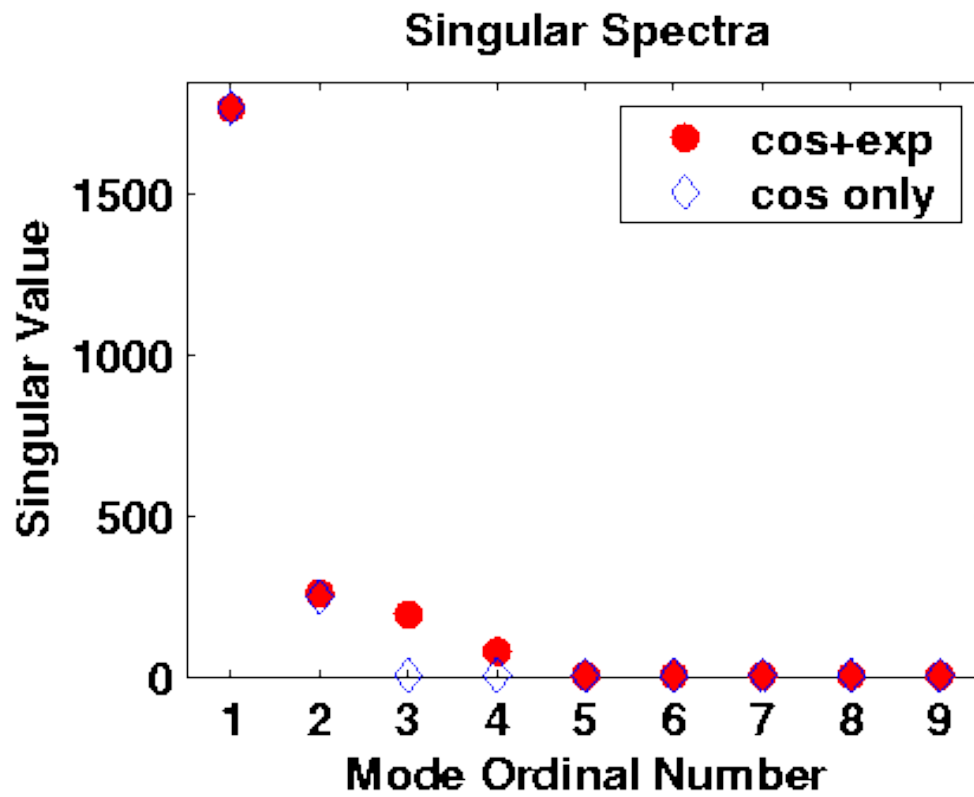
Figure 10.8: The singular spectra of the 2 synthetic cases discussed in the text. The empty blue diamonds are for the cosines-only case, while the solid red circles are for the case with all 4 generating patterns.

are statistically indistinguishable from zero.[3]

---

[3]Note that the decrease in amplitude with mode number (the falloff of the singular spectrum) is a property of the analysis, and does not always contain useful information. There is a substantial body of literature about the issue of the appropriate cutoff of the singular spectrum, beyond which, it is assumed, there is little or no useful information. The most commonly used cutoff rule in geophysics is the so-called 'Rule N', which basically retains only those modes whose amplitudes stand out above the population of singular spectra extracted from a large number of synthetic matrices of the same dimensions as the one being tested. We will not treat this issue here.
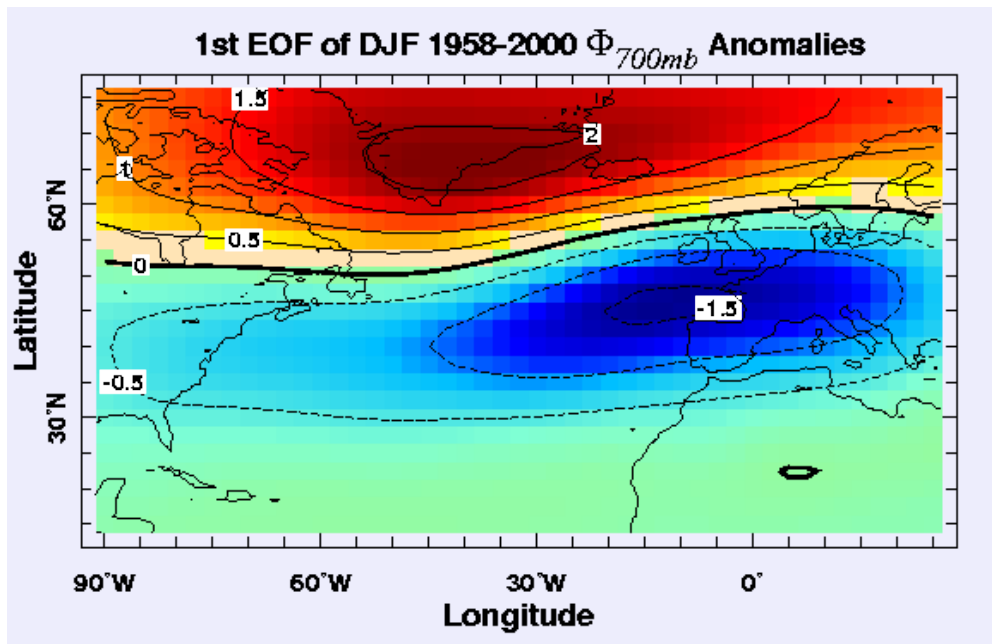
Figure 10.9: Leading mode of the winter 700 mb geopotential height anomalies over the Atlantic sector during the indicated period.

Example 2 of an EOF analysis   Fig. 10.9 shows an example of a real-life use of EOFs. Height anomalies within this domain (which encompasses ∼5000 grid points) obviously display a very rich spectrum of variability in time and space. And yet, when piped through the EOF algorithm, a very clear and coherent large scale structure emerges. This information is corroborated by Fig. 10.10, where the singular spectrum falloff clearly singles out the gravest mode as substantially more important than the 2nd mode, accounting for approximately twice as much variance.
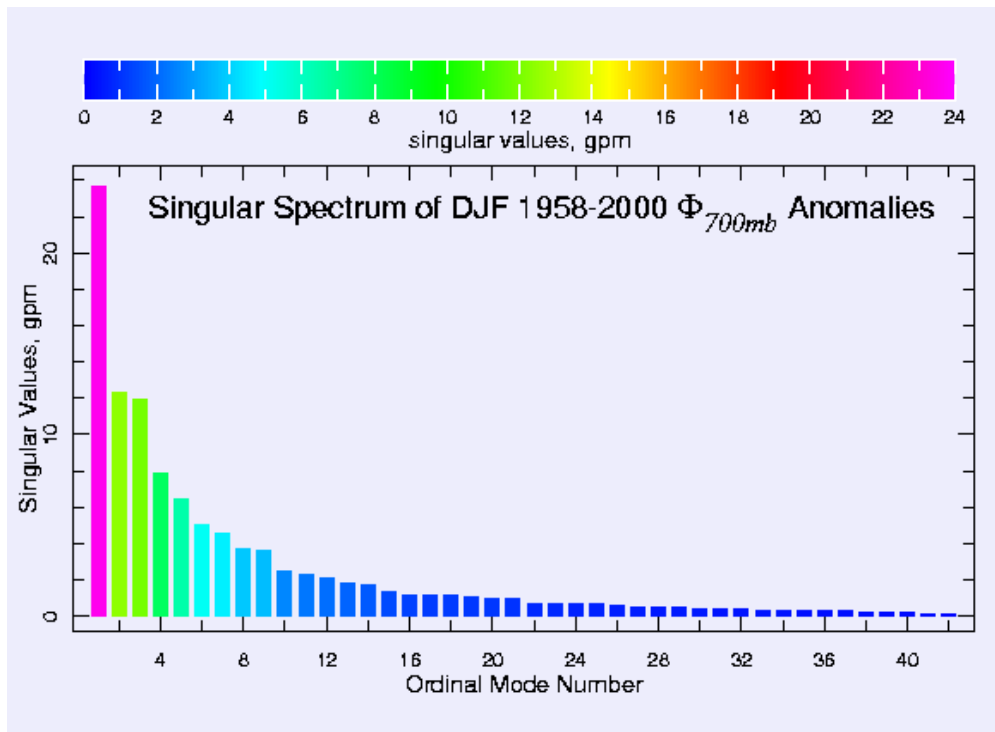
Figure 10.10: The singular spectrum of the observed winter (DJF) 700 mb geopotential height anomalies between 1958 and 2000.
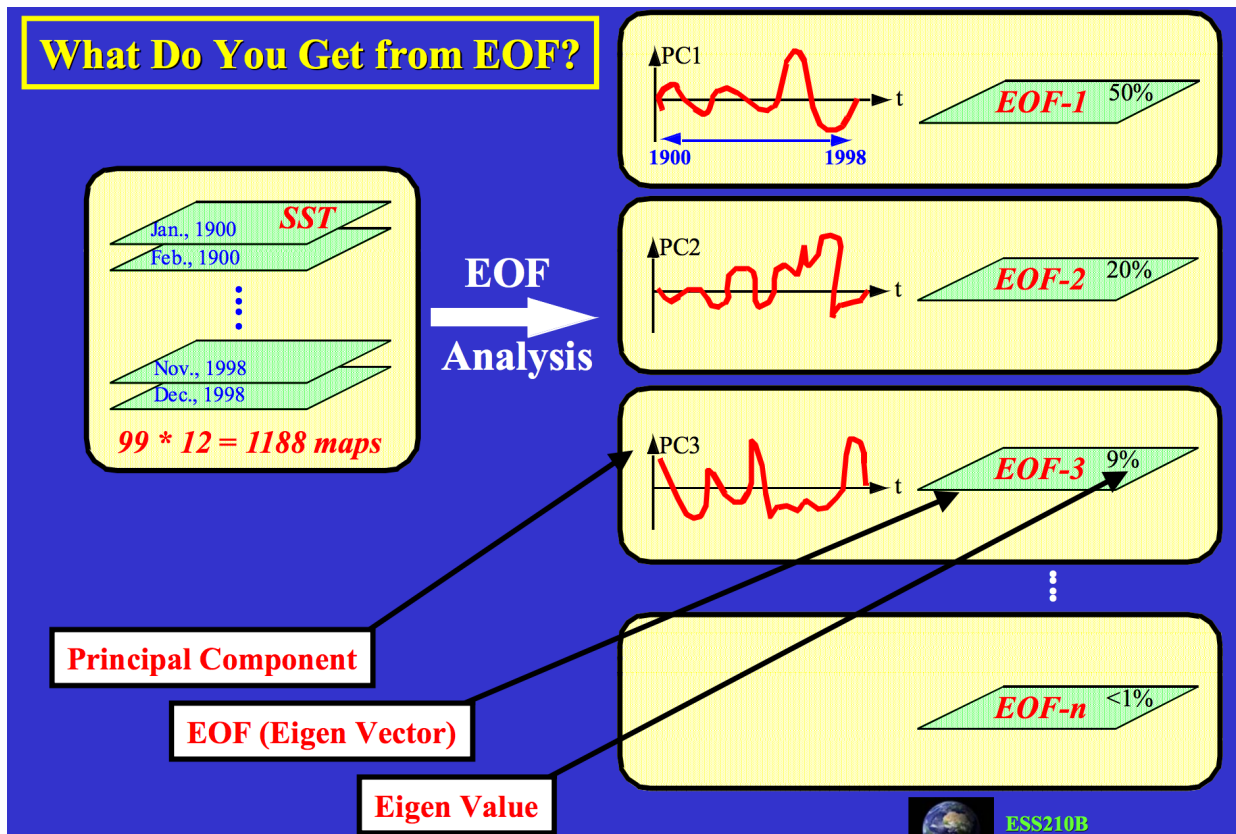
Figure 10.11: how to calculate EOFs.

Example 3 of an EOF analysis   The third example shows two different ways oh how to calculate

EOFs of time series.

Based on pca:

```
#EOF's and PCs from the pca
rm(list=ls())                               #clean the environment
a=read.table("ser1.txt")                     #read ser 1
b=read.table("ser2.txt")                     #read ser2
x=a[,1]
y=b[,1]
xa=x-mean(x)                                  #anomalies x
ya=y-mean(y)                                  #anomalies y
mdat=cbind(xa,ya)
aa=prcomp(mdat)
eof1=aa$rotation[1,]                          #eof1
eof2=aa$rotation[2,]                          #eof2
pc=predict(aa)
pc1=pc[,1]                                    #pc1
pc2=pc[,2]                                    #pc2
```

```
aa$sdev            # explained variance

heading = paste("pc1")
plot(xa, main=heading)
  lines(pc1)

ppc1=100*aa$sdev[1]/(aa$sdev[1]+aa$sdev[2]) # explained variance eof1
ppc2=100*aa$sdev[2]/(aa$sdev[1]+aa$sdev[2]) # explained variance eof2

xxx=eof1[1]*pc1 + eof2[1]*pc2

heading = paste("orig time ser1")
plot(xxx, main=heading)
lines(xa)


heading = paste("pc2")
plot(ya, main=heading)
  lines(pc2)
```

Here another approach based on covariance matrix:

```
#EOF's and PCs from the covariance matrix
rm(list=ls())                                 #clean the environment
a=read.table("ser1.txt")                 #read ser 1
b=read.table("ser2.txt")                 #read ser2
x=a[,1]
y=b[,1]
xa=x-mean(x)                             #anomalies x
ya=y-mean(y)                             #anomalies y
mdat=cbind(xa,ya)
n=length(x)
s=t(mdat)*mdat/(n-1)                     #covariance matrix
ee=eigen(s)                               #eigenvect val of cov. matrix.
e=ee$vectors                              #eigenvectors
val=ee$values                             #eigenvalues
proc=val/sum(val)*100                    #percent of explained variance
pc=t(e)*t(mdat)                         #pc matrix
eof1=e[,1]
eof2=e[,2]
pc1=pc[1,]
pc2=pc[2,]
```

10.4   Pattern of climate variability

A mode of variability is a climate pattern with identifiable characteristics, specific regional effects, and often oscillatory behavior. Many modes of variability are used by climatologists as indices

to represent the general climatic state of a region affected by a given climate pattern. Measured via an empirical orthogonal function analysis, the mode of variability with the greatest effect on climates worldwide is the seasonal cycle, followed by El Nino-Southern Oscillation, followed by the Atlantic Multidecadal Oscillation.

Other well-known modes of variability include:

- The Antarctic oscillation (AAO)

- The Arctic oscillation (AO)

- The Atlantic multidecadal oscillation (AMO)

- The Indian Ocean Dipole

- The Madden-Julian oscillation (MJO)

- The North Atlantic oscillation (NAO)

- The Pacific decadal oscillation (PDO)

- The Pacific-North American teleconnection pattern (PNA)

- The Quasi-biennial oscillation (QBO)

Links: https://en.wikipedia.org/wiki/Climate_pattern

10.4.1 ENSO

El Nino Southern Oscillation (ENSO), refers to the effects of a band of sea surface temperatures which are anomalously warm or cold for long periods of time that develops off the western coast of South America and causes climatic changes across the tropics and subtropics. The "Southern Oscillation" refers to variations in the temperature of the surface of the tropical eastern Pacific Ocean, with warming known as El Nino and cooling known as La Nina, and in air surface pressure in the tropical western Pacific.

The two variations are coupled: the warm oceanic phase, El Nino, accompanies high air surface pressure in the western Pacific, while the cold phase, La Nina, accompanies low air surface pressure in the western Pacific. The two phases relate to the Walker circulation, discovered by Gilbert Walker during the early twentieth century. The Walker circulation is caused by the pressure gradient force that results from a high pressure system over the eastern Pacific ocean, and a low pressure system over Indonesia. When the Walker circulation weakens or reverses, an El Nino results, causing the ocean surface to be warmer than average, as upwelling of cold water occurs less or not at all. An especially strong Walker circulation causes a La Nina, resulting in cooler ocean temperatures due to increased upwelling.

Mechanisms that cause the oscillation remain under study. The extremes of this climate pattern's oscillations cause extreme weather (such as floods and droughts) in many regions of the world. Developing countries dependent upon agriculture and fishing, particularly those bordering the Pacific Ocean, are the most affected.

Fig. 10.12: wind stress $\tau_x < 0$ quiet deep ocean with ρ_1, capped by a mixed layer with depth h and $\rho_2 < \rho_1$

Fig. 10.13: Ekman-relation on the near equator:

$$-\beta y \int_{-H}^{0} v \, dz = \tau_x / \rho \qquad (10.36)$$

with $\beta = 2\Omega/a = 2.3 \cdot 10^{-11} m^{-1} s^{-1}$.
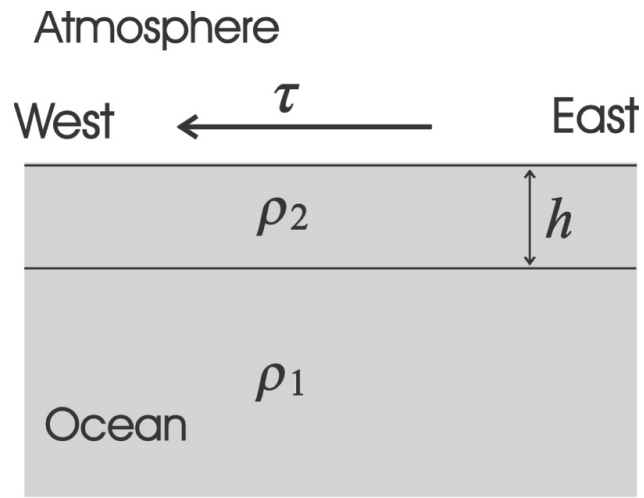
Figure 10.12: Schematic of a two layer ocean model.
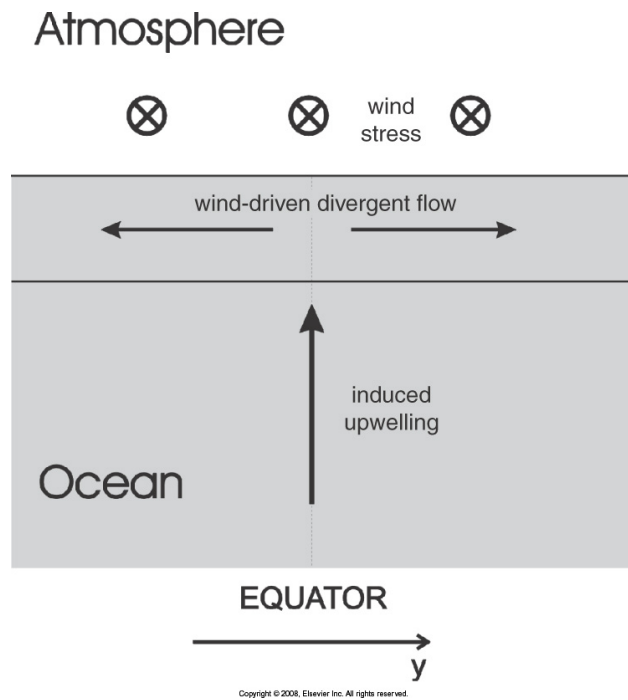
Figure 10.13: Schematic meridional section of near equatorial upwelling.

Links: https://www.youtube.com/watch?feature=player_embedded&v=huweohIh_Bw
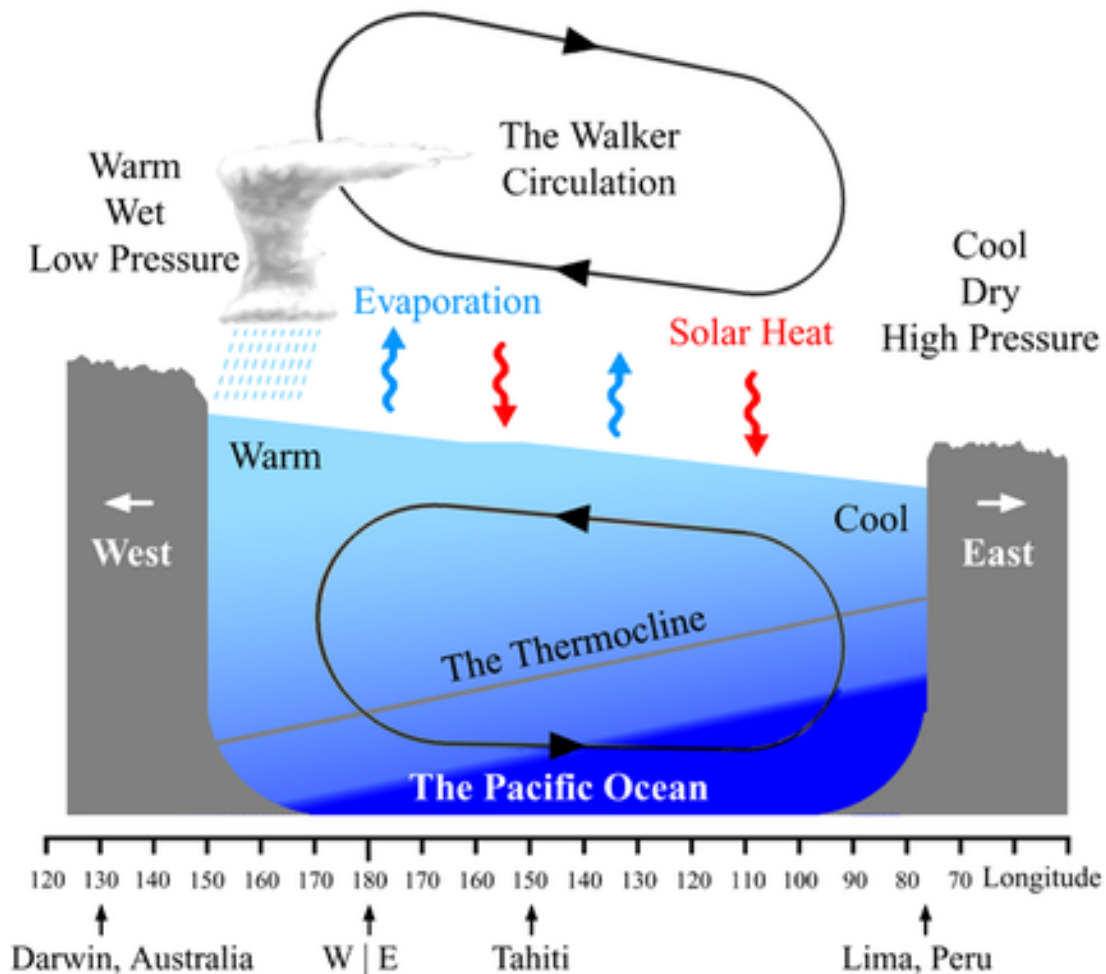
Figure 10.14:   Diagram of the quasi-equilibrium and La Nina phase of the Southern Oscillation. The Walker circulation is seen at the surface as easterly trade winds which move water and air warmed by the sun towards the west. The western side of the equatorial Pacific is characterized by warm, wet low pressure weather as the collected moisture is dumped in the form of typhoons and thunderstorms. The ocean is some 60 centimetres higher in the western Pacific as the result of this motion. The water and air are returned to the east. Both are now much cooler, and the air is much drier. An El Nino episode is characterised by a breakdown of this water and air cycle, resulting in relatively warm water and moist air in the eastern Pacific.
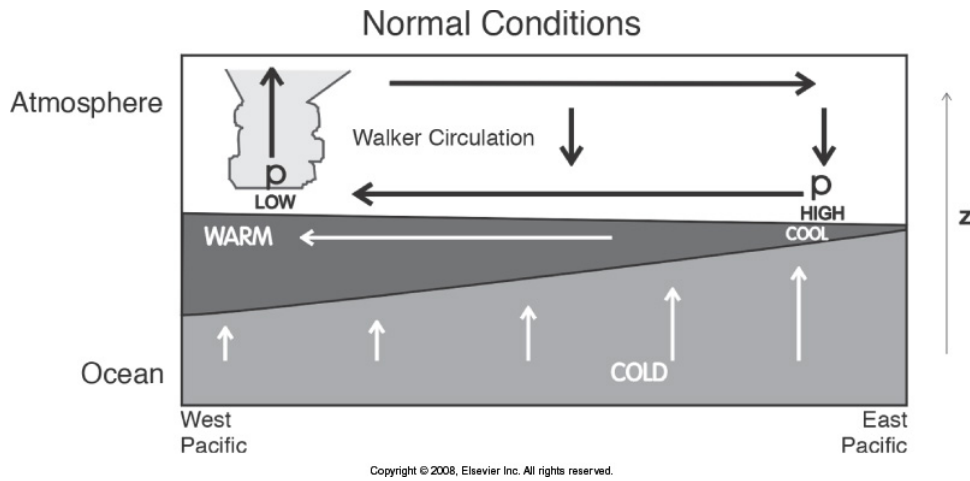
Figure 10.15: Schematic E-W cross section
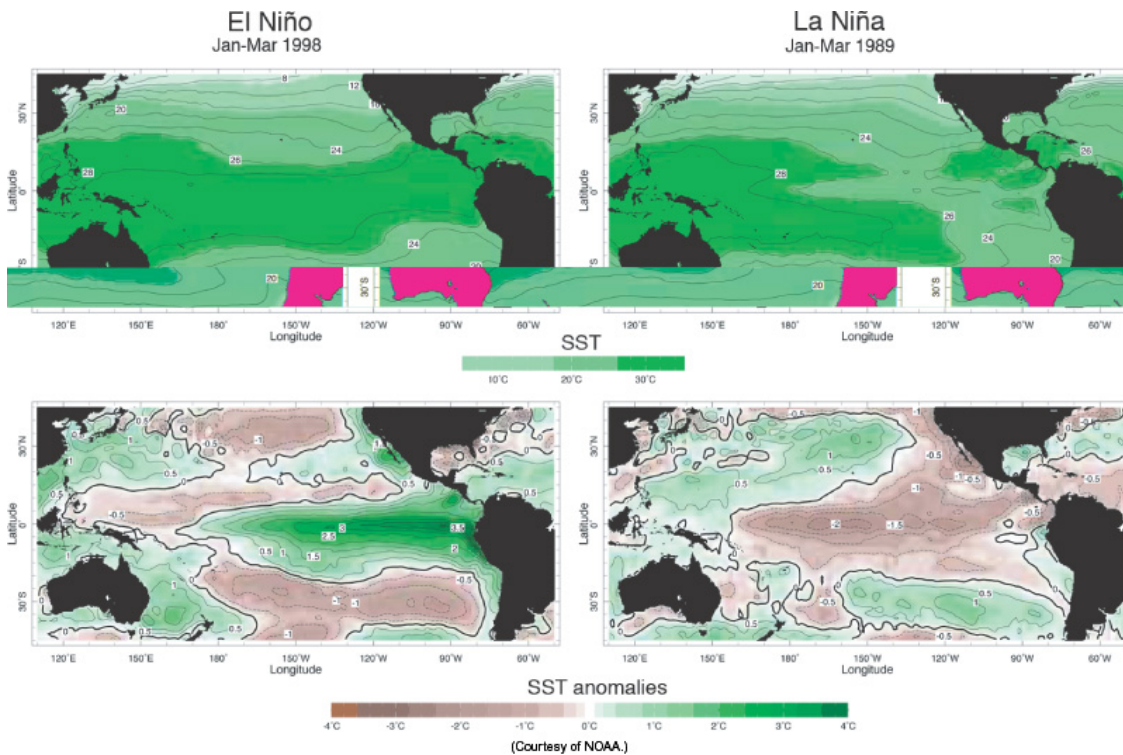
Figure 10.16: SSTs for an El Nino (left) and Lan Nina (right) event.
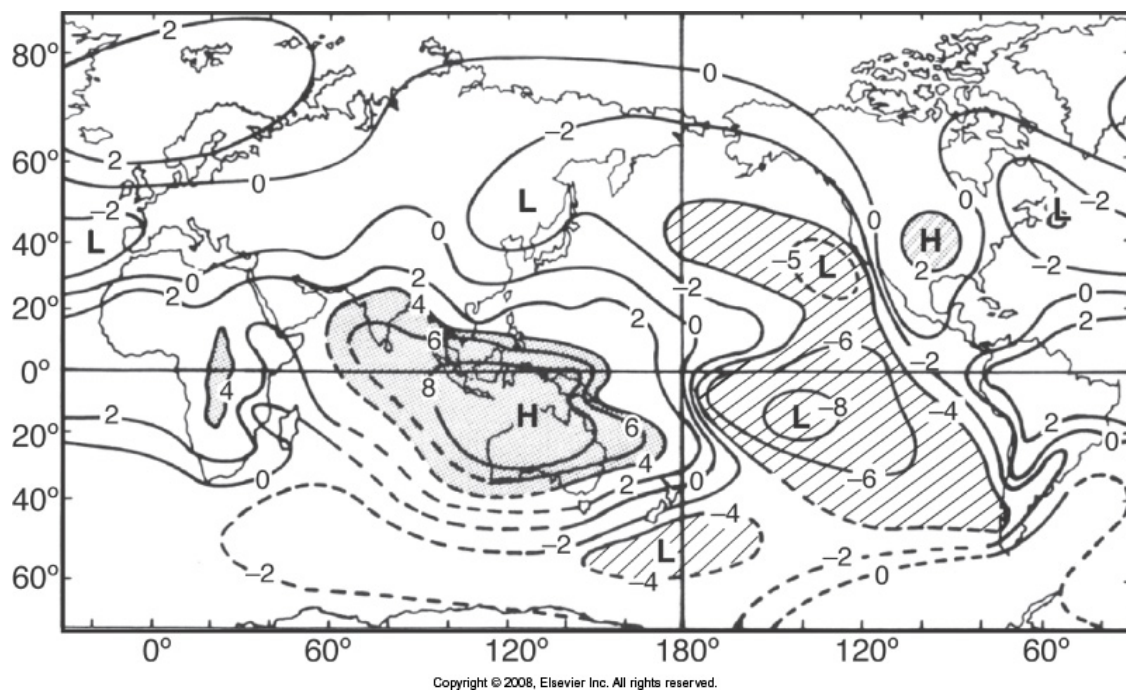
Figure 10.17: SLP signature Lan Nina event

10.4.2 NAO

The North Atlantic Oscillation (NAO) is a climatic phenomenon in the North Atlantic Ocean of fluctuations in the difference of atmospheric pressure at sea level between the Icelandic low and the Azores high. Through fluctuations in the strength of the Icelandic low and the Azores high, it controls the strength and direction of westerly winds and storm tracks across the North Atlantic. It is part of the Arctic oscillation, and varies over time with no particular periodicity.

The NAO was discovered in the 1920s by Sir Gilbert Walker. Unlike the El Nino-Southern Oscillation phenomenon in the Pacific Ocean, the NAO is a largely atmospheric mode. It is one of the most important manifestations of climate fluctuations in the North Atlantic and surrounding humid climates.

The North Atlantic Oscillation is closely related to the Arctic oscillation (AO) or Northern Annular Mode (NAM), but should not be confused with the Atlantic Multidecadal Oscillation (AMO).

10.4.3 Atlantic Multidecadal Oscillation

The Atlantic multidecadal oscillation (AMO) is a mode of natural variability occurring in the North Atlantic Ocean and which has its principle expression in the sea surface temperature (SST) field. While there is some support for this mode in models and in historical observations, controversy exists with regard to its amplitude, and in particular, the attribution of sea surface temperatures in the tropical Atlantic in areas important for hurricane development. The AMO assumedly runs through an 70-year cycle, and its current warm phase (after peaks in 1880 and 1950) is projected to peak in 2020.

The AMO signal is usually defined from the patterns of SST variability in the North Atlantic once any linear trend has been removed. This detrending is intended to remove the influence of greenhouse gas-induced global warming from the analysis. However, if the global warming signal is significantly non-linear in time (i.e. not just a smooth increase), variations in the forced signal will leak into the AMO definition. Consequently, correlations with the AMO index may alias
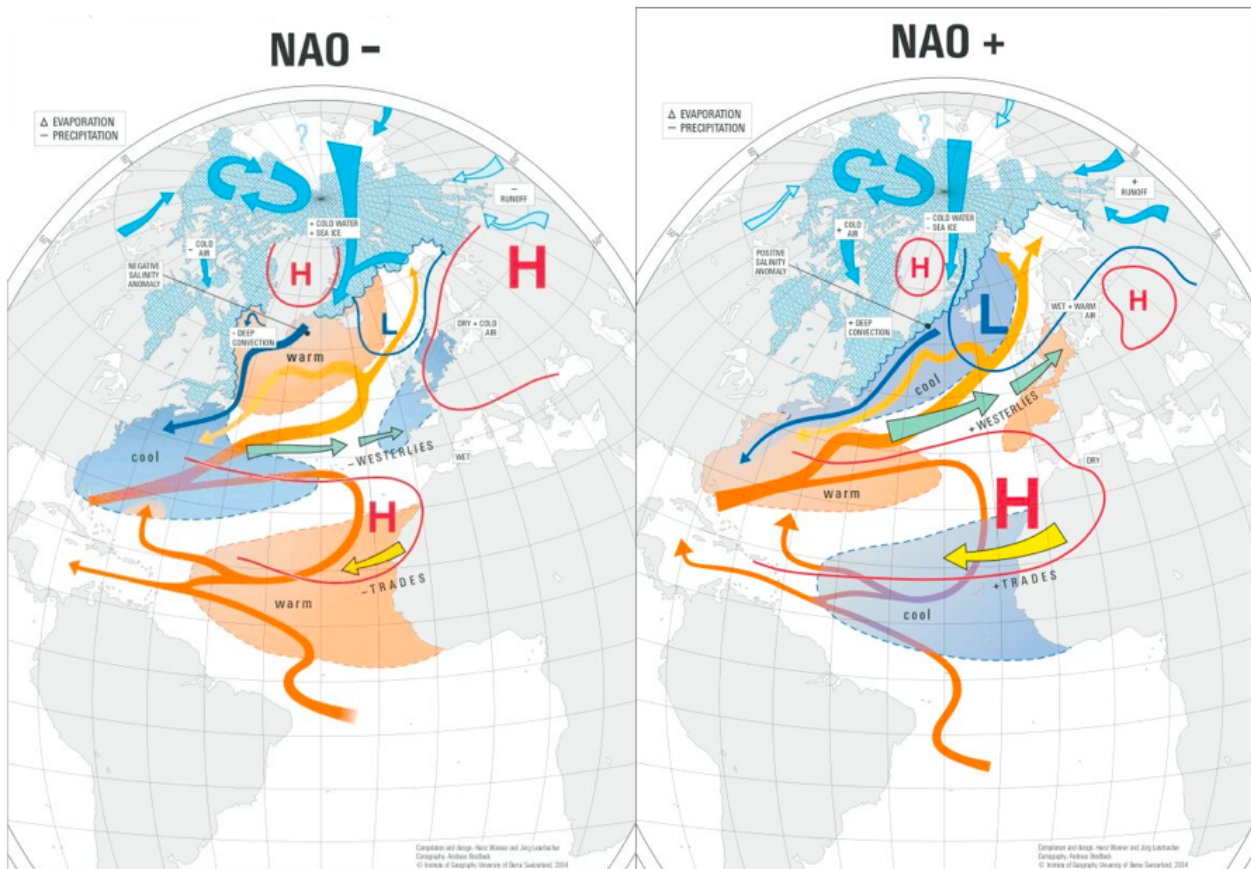
Figure 10.18: NAO plus and minus

effects of global warming. In 2008, new models revealed that global warming should reduce the frequency of hurricanes overall, while intensity might increase in some areas. Because reliable records of hurricane strength and frequency only extend back to approximately 1970, researchers have faced difficulty in developing reliable models.

In models, AMO-like variability is associated with small changes in the North Atlantic branch of the Thermohaline Circulation, however historical oceanic observations are not sufficient to associate the derived AMO index to present day circulation anomalies.

The AMO index is correlated to air temperatures and rainfall over much of the Northern Hemisphere, in particular, North America and Europe such as North Eastern Brazilian and African Sahel rainfall and North American and European summer climate. It is also associated with changes in
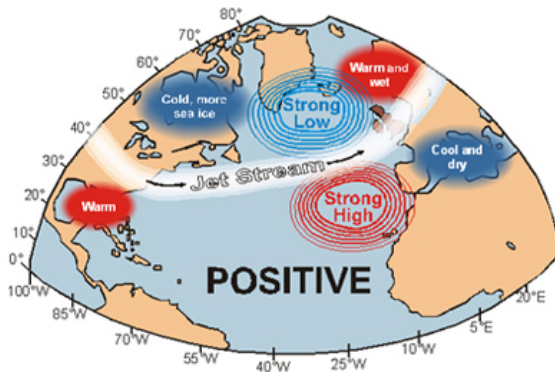
Fig. 2. Diagram showing the NAO in the positive mode. © Severn Estuary Partnership, 2012.
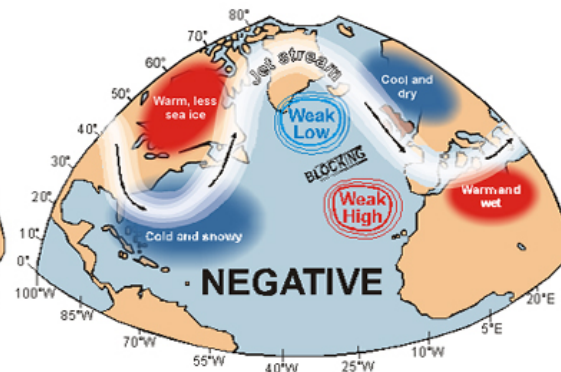
Fig.3. Diagram showing the NAO in the negative mode.© Severn Estuary Partnership, 2012

Figure 10.19: NAO

the frequency of North American droughts and is reflected in the frequency of severe Atlantic hurricanes. It alternately obscures and exaggerates the global increase in temperatures due to human-induced global warming.

Recent research suggests that the AMO is related to the past occurrence of major droughts in the Midwest and the Southwest. When the AMO is in its warm phase, these droughts tend to be more frequent or prolonged. Vice-versa for negative AMO (cool phase). Two of the most severe droughts of the 20th century occurred during the positive AMO between 1925 and 1965: The Dust Bowl of the 1930s and the 1950s drought. Florida and the Pacific Northwest tend to be the opposite warm AMO, more rainfall.

Climate models suggest that a warm phase of the AMO strengthens the summer rainfall over India and Sahel and the North Atlantic tropical cyclone activity.[3] Paleoclimatologic studies have confirmed this pattern increased rainfall in AMO warmphase, decreased in cold phase for the Sahel over the past 3,000 years.

10.4.4 Reconstructing past climates from high-resolution proxy data

It is difficult to forecast the long term climatic changes on the basis of the instrument records because of their short time series. Accordingly, how to apply the geological and phenological records to reconstruct paleo-environmental changes is very important for studying the global changes. Deep-sea sediments, ice cores, lake sediments, and corals have been widely used in the recent
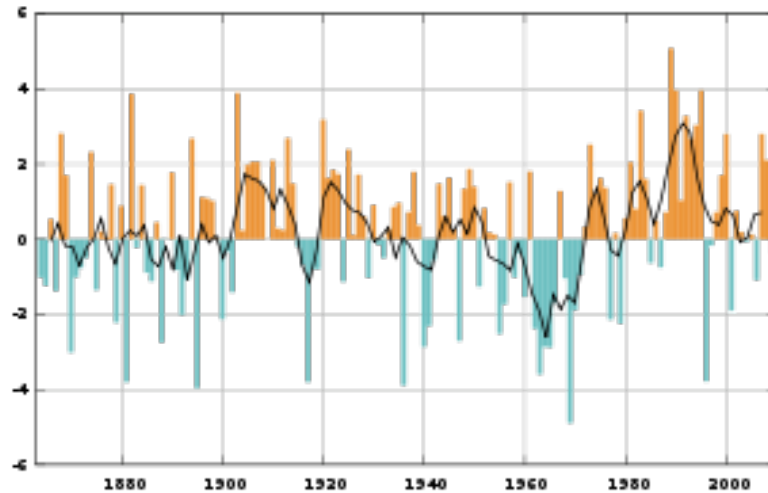
Figure 10.20: Winter index of the NAO based on the difference of normalized sea level pressure (SLP) between Lisbon, Portugal and Stykkisholmur/ReykjavÃk, Iceland since 1864, with a five year moving average (black)

years around the world, especially in the regions without other geological records.

Systematic measurements of climate using modern instruments have produced records covering the last 150 years. In order to reconstruct past variations in the climate system further back in time, scientists use natural archives of climatic and environmental changes, such as ice cores, tree rings, ocean and lake sediments, corals, and historical evidence. Scientists call these records proxies because, although they are not usually direct measures of temperature or other climatic variables, they are affected by temperature, and using modern calibrations, the changes in the proxy preserved in the fossil record can be interpreted in terms of past climate.

Ice core data, coral data, ring width of a tree, or information from marine sediments are examples of a proxy for temperature, or in some cases rainfall, because the thickness of the ring can be statistically related to temperature and/or rainfall in the past. The most valuable proxies are those that can be scaled to climate variables, and those where the uncertainty in the proxy can be measured. Proxies that cannot be quantified in terms of climate or environment are less useful in studying abrupt climate change because the magnitude of change cannot be determined. Quite often, the interpretation of proxy data is already a model of climate change since it involves

Atlantic Multidecadal Oscillation
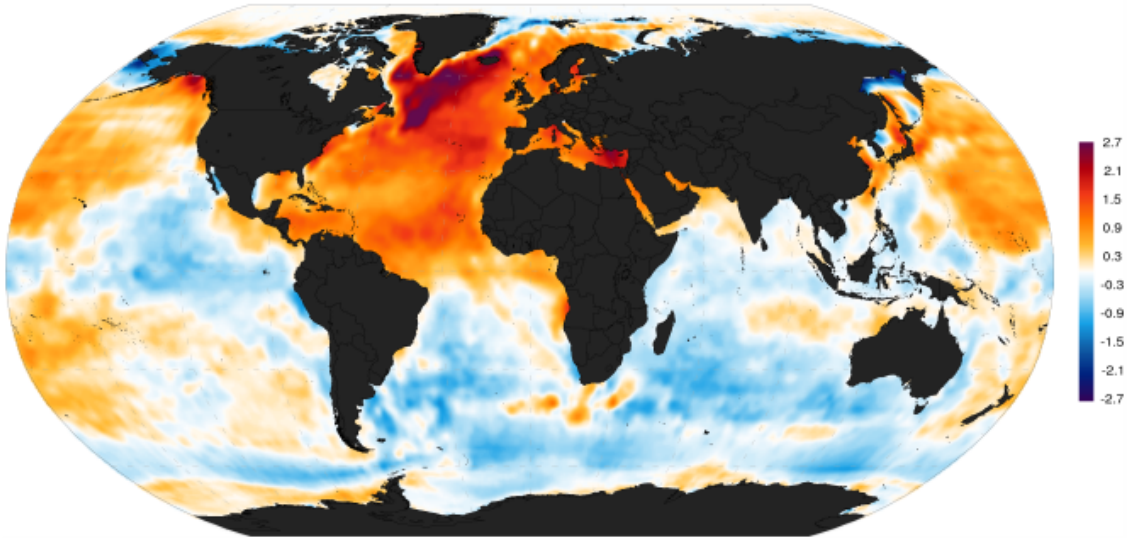
Figure 10.21: AMO spatial pattern.

Monthly values for the AMO index, 1856 -2013
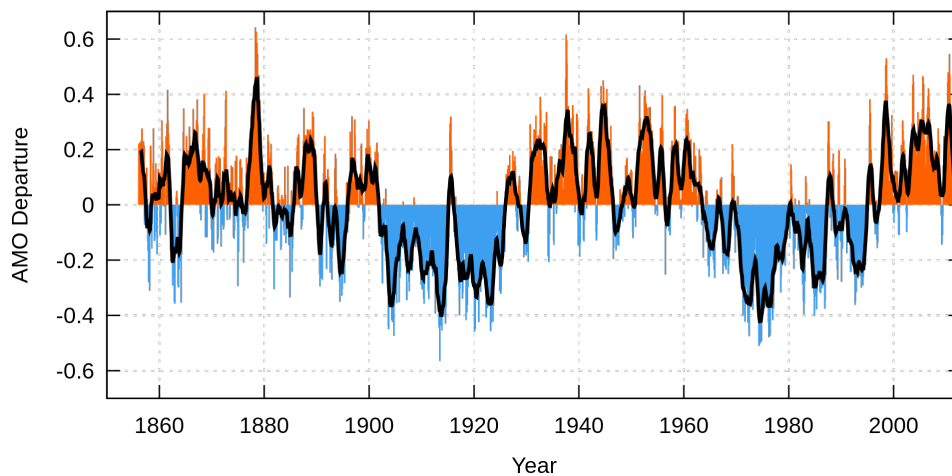
Figure 10.22: Atlantic Multidecadal Oscillation index computed as the linearly detrended North Atlantic sea surface temperature anomalies 1856-2013.
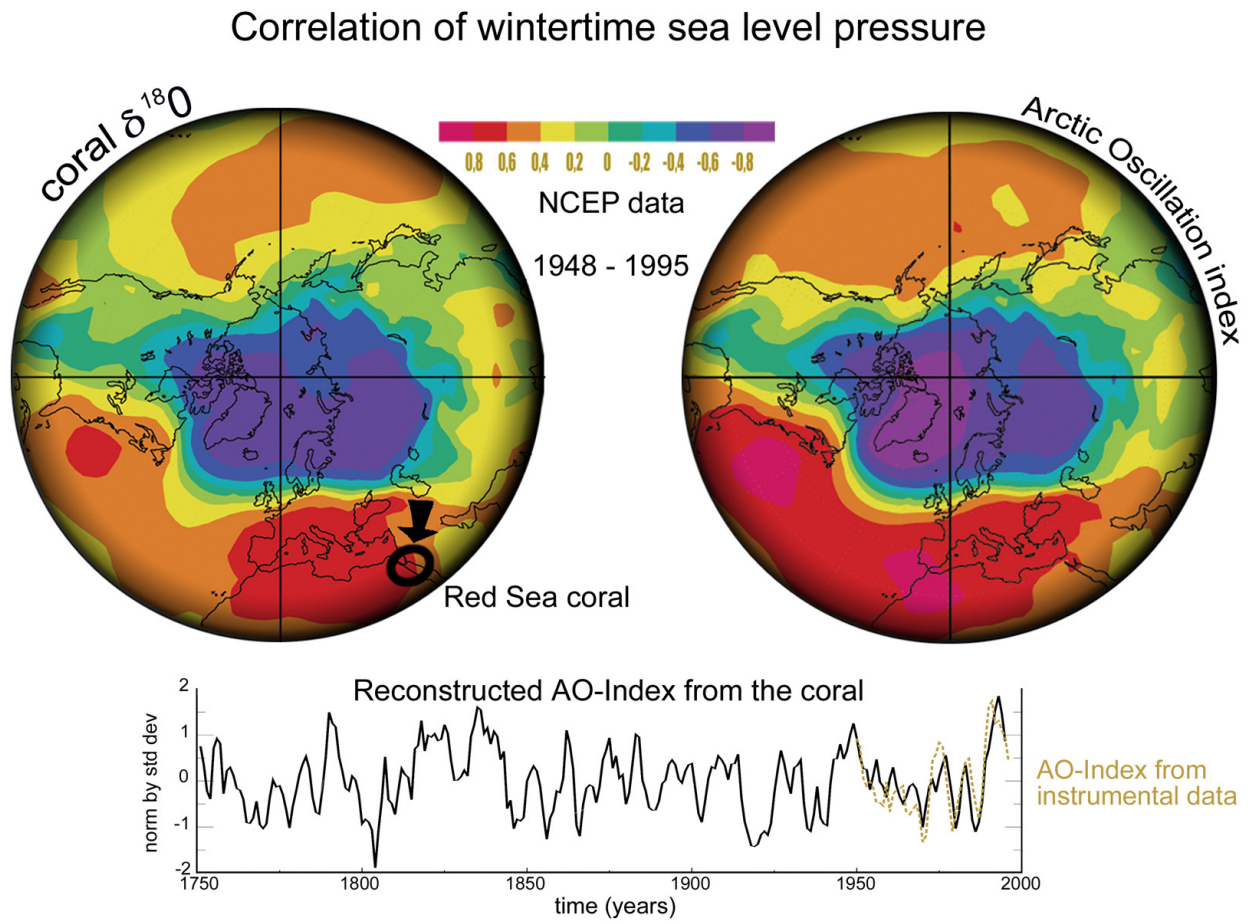
Figure 10.23: Corals as an important archive recording long term climatic changes

constraints (dating, representativeness etc.). Uncertainties in the proxies, and uncertainties in the dating, are the main reasons that abrupt climate change is one of the more difficult topics in the field of paleoclimatology.

Exercise 79   –   Interpretation of SST correlation maps and modes of variability

The two maps show correlations with local climate indices (Fig. 10.24).

a) Which climate modes correspond to the sea surface temperature correlation-maps?

b) Describe and draw schematically the associated sea level pressure patterns!

c) What are the dominant time scales and dominant seasons for these two modes in the climate system?
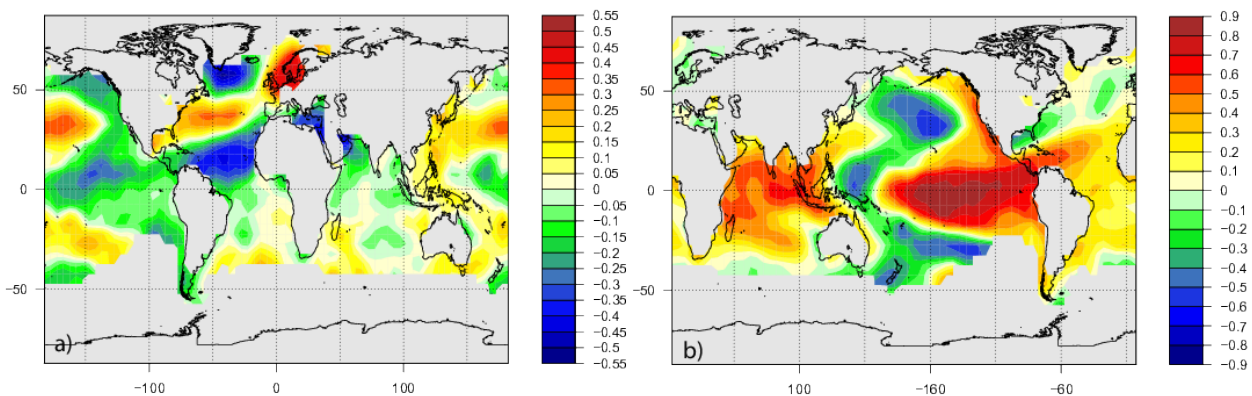
Figure 10.24: Correlation map of climate indices with global sea surface temperature.

Exercise 80   –   Temperature and precipitation of your home town

1. Calculate the temperature and precipitation of your home town. Create the timeseries (annual, DJF, JJA) for temperature and precipitation of your hometown. (Use the R-Shiny App "RunCorrelation" to create and download the data)

2. Calculate the teleconnection maps for temperature, precipitation and sea level pressure with the time series created in II (use R-Shiny App "Field Correlation").

3. Discuss the circulation via the sea level pressure teleconnection map.
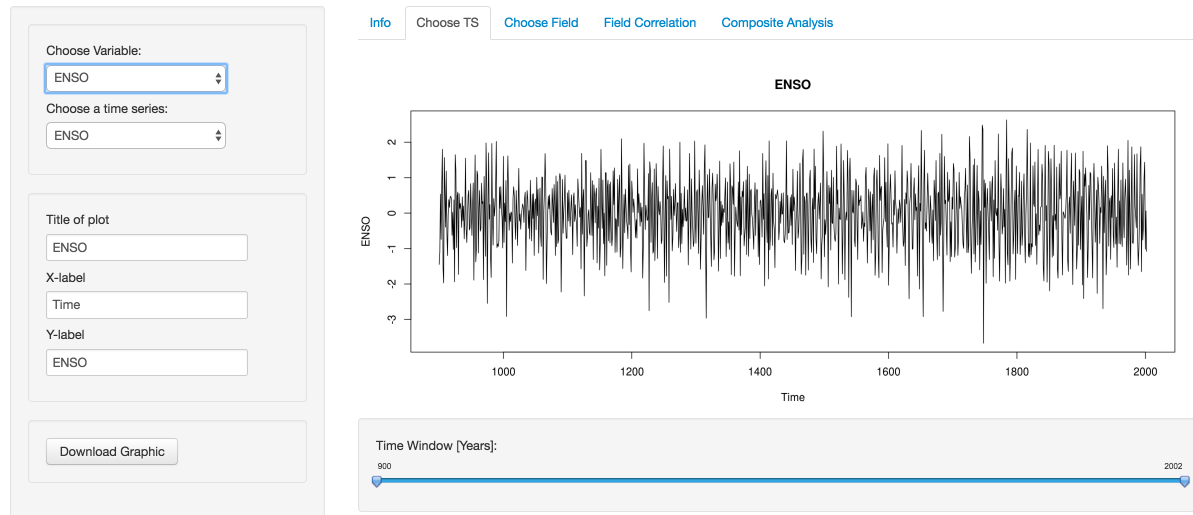
Figure 10.25: Correlation and composite analyses using the web application.

The R-Shiny Apps can be accessed via https://paleosrv2.awi.de/. The username is *student* and the Password is *EbJir5ow* ! More details on how to use these apps can be found in the description on the website.

Solution

Field Correlation – Correlation and Composite analysis of time series

This application allows the correlation and composite analysis of time series. In the first step, a timeseries has to be chosen in the tab 'Choose TS'. Note, that not all available time series data sets are currently supported by this application. Furthermore, in the drop-down field 'Choose Variable' a custom time series (e.g. a climate index derived with the app 'Runcorr') may be uploaded and selected for the analysis. The Field, to which the chosen time series shall be correlated, may be selected in the tab 'Choose Field'. Several options are available to adjust the analysis, including the physical quantity (dropdown field 'Choose Variable', e.g. land temperature), the reference data set (dropdown field 'Choose a data set', e.g. Berkely Temperature), the season for the analysis (dropdown field 'Choose seasons(s)', e.g. annual), and the area of interest for the analysis (dropdown

field 'Select Area', e.g. Europe). After all options have been set as wished, it can be selected what kind of metrics for the analysis to use (dropdown field 'Show', e.g. a plot or a histogram). After the application has finished some computations, a graphical representation of the chosen metric will be shown. If 'Plot' has been chosen, a geographical map will appear that presents the chosen field.

In the tab 'Field Correlation' the correlation of the two selected timeseries (refer to tab 'Choose TS' and 'Choose Field') can be computed. A respective plot is shown, potentially with significance testing.

In the tab 'Composite Analysis' a composite analysis of the two selected timeseries (refer to tab 'Choose TS' and 'Choose Field') can be computed. A respective plot is shown.

Note: if you get an error message that is similar to 'no time series or field chosen', just re-activate the tabs 'Choose TS' und 'Choose Field' once more before you select either one of the tabs 'Field Correlation' or 'Composite Analysis'. Furthermore, this application has not yet been extensively used, so bugs may occur from time to time.

Run Correlation and generation of climate indices – Running correlation of a timeseries and generation of climate indices (e.g. for use in the Field Correlation app)

This application allows the computation of running correlation of time series. Such time series may be uploaded via the tab 'Upload TS'. Alternatively, the time series may be extracted from already present historical climate data in the application's database (tab 'Create Index'). Such indices may be also downloaded and used in other applications, e.g. in the app "Field Correlation". The generation of a climate index from the application's climate data inventory for use within the Runcorr application, or for alternative download and later use in other apps, is in the following shortly illustrated.

Click on the tab 'Create Index' and select the desired climate variable (e.g. 'Land Temperature') from your preferred data set (e.g. 'Berkeley Temperature'). As soon as you select a season, a plot of the data will appear, and options for changing the appearance of the plot will become available. Furthermore, you can choose the spatial domain for the climate index. The options 'Area'

(creating an index via averaging over a specific latitude and longitude range) and 'Single Point' (for creating an index at a specific location) are available. After carefully selecting the options for the climate index, click on the button 'Create Index' in order to create a derived data set. Below the spatial representation of the climate data that was already present before, a second plot will appear that provides an illustration of the just created climate index. Now it is possible to either keep this index for further use in the Runncorr app (button 'Save Index', note down the attributed name to access the index in other tabs of the Runcorr app), or to download a text file that contains the time series for use in other apps (button 'Download Index').

Computation of the running correlation between two time series that are either a part of the data base of the app, but that can also include a custom index as created via the procedure illustrated above, may be performed in the tab 'Choose TS'. Further adjustments of the analysis (e.g. detrending and scaling) as well as custom modification of the plot appearance is available in the tab 'Running Correlation'. There, it is also possible to download a plot of the running correlation of the two selected time series.

10.4.5   Climate variability and bifurcation*

The internal free variations within the climate system are associated with both positive and negative feedback interactions between the atmosphere, oceans, cryosphere and biosphere. These feedbacks lead to instabilities or oscillations of the system on all time scales, and can either operate independently or reinforce external forcings. Investigations of the properties of systems which are far from equilibrium show that they have a number of unusual properties. In particular, as the distance from equilibrium increases, they can develop complex oscillations with both chaotic and periodic characteristics. They also may show bifurcation points where the system may switch between various regimes. Under non-equilibrium conditions, local events have repercussions throughout the whole system. These long-range correlations are at first small, but increase with distance from equilibrium, and may become essential at bifurcation points.

When applying (**??**), different concepts of climate change are in the literature. Quite often, the

dynamics is governed by the following stochastic differential equation

$$\frac{d}{dt}x(t) = -\frac{d}{dx}U(x) + \sigma\xi + F(t) \tag{10.37}$$

with potential

$$U(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x \quad . \tag{10.38}$$

If the potential is quadratic and $F(t) = 0$, the Ornstein-Uhlenbeck process is retained. In contrast, a bistable non-linear system with two minima in $U(x)$ has been assumed in which shifts between the two distinctly different states are triggered randomly by stochastic forcing, e.g. **?**.

From the online malterial, please see the browngui directory: BrownianMotion.zip We can consider many Brownian particles in a potential.

```
#brownian motion, multiple particle: Brown_mult.R
#forward modelling

#the function dy/dt<-f(y,a,b,c,d)
f<-function(y,a,b,c,d)
{
  return(d*y^3+c*y^2+b*y-a)
}

#constants
Ca<-10
a<-1
b<- 0.8
c<- 0
d<- -0.001

Nparticle<-1000 #number of particles
T<-  500    #integration time in time units
h<-  0.5    #step size in time units
N<-T/h
t<-(0:(N-1))*h

x<-matrix(10,Nparticle,N) # Initial condition, all = 0
# Initial condition,

for (i in 1:(N-1)) {
  x[,i+1]<- x[,i]+h*f(x[,i],a,b,c,d)   + Ca*rnorm(Nparticle)*sqrt(h)
}
```
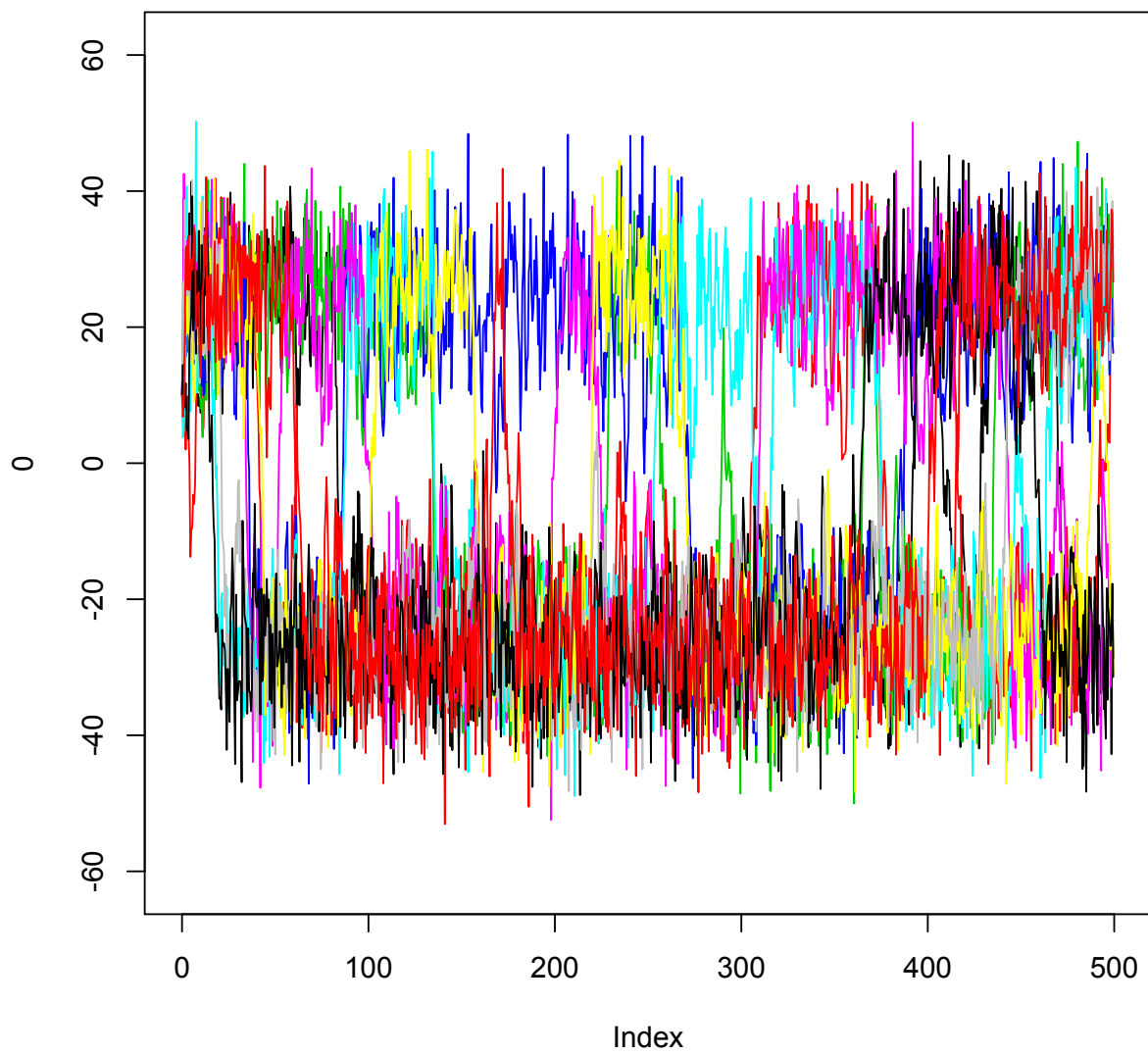
Figure 10.26: Brownian motion in a potential

```
ama2=max(x,2)
ami=min(x,-2)
ama=max(ama2,-ami)
plot(0,xlim=c(0,T),ylim=c(ami,ama),type="n")
for (i in 1:10) lines (t,x[i,],col=i)

#analyse the densities
h<-matrix(0,N,40)
for (i in 1:(N-1)) { h[i,]<-hist(x[,i],breaks=
      c(-20:20)*ama/10,freq=FALSE,ylim=c(0,0.04))$counts     }
hstat<-matrix(0,N)
for (i in N/2:(N-1)) hstat[]<-h[i,]+hstat[]
hstat[]<-hstat[] *2/Nparticle/N
#plot(t,hstat[],type="l")
plot(table(hstat[]), type = "h", col = "red")

# to show the time evolution, 1, 2, 4, 8, ....  time step
 op <- par(mfrow = c(3, 2))
plot(h[1,]/Nparticle,type="l")
plot(h[2,]/Nparticle,type="l")
plot(h[4,]/Nparticle,type="l")
plot(h[8,]/Nparticle,type="l")
plot(h[N/2,]/Nparticle,type="l")
plot(h[N-1,]+h[N-2,]/Nparticle/2,type="l")

filled.contour(t,(-19:20)*ama/10-ama/20,h,
      color.palette=rainbow,xlab="time",ylab="space")
```

Exercise 81   –   Brownian motion in a potential

1. Calculate the time series created in the R-Shiny App "Brownian Motion".

2. Discuss the results using the graphical interface (Fig. 10.27) for different forms of the potential.

3. Discuss the results using the graphical interface (Fig. 10.27) for a time-dependent potential.

4. Can you induce noise-induced transitions?

The R-Shiny Apps can be accessed via https://paleosrv2.awi.de/. The username is *student* and the Password is *EbJir5ow* !

More details on how to use these apps can be found in the description on the website.

Brownian Motion GUI
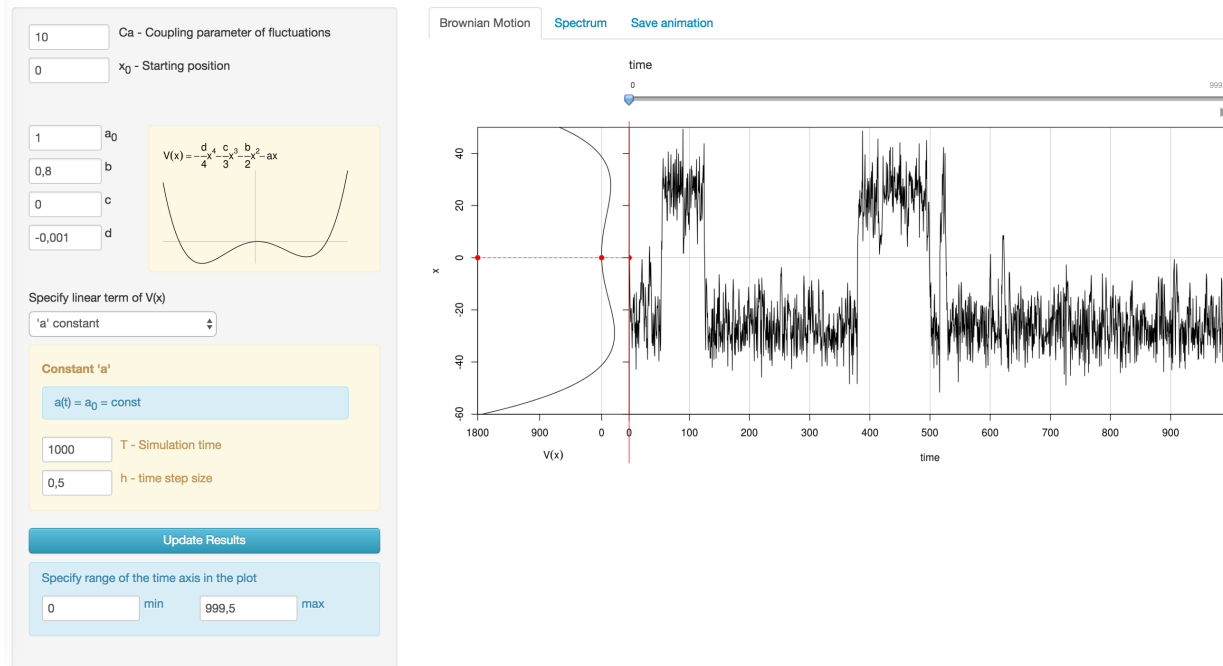
Figure 10.27: Potential curves and web application.

Brownian Motion – Simulation of the brownian motion of particles.

This application allows the simulation of brownian motion of particles. The dynamics may be altered by adjusting various physical parameters. These include a coupling parameter Ca, the starting position x0, and the coefficients of the potential V(x) that describes the interaction of the particles. Characteristics of V(x) may be further adjusted by specifiying the nature of the linear term a. The graphical output may be adjusted by specifying the range of the time axis of the plot. In order to update the simulation after adjusting any parameters, press the button 'Update Results'.

At the right hand side of the browser window you find three tabs: 'Brownian Motion', 'Spectrum', and 'Save animation'. The tab 'Brownian Motion' visualizes the potential V(x) as well as the time evolution of the particle's position. The tab 'Spectrum' provides a periodogram of the particle's motion. The tab 'Save animation' lets you export the time evolution of the particle's position.

10.4.6 Millennial climate variability*

Within glacial periods, and especially well documented during the last one, spanning from around 110 to 11.6 ky ago, there are dramatic climate oscillations, including high latitude temperature changes approaching the same magnitude as the glacial cycle itself, recorded in archives from the polar ice caps, high to middle latitude marine sediments, lake sediments and continental loess sections. These oscillations are usually referred to as Dansgaard-Oeschger Cycle and occur mostly on 1 to 2 ky time scales, e.g. **?**, although regional records of these transitions can show much more rapid change. The termination of the Younger Dryas cold event, for example, is manifested in ice core records from central Greenland as a near doubling of snow accumulation rate and a temperature shift of around 10°C occurring within a decade with world-wide teleconnections. One hypothesis for explaining these climatic transitions is that the ocean thermohaline circulation flips between different modes, with warm intervals reflecting periods of strong deep water formation in the northern North Atlantic and vice versa **?**. As an alternative approach, one can estimate the underlying dynamics (10.37, 10.38) directly from data **?**. The method is based on the unscented Kalman filter, a non-linear extension of the conventional Kalman filter. This technique allows to consistently estimate parameters in deterministic and stochastic non-linear models. The optimization yields for the coefficients $a_4 = 0.13 \pm 0.01$, $a_3 = -0.27 \pm 0.02$, $a_2 = -0.36 \pm 0.08$, and $a_1 = 1.09 \pm 0.23$. The dynamical noise level of the system σ is estimated to be 2.4. The potential is highly asymmetric and degenerate (that is, close to a bifurcation): there is one stable cold stadial state and one indifferently stable warm interstadial state (Fig. 10.28). This seems to be related to the fact that the warm intervals are relatively short-lasting.

Coming back to the ice cores and a potential linkage of the hemispheres, Stocker and Johnson **?** proposed a conceptual model linking the isotopic records from Antarctica and Greenland. The basis is an energy balance with temperatures in the North and South Atlantic Ocean, as well as a "southern heat reservoir". It is assumed that the change in heat storage of a "southern heat reservoir" T_S is given by the temperature difference between the reservoir T_S and the Southern
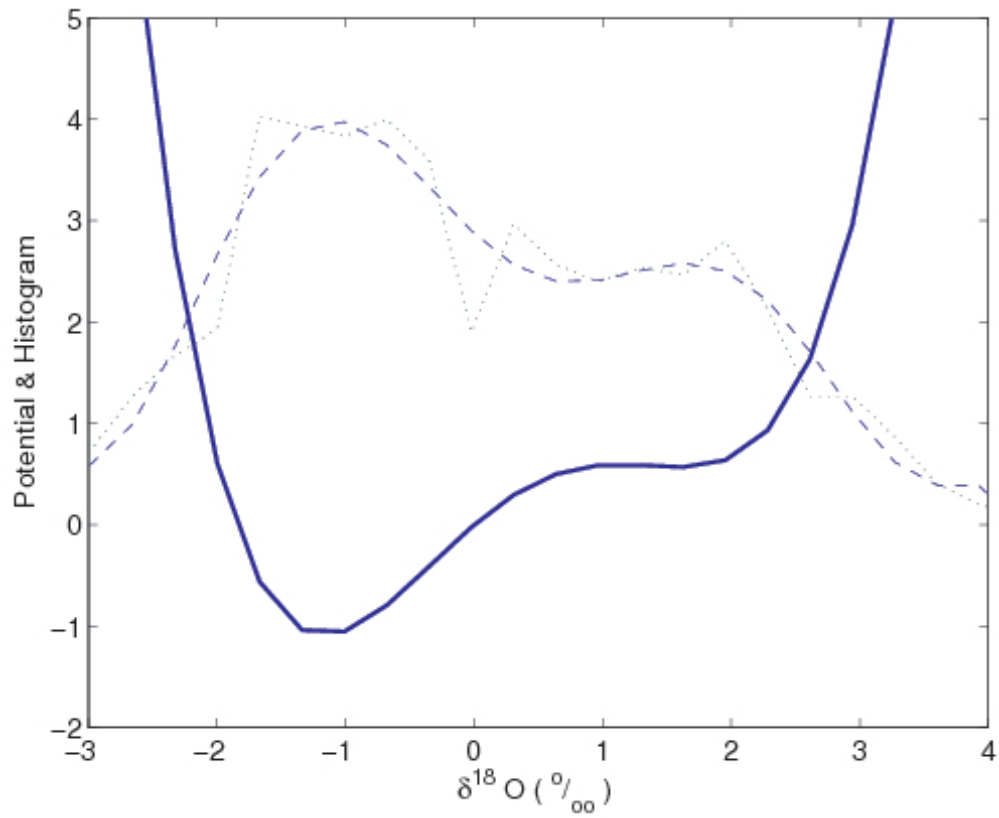
Figure 10.28: Potential derived from the data (solid) together with probability densities of the model (dashed) and the data (dotted).

Ocean temperature T , with a characteristic time scale τ :

$$\frac{d}{dt}T_S(t) \;=\; \frac{1}{\tau}[T - T_S] \tag{10.39}$$

T_N denotes the time-dependent temperature anomaly of the North Atlantic. The Southern Ocean temperature T is assumed to be $-T_N$ according to the bipolar seesaw (North Atlantic cold $\leftrightarrow$ South Atlantic warm). Using Laplace transform, one can solve for T_S

$$T_S = -\frac{1}{\tau}\int_0^t T_N(t - t')\exp(-t'/\tau)dt' + T_S(0)\exp(-t/\tau) \tag{10.40}$$

The reservoir temperature is therefore a convolution of the northern temperature using the time scale τ ranging from 100 to 4000 years. Equation (10.40) demonstrates that T_S and T_N will have entirely different time characteristics. Abrupt changes in the north appear damped and integrated in time in the southern reservoir. A sudden reduction in the thermohaline circulation causes a cooling in the North Atlantic and a warming in the South, a situation similar to the Younger Dryas period **?**, see also Fig. 9.10.

10.4.7 Noise induced transitions*

More specifically, one considers one-variable bistable dynamical systems subjected simultaneously to noise and to a weak periodic forcing:

$$\frac{dx}{dt} = -\frac{\partial U}{\partial x} + F(t) + \epsilon h(x)\cos(\omega_0 t + \phi) \tag{10.41}$$

Here x is the state variable (e.g., the global temperature or the global ice volume in the context of the Quaternary glaciations); U is the "potential" driving the internal dynamics, taken to possess two minima x_+ and x_- associated to the two stable states, separated by a maximum corresponding to an intermediate unstable state x_0; $F(t)$ is a "random force" accounting for internal variability or environmental noise and modeled classically as a Gaussian white noise of zero mean and strength

equal to q^2; and ϵ, ω_0 and ϕ are, respectively, the amplitude, frequency and phase of the periodic forcing. Actually, the forcing contribution can be cast in a form similar to the first term in the right hand side of (10.41) by introducing a generalized time-dependent potential

$$W(x, t) = U(x) - \epsilon g(x) \cos(\omega_0 t + \phi) \tag{10.42}$$

with $dg(x)/dx = h(x)$.

According to the theory of stochastic processes the stochastic differential equation for the random process x(t) in (10.41) is equivalent to a Fokker-Planck equation for the probability distribution function P(x,t) of values of x. In the absence of periodic forcing this latter equation defines a particular type of Markov process known as diffusion process: The variable x realizes, for most of the time, small scale excursions around x_+ or x_-, which are interrupted every now and then by noise-driven abrupt transitions from x_+ to x_- or vice versa across the unstable state x_0, which constitutes a barrier of some sort. The kinetics of these transitions are determined by two quantities: The noise strength q^2 and the potential barrier $\Delta U_\pm$, defined by

$$\Delta U_\pm = U(x_0) - U(x_\pm) \tag{10.43}$$

In the limit where q^2 is much smaller than $\Delta U_\pm$ the mean value of the transition time is given by the celebrated Kramers formula

$$\tau_\pm^{-1} = r_\pm = \frac{1}{2\pi}(-U''(x_0)U''(x_\pm))^{1/2} \exp\left(-\frac{\Delta U_\pm}{q^2/2}\right) \tag{10.44}$$

where the double prime designates the second derivative. The transitions themselves occur in an incoherent fashion, as their dispersion around the above mean value is comparable to the mean itself.

When the periodic forcing is switched on U is replaced by the generalized potential W. The corresponding barrier $\Delta W_\pm$ is now modulated in time leading periodically to situations where

states $x_\pm$ are found at the bottom of wells that are, successively, less shallow and more shallow than those in the forcing-free system. One is thus led to expect that the transitions will be facilitated during a part of this cycle, provided the periodicity of the forcing matches somehow the Kramers time τ. As it turns out this intuitive idea is fully justified in the asymptotic limit of small q^2 in which the Fokker-Planck equation can be reduced, using an adiabatic approximation, to a closed equation for the probability $p_\pm$ to be in the attraction basin of state x_+ or x_- :

$$\frac{dp_+(t)}{dt} = r_-(t)\, p_-(t) - r_+(t)\, p_+(t) \tag{10.45}$$

with $p_+ + p_- = 1$ and $r_\pm$ given by an expression in which U is replaced by the generalized potential W. This equation can be solved straightforwardly. In most of the quantitative studies of stochastic resonance the result is further expanded to the first non-trivial order in the forcing amplitude ϵ. A popular minimal model capturing the essence of the results is to set $h(x) = 1$ (and hence $g(x) = x$) and to consider a symmetric quartic potential $U(x) = -\lambda x^2/2 + x^4/4 (\lambda > 0)$, corresponding to $x_\pm = \pm\lambda^{1/2}$ and $x_0 = 0$. This leads to the following expression for the periodic component $\delta p(t)$ of the response,

$$\delta p(t) = A\cos(\omega_0 t + \phi + \psi) \tag{10.46}$$

Here the amplitude A and phase shift ψ are given by

$$A = \epsilon\frac{\lambda}{q^2}\frac{r(q^2)}{(r^2(q^2) + \omega_0^2/4)^{1/2}} \qquad \psi = -\arctan\left(\frac{\omega_0}{2r}\right) \tag{10.47}$$

where $r(q^2) = r_+ = r_- = (\sqrt{2}\pi)^{-1}\lambda\exp\{(-\lambda^2/(2q^2))\}$ for the symmetric potential model. The essential point is now that

- the transitions across the barrier have been synchronized to follow, in the mean, the periodicity of the external forcing;

- the response is negligible unless the period of the forcing comes close to the (noise intensity-
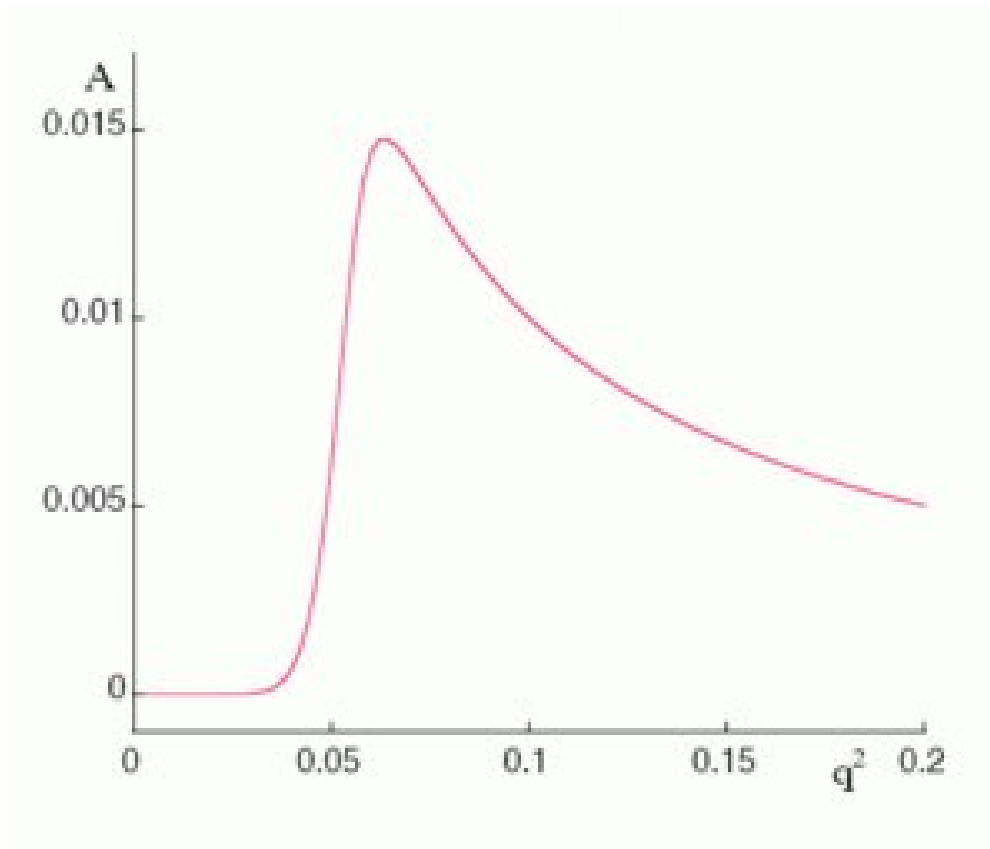
Figure 10.29: Amplitude of the periodic component of the response of a bistable system described by a symmetric quartic potential subjected simultaneously to noise and to a weak periodic forcing, against the variance q^2 of the noise strength. Parameter values are $\lambda = 1, \omega_0 = 2\pi/10^5$ and $\epsilon = 0.001$. The existence of a sharp maximum is one of the principal signatures of stochastic resonance.

dependent!) Kramers time; and

- for given ω_0 and ϵ, A goes through a sharp maximum for an intermediate (finite) value of q^2 (see Fig. 10.29), thereby enhancing considerably the response to the (weak) periodic signal. This latter property is the principal signature of stochastic resonance and should be clearly differentiated from the mechanisms underlying classical resonance. More refined studies based on Floquet theory or on a spectral decomposition of the full Fokker-Planck equation confirm fully the validity of these conclusions.

In such a system, climate variability and change in the potential can interact due to stochastic

resonance **?**; **?**. Stochastic resonance occurs when the signal-to-noise ratio of a non-linear device is maximized for a moderate value of noise intensity σ. It often occurs in bistable and excitable systems with sub-threshold inputs. For lower noise intensities, the signal does not cause the device to cross threshold, so little signal is passed through it. For large noise intensities, the output is dominated by the noise, also leading to a low signal-to-noise ratio. For moderate intensities, the noise allows the signal to reach threshold, but the noise intensity is not so large as to swamp it.

Strictly speaking, stochastic resonance occurs in bistable systems, when a small periodic force $F(t)$ (which is external) is applied together with a large wide-band stochastic force $\sigma\xi$ (which is internal). The system response is driven by the combination of the two forces that compete/co-operate to make the system switch between the two stable states. The degree of order is related to the amount of periodic function that it shows in the system response. When the periodic force is chosen small enough in order not to make the system response switch, the presence of a non-negligible noise is required for it to happen. When the noise is small very few switches occur, mainly at random with no significant periodicity in the system response. When the noise is very strong a large number of switches occur for each period of the periodic force and the system response does not show remarkable periodicity. Quite surprisingly, between these two conditions, there exists an optimal value of the noise that cooperatively concurs with the periodic forcing in order to make almost exactly one switch per period (a maximum in the signal-to-noise ratio).

Furthermore, non-linear oscillators have been proposed where the timing of the deterministic external forcing is crucial for generating oscillations **?**; Lorenz [1976]; **?**. Some aspects of non-equilibrium systems can be found in the climatic system. On the climatological scale, it exhibits abrupt jumps in the long-term rate of temperature change, which are often associated with changes in circulation patterns.

Chapter 11

Future Directions

Until now, details of abrupt climate change are not well known to be able to accurately predict it. With better information, the society could take more confident action to reduce the potential impact of abrupt changes on agriculture, water resources, and the built environment, among other impacts. A better understanding of sea-ice and glacier stability, land-surface processes, and atmospheric and oceanic circulation patterns is needed. Moreover, to effectively use any additional knowledge of these and other physical processes behind abrupt climate change, more sophisticated ways of assessing their interactions must be developed, including:

Better models. At present, the models used to assess climate and its impacts cannot simulate the size, speed, and extent of past abrupt changes, let alone predict future abrupt changes. Efforts are needed to improve how the mechanisms driving abrupt climate change are represented in these models and to more rigorously test models against the climate record.

More theory. There are concepts to find the underlying dynamical system, to derive a theory from a high-order to low-order description similar to what is done in statistical physics (Mori-Zwanzig approach **?**; **?**, Master equation), or in stochastic differential equations. A systematic reduction of the complex system into fewer degrees of freedom shall bring a deeper level of understanding about the underlying physics. A systematic approach was suggested by Saltzman **?**. Spectral and pseudo-spectral concepts have not been used too much in climate theory. There is

a variety of phenomenological stochastic models in which non-linearity and fluctuations coexist, and in which this coexistence leads to interesting phenomena.

Earth System Modeling and Analysis Modeling is necessary to produce a useful understanding of abrupt climate processes. Model analyses help to focus research on possible causes of abrupt climate change, such as human activities; on key areas where climatic thresholds might be crossed; and on fundamental uncertainties in climate-system dynamics. Improved understanding of abrupt climatic changes that occurred in the past and that are possible in the future can be gained through climate models. A comprehensive modeling strategy designed to address abrupt climate change includes vigorous use of a hierarchy of models, from theory and conceptual models through models of intermediate complexity, to high-resolution models of components of the climate system, to fully coupled earth-system models. The simpler models are well-suited for use in developing new hypotheses for abrupt climate change. Model-data comparisons are needed to assess the quality of model predictions. It is important to note that the multiple long integrations of enhanced, fully coupled Earth system models required for this research are not possible with the computer resources available today, and thus, these resources are currently enhanced.

One particularly convincing example showing that the feedbacks in the climate system are important is the drying of the Sahara about 5000 years before present which is triggered by variations in the Earth's orbit around the sun. Numerous modeling studies, e.g. **?**, suggest that the abruptness of the onset and termination of the early to mid-Holocene humid period across much of Africa north of the equator depends on the presence of non-linear feedbacks associated with both ocean circulation and changes in surface hydrology and vegetation, e.g. **?**. Without including these feedbacks alongside gradual insolation forcing, it is impossible for existing models to come even close to simulating the rapidity or the magnitude of climatic change associated with the extension of wetlands and plant cover in the Sahara/Sahel region prior to the onset of extreme dryness around 5000 years before present.

Paleoclimatic data. More climate information from the distant past would go a long way toward strengthening our understanding of abrupt climate changes and models of past climate. In

particular, an enhanced effort is needed to expand the geographic coverage, temporal resolution, and variety of paleoclimatic data. Although the present climate has no direct analogon to the past **?**, the dynamical interpretation of data will improve the understanding of thresholds and non-linearities in the Earth system.

Systematic measurements of climate using modern instruments have produced records covering the last 150 years. In order to reconstruct past variations in the climate system further back in time, scientists use natural archives of climatic and environmental changes, such as ice cores, tree rings, ocean and lake sediments, corals, and historical evidence. Scientists call these records proxies because, although they are not usually direct measures of temperature or other climatic variables, they are affected by temperature, and using modern calibrations, the changes in the proxy preserved in the fossil record can be interpreted in terms of past climate.

Ice core data, coral data, ring width of a tree, or information from marine sediments are examples of a proxy for temperature, or in some cases rainfall, because the thickness of the ring can be statistically related to temperature and/or rainfall in the past. The most valuable proxies are those that can be scaled to climate variables, and those where the uncertainty in the proxy can be measured. Proxies that cannot be quantified in terms of climate or environment are less useful in studying abrupt climate change because the magnitude of change cannot be determined. Quite often, the interpretation of proxy data is already a model of climate change since it involves constraints (dating, representativeness etc.). Uncertainties in the proxies, and uncertainties in the dating, are the main reasons that abrupt climate change is one of the more difficult topics in the field of paleoclimatology.

Appropriate statistical tools. Because most statistical calculations at present are based on the assumption that climate is not changing but is stationary, they have limited value for non-stationary (changing) climates and for climate-related variables that are often highly skewed by rapid changes over time such as for abrupt-change regimes. Available statistical tools themselves need to be adapted or replaced with new approaches altogether to better reflect the properties of abrupt climate change.

Synthesis. Physical, ecological, and human systems are complex, non-linear, dynamic and imperfectly understood. Present climate change is producing conditions outside the range of recent historical experience and observation, and it is unclear how the systems will interact with and react to further climate changes. Hence, it is crucial to be able to better understand and recognize abrupt climate changes quickly. This capability will involve improved monitoring of parameters that describe climatic, ecological, and economic systems. Some of the desired data are not uniquely associated with abrupt climate change and, indeed, have broad applications. Other data take on particular importance because they concern properties or regions implicated in postulated mechanisms of abrupt climate change. Research to increase our understanding of abrupt climate change should be designed specifically within the context of the various mechanisms thought to be involved. Focus is required to provide data for process studies from key regions where triggers of abrupt climate change are likely to occur, and to obtain reliable time series of climate indicators that play crucial roles in the postulated mechanisms. Observations could enable early warning of the onset of abrupt climate change. New observational techniques and data-model comparisons will also be required.

Part IV

Fourth part: Numerical applications and further exercises

Chapter 12

Appendix: Numerical examples

12.1 Examples in matlab

How to plot in matlab:

```
% graphlab.m
% sin(x)/x example
%
x = -10.5:.2:10.5;
plot(x,sin(x)./x)
set(gca,'FontName','Times-Roman','FontSize',16);
print -deps graphlab.eps
```

12.1.1 Covariance

```
% calculate Covariance in matlab\\
%
% pca.m
%
clear all
eps = 1e-10;
npts = 100000;
x(:,1) = randn(npts,1);
x(:,2) = randn(npts,1);
x(:,3) = x(:,1)+x(:,2);
Cx = cov(x);
fprintf('covariance of x:\n')
fprintf('   %f %f %f\n',Cx)
[M,Ex] = eig(Cx);
```

```
fprintf('eigenvalues of covariance of x:\n')
fprintf('  %f %f %f\n',Ex)
M = M';
index = (diag(Ex) > eps);
M = M(index,:);
y = (M*x')';
Cy = cov(y);
fprintf('measured covariance of y:\n')
fprintf('  %f %f\n',cov(y))
```

12.1.2   Random walk

Running stochwalk.m will open a window showing the results of 10 random walks. Every time

you run the script the output will be different, but a typical run will produce something like this.

```
npts = 1000;
nplot = 10;
seed = 1;
for i = 1:nplot
   xplot = [];
   x = 0;
   for j=1:npts
      seed = rem(8121 * seed + 28411, 134456);
      if (seed > (134456/2))
         x = x + 1;
      else
         x = x - 1;
         end
      xplot = [xplot x];
      end
   plot(xplot,'g')
   hold on
   end
t = 1:(npts/10):npts;
error = sqrt(t)*1.5;
errorbar(t,zeros(size(t)),error,error,'.')
xlabel('t')
ylabel('x')
axis([0 npts -2*sqrt(npts) 2*sqrt(npts)])
hold off
print -deps plot.eps
```

Two dimensional random walk

```
npts = 10000;
seed = 1;
xplot = [];
yplot = [];
```
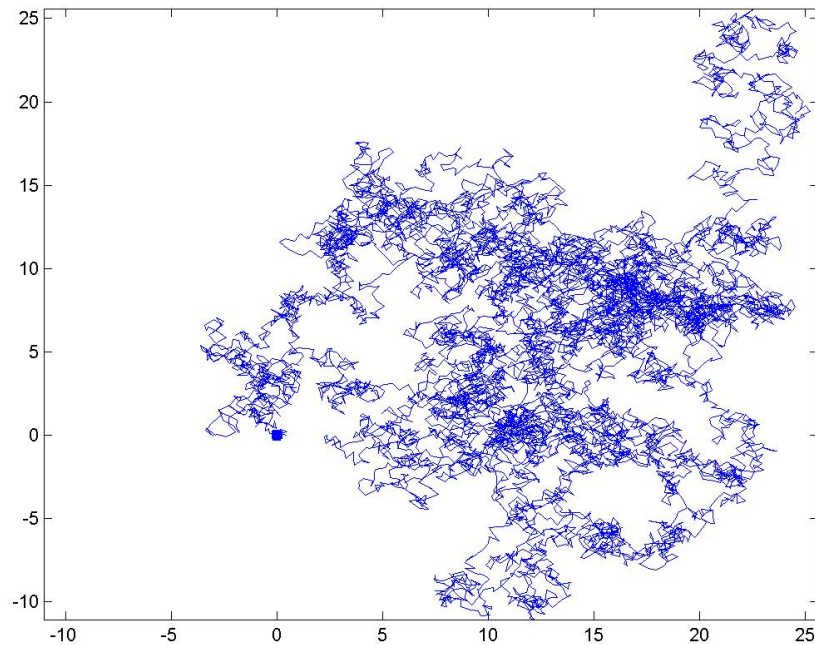
Figure 12.1: Numerical solution of the brownian motion.

```
x = 0;
y = 0;
for i = 1:npts
   x = x + (rand-0.5);
   xplot = [xplot x];
   y = y + (rand-0.5);
   yplot = [yplot y];
   end
plot(xplot,yplot)
axis([min(min(xplot),min(yplot)) max(max(xplot),max(yplot)) ...
      min(min(xplot),min(yplot)) max(max(xplot),max(yplot))])
%axis('off')
hold on
plot(0,0,'+','LineWidth',5)
print -djpeg twoDrandom.jpg
```

12.1.3   Carbon cycle

```
Zeebe, R. E. and D. A. Wolf-Gladrow, \COz in Seawater:

Equilibrium, Kinetics, Isotopes, Elsevier Oceanography Series, 65, pp. 346,
```

Amsterdam, 2001. Reprinted in 2003, 2005.

ReadMe file for csys3.m

This file contains instructions on how to run the MATLAB file with the user-i

The file csys.m contains numerical routines to calculate carbonate system par
from any two given quantities.
The file equic.m contains equations to calculate the various
equilibrium constants used in csys.m.

Table of Contents:
----

vii) Location of Output Data File

II) NON-INTERFACE MODE

1) Required System Variables

i) runUserInterface

ii) numOfInputs

iii) flag

iv) phflag

v) k1k2flag

2) Inputting Values

i) TC, S, and P

ii) ph1, s1, hco31, co31, alk1, dic1, p\COz1

```
% bern_lgm.m
% BERN  4-box model of the terrestrial carbon cycle, LGM version,
% assuming that the glacial terrestrial biosphere contained
% 500 Pg less carbon than during the Holocene.
% Carbon storage in the various reservoirs:
%   atmosphere:    425 GtC (equiv. to p\COz = 200 uatm)
%   ground vegetation:   77 GtC
%   woody vegetation:   387 GtC
%   detritus:           93 GtC
%   humus             1162 GtC


dt=1;yrps=1;decay=1/8267;
% exchange coefficients
kag=0.0592*yrps;  % atmosphere -> ground vegetation
kaw=0.0423*yrps;  % atmosphere -> wood
kgd=0.3250*yrps;  % ground vegetation -> detritus
kwd=0.0371*yrps;  % wood -> detritus
kwh=0.0093*yrps;  % wood -> humus
kda=0.3482*yrps;  % detritus -> atmosphere
kdh=0.0774*yrps;  % detritus -> humus
kha=0.0093*yrps;  % humus -> atmosphere
% reservoir size ratios
vag=5.4878;  % atmosphere / ground vegetation
vaw=1.0976;  % atmosphere / wood
vgd=0.8333;  % ground vegetation / detritus
vwd=4.1667;  % wood / detritus
```

```matlab
vwh=0.3333;   % wood / humus
vda=0.2187;   % detritus / atmosphere
vdh=0.0800;   % detritus / humus
vha=2.7333;   % humus / atmosphere

c14gr=0.9;
c14wo=0.9;
c14de=0.9;
c14hu=0.9;
c14at=1.0;
alpha=1;%alpha=0.9641;
for i=1:10000,
%   c14at=c14at+dt*(pc14-decay*c14at-kas*(c14at-c14se))
%   c14at=c14at+dt*(kda*vda*c14de+kha*vha*c14hu-(kag+kaw)*c14at)
  c14gr=c14gr+dt*(kag*vag*c14at*alpha-(decay+kgd)*c14gr);
  c14wo=c14wo+dt*(kaw*vaw*c14at*alpha-(decay+kwd+kwh)*c14wo);
  c14de=c14de+dt*(kgd*vgd*c14gr+kwd*vwd*c14wo-(decay+kda+kdh)*c14de);
  c14hu=c14hu+dt*(kdh*vdh*c14de+kwh*vwh*c14wo-(decay+kha)*c14hu);
  c14grnd(i)=c14gr;
  c14wood(i)=c14wo;
  c14dets(i)=c14de;
  c14hums(i)=c14hu;
end;
```

12.1.4   Earth orbital parameters and insolation

```matlab
% Milankovitch Daily Insolation For 1Myr Ago to Present
%
%   function BERGER - ORBITAL MECHANICS: ECCENTRICITY, OBLIQUITY, PRECESSION
%                   - DETERMINE LENGTH OF DAY, LONGITUDE OF PARHELION
%       function NDAY - CALCULATE JULIAN DAY OF YEAR ON A 365 BASIS
%           function DAYINS - CALCULATE DAILY SOLAR INSOLATION
%
%
latitude=65.0;     % North is positive
month = 6.0;       % June
mday = 15;         % Day of Month
past_yr = 1.e6;    % years into the past, 1 million
future_yr = 0.0e0; % years into the future, 0
nres=100.;         % time resolution = 100 years:
nnlk=(past_yr+future_yr)/nres; % number of calculation points
yrr=[-past_yr+nres:nres:future_yr];
tops=zeros(1,length(yrr));

for step=1:length(yrr)
    tops(step) = 86.4 * BERGER_1(latitude,yrr(step),month,mday);
end

save('t.dat', 'yrr' , 'tops','-ascii');
```

```matlab
%%%%%%%%%%%%
function BERGER = my_BERGER(PHI,T,MA,JA)
%
%       PROGRAM ORIGINALLY DEVELOPED BY A. BERGER (1978)
%
%       DECRIPTION OF PROGRAM :
%       A.BERGER : A SIMPLE ALGORITHM TO COMPUTE LONG TERM VARIATIONS OF
%       DAILY OR MONTHLY INSOLATION
%       INSTITUT D'ASTRONOMIQUE ET DE GEOPHYSIQUE
%       UNIVERSITE CATHOLIQUE DE LOUVIAN
%       CONTRIBUTION NO. 18 (1978)
% %
%       INPUT :
%       PHI - LATITUDE IN DEGREES
%             NORTHERN HEMISPH. : POSITIVE VALUES
%             SOUTHERN HEMISPH. : NEGATIVE VALUES
%       T   - TIME IN YEARS, NEGATIVE FOR THE PAST
%             (REFERENCE YEAR : 1950 A.D.)
%       MA  - MONTH IN SPECIFIED YEAR
%       JA  - DAY IN SPECIFIED MONTH
% %
%       OUTPUT :
%       FUNCTION 'BERGER' GIVES DAILY INSOLATION IN KJ M(-2) DAY(-1)
%
%       EXAMPLE : BERGER(45.,0.,7,15) = 40304.05 KJ M(-2) DAY(-1)
%
%       ORBITAL PARAMETERS:
%       ADDITIONAL OUTPUT OF ORBITAL PARAMETERS IS POSSIBLE, SEE WRITE-
%       STATEMENTS MARKED BY 'CPPP'
%
%
%   CONSTANTS
%
PIR = pi/180.0;
PIRR = PIR/3600.0;
STEP = 360.0/365.25;
TEST = 0.0001;
%
%   1.EARTH ORBITAL ELEMENTS : ECCENTRICITY           ECC   TABLE 1
% **************************   PRECESSIONAL PARAMETER PRE
%                             OBLIQUITY               XOB   TABLE 2
%                             GENERAL PRECESSION      PRG
%                             LONGITUDE PERIHELION   PERH   TABLE 3
%
%             AMPLITUDE A   MEAN RATE B   PHASE C
%             THEY ARE IMMEDIATELY CONVERTED IN RADIANS
%
%
%     ECCENTRICITY
```

```
%
NEF=19;
AE = ones(19,1);
BE=ones(19,1);
CE=ones(19,1);

AE(1:5)   = [0.01860798  0.01627522 -0.01300660  0.00988829 -0.00336700];
AE(6:10)  = [0.00333077 -0.00235400  0.00140015  0.00100700  0.00085700];
AE(11:15) = [0.00064990  0.00059900  0.00037800 -0.00033700  0.00027600];
AE(16:19) = [0.00018200 -0.00017400 -0.00012400  0.00001250];

BE(1:5)   = [ 4.2072050  7.3460910 17.8572630 17.2205460 16.8467330];
BE(6:10)  = [ 5.1990790 18.2310760 26.2167580  6.3591690 16.2100160];
BE(11:15) = [ 3.0651810 16.5838290 18.4939800  6.1909530 18.8677930];
BE(16:19) = [17.4255670  6.1860010 18.4174410  0.6678630];
BE = BE.*PIRR;

CE(1:5)   = [ 28.620089 193.788772 308.307024 320.199637 279.376984];
CE(6:10)  = [ 87.195000 349.129677 128.443387 154.143880 291.269597];
CE(11:15) = [114.860583 332.092251 296.414411 145.769910 337.237063];
CE(16:19) = [152.092288 126.839891 210.667199  72.108838];
CE = CE.*PIR;
%
%   OBLIQUITY
%
XOD=23.320556;
NOB=18;
AOB=ones(18,1);
BOB=ones(18,1);
COB=ones(18,1);

AOB(1:5)   = [-2462.2214466 -857.3232075 -629.3231835 -414.2804924 -311.7632587];
AOB(6:10)  = [  308.9408604 -162.5533601 -116.1077911  101.1189923  -67.6856209];
AOB(11:15) = [   24.9079067   22.5811241  -21.1648355  -15.6549876
15.3936813];
AOB(16:18) = [   14.6660938  -11.7273029   10.2742696];

BOB(1:5)   = [31.609974 32.620504 24.172203 31.983787 44.828336];
BOB(6:10)  = [30.973257 43.668246 32.246691 30.599444 42.681324];
BOB(11:15) = [43.836462 47.439436 63.219948 64.230478  1.010530];
BOB(16:18) = [ 7.437771 55.782177  0.373813];
BOB = BOB.*PIRR;

COB(1:5)   = [251.9025 280.8325 128.3057 292.7252  15.3747];
COB(6:10)  = [263.7951 308.4258 240.0099 222.9725 268.7809];
COB(11:15) = [316.7998 319.6024 143.8050 172.7351  28.9300];
COB(16:18) = [123.5968  20.2082  40.8226];
COB = COB.*PIR;
%
%   GENERAL PRECESSION IN LONGITUDE
%
XOP=3.392506;
```

```
PRM=50.439273;
NOP=9;
AOP=ones(9,1);
BOP=ones(9,1);
COP=ones(9,1);

AOP(1:5) = [7391.0225890 2555.1526947 2022.7629188 -1973.6517951  1240.2321818];
AOP(6:9) = [ 953.8679112 -931.7537108  872.3795383   606.3544732];

BOP(1:5) = [  31.609974  32.620504  24.172203   0.636717  31.983787];
BOP(6:9) = [   3.138886  30.973257  44.828336   0.991874];
BOP = BOP.*PIRR;

COP(1:5) = [251.9025 280.8325 128.3057 348.1074 292.7252];
COP(6:9) = [165.1686 263.7951  15.3747  58.5749];
COP = COP.*PIR;
%
%    3.NUMERICAL VALUE FOR ECC PRE XOB
%
%        T IS NEGATIVE FOR THE PAST   T IS IN YEARS
%
%
     ARG = BE .* T + CE;
     XES = sum(AE .* sin(ARG));
     XEC = sum(AE .* cos(ARG));
     ECC = sqrt(XES2 + XEC2);
     TRA = abs(XEC);
%
if(TRA <= 1.0E-08 | XEC == 0.0)

    if(XES<0.0)
        RP = 1.5 * pi;
    elseif(XES == 0.0)
        RP = 0.0;
    else
        RP = pi/2.0;
    end
else
    RP=atan(XES/XEC);
    if (XEC<0.0)
        RP = RP + pi;
    elseif(XEC == 0.0)
        if(XES<0.0)
            RP = 1.5 * PI;
        elseif(XES==0.0)
            RP = 0.0;
        else
            RP = pi/2.0;
        end
    else
        if(XES<0.0)
            RP = RP + 2.0 * pi;
```

```matlab
        end
    end
end
PERH = RP/PIR;

PRG  = PRM*T + sum(AOP .* sin(BOP .* T + COP));
PRG  = PRG/3600.0 + XOP;
PERH = PERH + PRG;
if(PERH<0.0)
    PERH = PERH + 360.0;
elseif(PERH>=360.0)
    PERH = PERH - 360.0;
end

%
PRE=ECC * sin(PERH * PIR);
%
XOB = XOD + sum(AOB ./ 3600.0 .* cos(BOB .* T + COB));
%
%    4.DAILY INSOLATION
%
%
%        DAILY INSOLATION IN KJ M(-2) DAY(-1)
%                 IF SS IN W M(-2)
%               AND  TAU = 24*60*60 SEC / 1000
%
SS=1353.0;
TAU=86.4;
SF=TAU * SS / pi;
SO=sin(XOB * PIR);
XL=PERH + 180.0;
%
%   4.2 CALENDAR DATE  MA-JA
%---------------------
%      ND  NUMBER OF THIS DAY IN A YEAR OF 365 DAYS
%      XLAM = MEAN LONG. SUN FOR TRUE LONG. = 0
%      DLAMM = MEAN LONG. SUN FOR MA-JA
%
ND=NDAY(MA,JA); % NDAY user defined MatLab function
XLLP=XL * PIR;
XEE=ECC2;
XSE=sqrt(1.0 - XEE);
XLAM=(ECC/2.0 + ECC * XEE/8.0) * (1.0 + XSE) * sin(XLLP)...
     -XEE/4.0 * (0.5 + XSE) * sin(2.0 * XLLP)...
        + ECC * XEE/8.0 * (1.0/3.0 + XSE) * sin(3.0 * XLLP);
XLAM=2.0 * XLAM/PIR;
DLAMM=XLAM + ( ND - 80) * STEP;
ANM=DLAMM - XL;
RANM=ANM * PIR;
XEE=XEE.*ECC;
RANV=RANM + (2.0 * ECC - XEE / 4.0) * sin(RANM)...
     + 5.0/4.0 * ECC2 * sin(2.0 * RANM) + 13.0/12.0 * XEE * sin(3.0 * RANM);
```

```matlab
ANV=RANV/PIR;
TLS=ANV+XL;
[WW,DAYL] = DAYINS_1(ECC,XL,SO,TLS,PHI,PIR,TEST,SF); % DAYINS user defined MatLab
BERGER=WW;
end

%%%

function [WW,DAYL] = DAYINS(ECC,XL,SO,DLAM,PHI,PIR,TEST,SF)
% INPUT
%     ECC = eccentriciy
%      XL =
%      SO = solar constant kJ/m2/day Top Of The Atmosphere
%    DLAM =
%     PHI = latitude, deg N
%     PIR =
%    TEST =
%      SF =
%
% OUTPUT :
%      WW = LY/DAY   OR   KJ M(-2) DAY(-1)
%    DAYL = LENGTH OF DAY (HOURS)

      RPHI = PHI * PIR;
      RANV = (DLAM-XL) * PIR;
      RAU = (1.0 - ECC2) / (1.0 + ECC * cos(RANV));
      S=SF / RAU / RAU;
      RLAM=DLAM * PIR;
      SD=SO * sin(RLAM);
      CD=sqrt(1.0 - SD * SD);
      RDELTA = atan(SD/CD);
      DELTA = RDELTA/PIR;
      SP = SD * sin(RPHI);
      CP = CD * cos(RPHI);
      APHI = abs(PHI);
      ADELTA = abs(DELTA);
%
%    SINGULARITY FOR APHI=90 AND DELTA=0
%    PARTICULAR CASES FOR PHI=0   OR   DELTA=0
%
      TT = abs(APHI-90.0);
      SPD = PHI*DELTA;
      if ((TT<=TEST) | (ADELTA<=TEST) | (SPD<0.0))  % CASE 2 : POLAR CONTINUAL NIGH
DAYL=0
          DAYL = 0.00;
          WW = 0.00;
      elseif(ADELTA<=TEST)                          % CASE 6 : EQUINOXES
          DAYL = 12.0;
          WW = S*cos(RPHI);
      elseif(APHI<=TEST)                            % CASE 7 : EQUATOR
          DAYL = 12.0;
          WW = S*cos(RDELTA);
```

```matlab
    else
        AT = 90.0 - ADELTA;
        if (APHI<=AT | spd==0.0)                    % CASE 3 DAILY SUNRISE & SUNSET
            TP = -SP/CP;
            STP = sqrt(1.0 - TP2);
         RDAYL = acos(TP);
            DAYL = 24.0 * RDAYL/pi;
            WW = S * (RDAYL * SP + CP * STP);
        elseif(SPD>0.0)                             % CASE 4 POLAR CONTINUAL DAY
            DAYL = 24.00;
          WW = S * SP * pi;
        else                                        % LOGIC ERROR TRAP
            WW = NaN;
            DAYL = NaN;
    end
  end

%%%

function ND = NDAY(MA,JA)
% calculate the Julian day of the year from month and day of month
% assuming 365 days per year
% MA = month of year
% JA is day of the month
% NJM is a vector of julian day offsets by month;
%        0 for January up to
%           ...
%       334 for December
  %NJM = [31 59 90 120 151 181 212 243 273 304 334 365];
  NJM = [0 31 59 90 120 151 181 212 243 273 304 334];
  ND = NJM(MA)+JA;
end
```

12.2   The paleoLibrary in R

The paleoLibrary is a tool box with over 200 functions written in the programming language R that serve for climate data analysis. R is freely available at http://www.r-project.org/. In particular, R allows for many statistical calculations, so that it is by now the standard for developing statistical software and analyzing data from a mathematical viewpoint (see also Rwiki). The paleoLibrary uses basic functions implemented in R to provide software for relevant problems in climate science.

This is to announce the release of version 2.0 of the R contributed package 'climatol', which introduces a function for a completely automatic homogenization of a climatological dataset. This includes missing data filling and detection/correction of outliers and shifts in the mean of the se-

ries. (Trends are diagnosed but not corrected). This package is available at the contributed package CRAN repositories<http://cran.r-project.org/web/packages/index.html>, but before actually trying the software, potential users may want to look at the climatol guide, to see if it can meet their requirements. It can be downloaded from the climatol home page: <http://webs.ono.com/climatol/climatol.html> Help is offered if something is not well explained in the guide, and any feed-back will be welcome.

The main paleoLibrary folder has subfolders "data", "doc" and "src". In the "data" folder, one finds a couple of standard data sets (that are relatively small in size). The "doc" folder contains the tutorial "paleoLibrary.documentation.pdf" and the file "info.functions.pdf" describing the paleoLibrary functions. In the "src" folder, paleoLibrary functions are stored according to the category they belong to. For convenience, there is also a file called "all.functions.R" which incorporates all paleoLibrary functions. This might be helpful in case one is interested in the code of a certain function (search inside the file for "(functionname)<"). Note that the "header.R" has to be executed in R in order to load all paleoLibrary functions.

When first working with the library, it is necessary to have R installed on the work station (free download from http://www.r-project.org/) and to know about its basic functionality (see http://cran.r-project.org/manuals.html and especially http://cran.r-project.org/doc/manuals/R-intro.pdf for an introductory course). The data to be analyzed should be stored in one of the usual formats (e.g., netcdf files (.nc), text files (.txt)). Note that many standard data sets (e.g., downloaded from http://www.esrl.noaa.gov/psd/data/gridded/) and model output are already available on the AWI intranet (server paleo5 ("/csys/paleo5/data")). Concerning the paleoLibrary tools available for data analysis, the file "/docu/info.functions.pdf" contains elementary information. Here, functions are listed, briefly described and categorized according to the task they perform. Before loading the functions into the R console by executing "/src/header.R" and starting with first experiments, it is recommended to read the tutorial "paleoLibrary.documentation.pdf".

Learn about the concept and Get started

Data and metadata   In the paleoLibrary, the data are always encapsulated together with their metadata as position, time, name, etc. Additionally, the history of the data processing is saved, such that one can easily keep track of the changes made / experiments carried through.

Object orientation   The object orientation of R is used to make the work more intuitive. For instance, if you execute plot(data) with data being a time series, you will create a time series plot; if data is a 2D field ($\equiv$ matrix) instead, you will get a map.

   After having installed R etc (see readme file), you need to load the paleoLibrary functions into R. If path is the directory the paleoLibrary files are stored in (e.g., path="C:/Dokumente und Einstellungen/Micha/Eigene Dateien/uni/hiwi awi 2011/paleoLibrary/src/"), this can be done by typing

```
source(paste(path,"header.R",sep=""))
```

into the opened R console. It may occur that you get an error message like

```
Error in library(...)  :  there is no package called '...'
```

in which case you have to install the missing package(s) by hand. For Windows users, this is straightforward:

- click 'Packages'

- click 'Install Package(s)'

- select a CRAN mirror

- download the package of interest (see Figure 12.2)

If no more error messages pop up when loading the functions via

```
source(paste(path,"header.R",sep="")),
```

the paleoLibrary can be used.
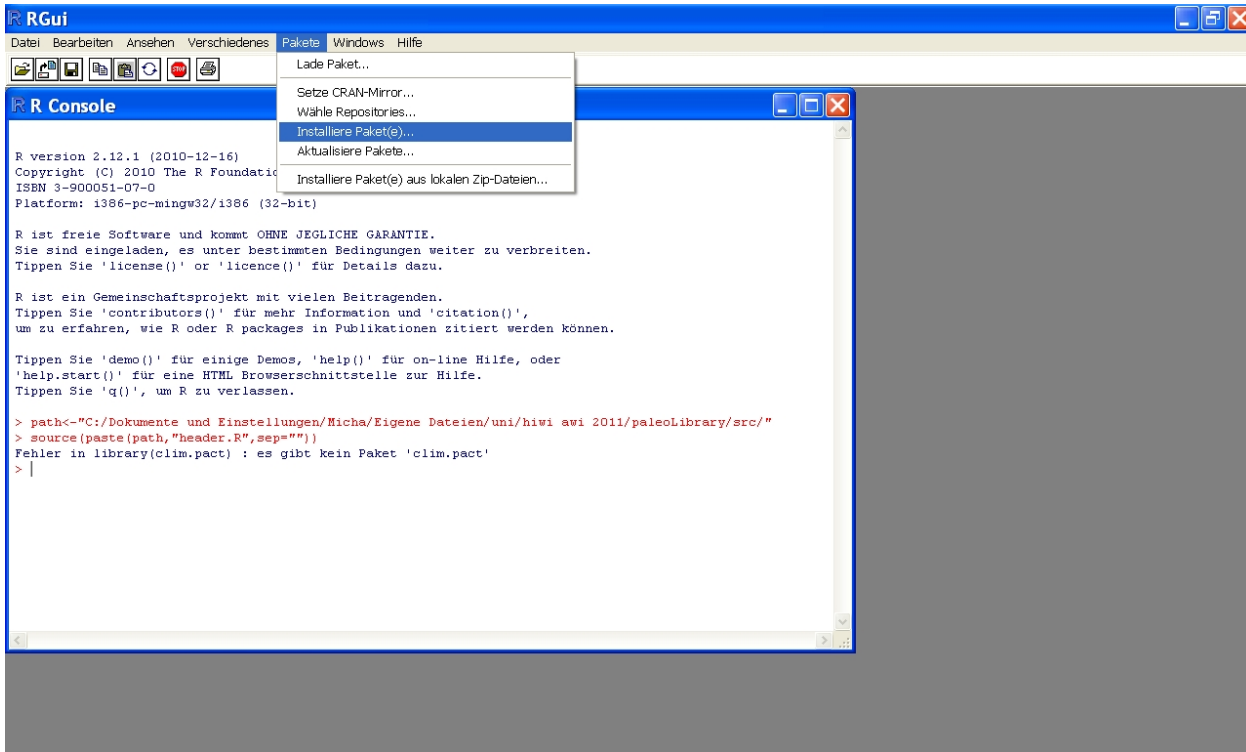
Figure 12.2: Install missing packages.

Get to know the paleoLibrary data objects   There are two paleoLibrary class objects: "pTs" and "pField".

An object with class "pTs" is used for data that have time series format. It extends the R class "ts" and, next to a time base (where the time steps have to be equidistant), allows to assign attributes as latitude, longitude to a time series vector.

Example   Consider the *burgundy2004.harvest.txt* file in the paleoLibrary data folder. This file contains grape harvest dates (counting the number of days after September 1) and reaches back to

the year 1370. You can read in the data as follows:

```
basedrive<-"..." (e.g., "C:/Dokumente und Einstellungen/Micha/
     Eigene Dateien/uni/hiwi awi 2011/paleoLibrary/data/")
grape.data<-read.table(paste(basedrive,"burgundy2004.harvest.txt",
     sep=""),skip=7)
# extract further information from the txt file
grape.lat<-47.0525047
grape.lon<-4.3837215

# generate a pTs object (times are stored in the first,
# grape harvest dates in the second column)
grape_pTs<-pTs(data=grape.data[,2],time=grape.data[,1],
     lat=grape.lat,lon=grape.lon,
     name="Grape harvest dates (counted in days from Sep 1st)",
     history="",date=TRUE)
```

If the grape harvest had started at the same day every year (which is not the case), we would have a constant time series and could simplify the pTs generation:

```
grape_pTs<-pTs(data=const,time=grape.data[,1],
     lat=grape.lat,lon=grape.lon,
     name="Grape harvest dates (counted in days from Sep 1st)",
     history="",date=TRUE)
```

It is also possible to store more than one time series in a pTs object. If there was a second data string in the grape harvest data set (e.g., somehow quantifying the crop yield), then the pTs object could be easily extended:

```
grape_pTs<-pTs(data=cbind(grape.data[,2],grape.data[,3]),
      time=grape.data[,1],lat=grape.lat,lon=grape.lon,
      name=c("Grape harvest dates (counted in days from Sep 1st)",
      "Crop yield"),history="",date=TRUE)
```

Furthermore, in case the data would stem from different sites, the latitude/longitude information could also be defined as a vector:

```
grape_pTs<-pTs(data=cbind(grape.data[,2],grape.data[,3])
      ,time=grape.data[,1],
      lat=c(grape.lat,grape.lat.b),lon=c(grape.lon,grape.lon.b)
      name=c("Grape harvest dates (counted in days from Sep 1st)",
      "Crop yield"),history="",date=TRUE)
```

Note that, however, the time base has to be the same for all data strings.

pField objects Especially climate models often have gridded data output. Thereby, a set of (equidistant) latitude and longitude values defines the grid (spanning in many cases the whole globe), such that climate variables as temperatures are computed for each latitude/longitude pair; a 2D field develops. If the model allows for variations in time (as it normally does), one such 2D field is obtained for every time step, and each latitude/longitude pair is associated with a time series. The paleoLibrary "pField" class is designed for this kind of three-dimensional data array. As with the pTs objects, it stores additional information (name, latitudes, longitudes) in the data

variable.

Example   Look into the *KAPLANsst.DJF.mean.nc* file in the paleoLibrary data folder, a file in which annual SST (Sea Surface Temperature) data are stored for the time from 1856 to 2007. The grid has a 5×5 resolution, i.e., the distance between adjacent both latitudes and longitudes is 5 degrees.

```
basedrive<-"..." (e.g., "C:/Dokumente und Einstellungen/Micha/
      Eigene Dateien/uni/hiwi awi 2011/paleoLibrary/data/")
# load the netcdf file with the open.ncdf command
temp.nc<-open.ncdf(paste(basedrive,"KAPLANsst.DJF.mean.nc",sep=""))
# extract time and data (here, variables are named "time" and "sst")
# here, "sst" is a 72×36×152 array (lon×latitude×time)
temp.time<-get.var.ncdf(temp.nc,"time")
temp.data<-get.var.ncdf(temp.nc,"sst")
# preprocess the time variable (to get year format)
temp.time<-floor(temp.time/10000)
# extract latitude and longitude info
temp.lat<-get.var.ncdf(temp.nc,"lat")
temp.lon<-get.var.ncdf(temp.nc,"lon")

# generate pField object
kaplan.sst_pField<-pField(data=temp.data,time=temp.time,
      lat=temp.lat,lon=temp.lon,name="Kaplan SST",history="")
```

The loading scheme has to be individually adjusted for each ncdf file, as different authors also have different preferences regarding, for instance, time format and variable naming. For this purpose, there is a universal reading tool in the paleoLibrary that copes with the various input styles (see the function `read_data` in the load.datasets.R file), requiring only the path to the ncdf file and the variable name(s) if known. More specifically, the load.datasets.R file also provides import tools for some standard data sets.

Extract information from paleoLibrary objects Functions `is_pTs` and `is_pField` allow for testing objects with regard to the pTs and pField class, respectively. With a pTs or pField object at hand (let's name it `pL.object`), one can easily retrieve the additional attributes (latitude, longitude, name, history):

```
# latitude scalar/vector
getlat(pL.object)
# longitude scalar/vector
getlon(pL.object)
# name(s) of the pTs/pField object
getname(pL.object)
# history of the pTs/pField object
gethistory(pL.object)
```

It is also possible to print out a summary via `summary(pL.object)`, stating the object's details.

Concerning subsetting, the [ ] operator works as follows for pTs objects:

- `pts[i]` = pTs object at time(s) `i` (considering only one series in the multivariate case)

- `pts[,j]` = pTs object: j^{th} time series (multivariate case)

- `pts[i,j]` = pTs object at time(s) `i` and `j`[th] time series

and likewise for pField objects:

- `pfield[i]` = scalar/vector: with `n` time steps, `pfield[1:n]` is the first grid point's time series, `pfield[(n+1):2n]` the second grid point's time series, ...

- `pfield[i,]` = pField object: `pfield` at time(s) `i` (2D if `i` is a scalar)

- `pfield[,j]` = pTs object: time series at `j`[th] grid point (see Figure 12.3)

- `pfield[i,j]` = scalar/vector/matrix: `pfield` at time(s) `i`, grid point(s) `j`
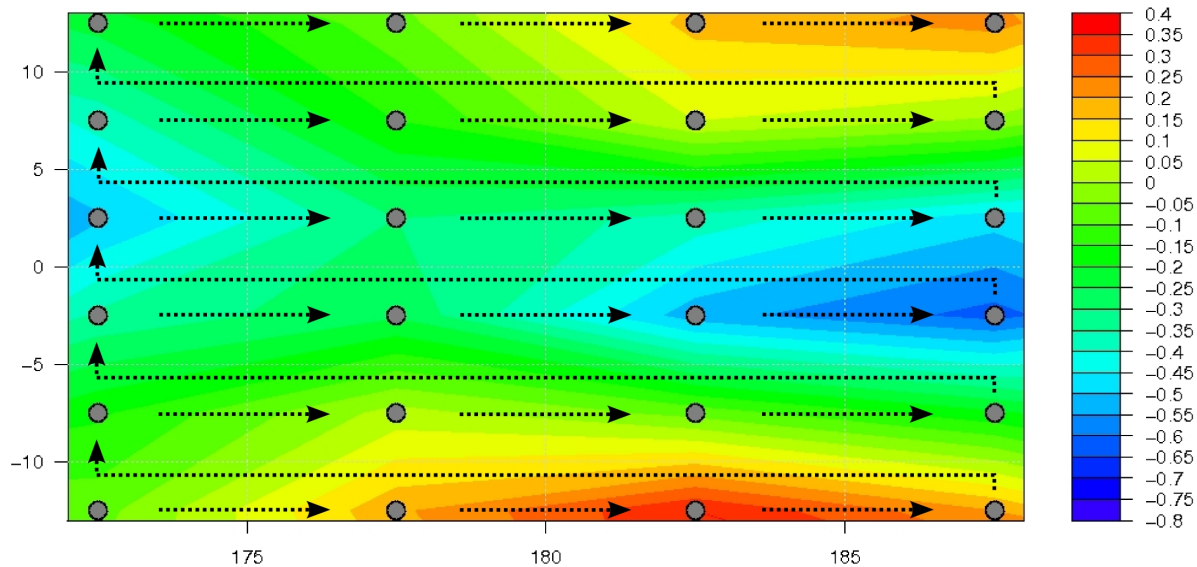
Figure 12.3: 2D field grid points: the ordering starts at the bottom left and ends at the top right.

Zoom into paleoLibrary data objects   Often, the spatial and / or temporal extent of a data set exceeds one's actual needs. In this case, functions `selspace` (for pField objects) and `window`

(for both pField and pTs objects) help to focus on sections of interest. Getting back to the Kaplan SST data, an idea could be to select the region around Europe for later experiments. With the original pField object introduced above,

```
# generate pField object
kaplan.sst_pField<-pField(data=temp.data,time=temp.time,
    lat=temp.lat,lon=temp.lon,name="Kaplan SST",history="")
```

and some latitude/longitude boundaries for Europe, this is achieved via

```
kaplan_europe<-selspace(data=kaplan.sst_pField,
                lat1=0,lat2=80,lon1=260,lon2=60)
```

The grape harvest data, stored in the pTs object

```
grape_pTs<-pTs(data=grape.data[,2],time=grape.data[,1],
    lat=grape.lat,lon=grape.lon,
    name="Grape harvest dates (counted in days from Sep 1st)",
    history="",date=TRUE)
```

reaches back to year 1370, while the Kaplan SST data spans from 1856 to 2007. As does Moritz Krieger in his work ("Climate signatures of grape harvest dates", 2010), some experiments require to determine the common time window of different data sets. Here, this is realized by

```
grape_pTs.common<-window(grape_pTs,start=1856,end=2003)
kaplan_europe.common<-window(kaplan_europe,start=1856,end=2003)
```

When doing calculations that involve pField and / or pTs objects, it is sometimes easier to let

the data come along in standard R arrays / vectors. Two commands are helpful in this regard:

$$c(pL.object)$$

outputs the data as a vector (also for pField objects, for which the output structure is as described in the first item of the [ ] enumeration (see previous subsection)), whereas

$$unclass(pL.object)$$

returns a matrix (with respect to pField objects, each column being associated with a certain grid point).

Process paleoLibrary data objects   All regular operators $(+,-,*,\nabla\cdot)$ work with pField and pTs objects. For instance, scalar multiplication can be done by `scalar*pL.object` (affecting each element of the pField/pTs object `pL.object`). Elementwise operations (more generally indicated by the '·' symbol) between two pTs/pField objects of the same length are performed as for regular R objects:

$$pL.object1 \cdot pL.object2.$$

More complex tools exist in particular for pField objects. To start with, some input function `FUN` can be applied to a 3D field either by fixing the time steps

- `applyspace`: e.g., in order to compute the field average for each time step with `FUN=mean`; the output then is a pTs object, yielding one mean value per time step

or by fixing the grid points

- `applytime`: e.g., in order to get internal time series correlations associated with the grid points (using `FUN=cor`); the output then is a 2D field with one correlation coefficient per grid point

Example Again considering the Kaplan SST data, one could be interested in comparing the global mean sea surface temperatures for different time steps. Averaging over all grid points,

```
kaplan.sst_spatial.mean<-applyspace(data=kaplan.sst_pField,FUN=mean)
```

the data show a clear temperature increase (see Figure 12.4), where the graphic was produced entering

```
plot(kaplan.sst_spatial.mean,lwd=2)
abline(h=mean(kaplan.sst_spatial.mean),col="red",lty=3,lwd=2)
legend("topleft",
       c("annual temperature means","overall temperature mean"),
       lty=c(1,3),lwd=c(3,3),col=c("black","red")).
```

The `applyspace` and `applytime` functions exhibit in fact only a small fraction of the paleoLibrary potential. One can choose among approximately 200 functions that cover a large variety of topics:

- correlation and regression analysis

- frequency domain analysis and filtering

- principal component analysis

- Monte Carlo simulation

- insolation

- other basics as detrending, scaling, rolling/running functions

- …
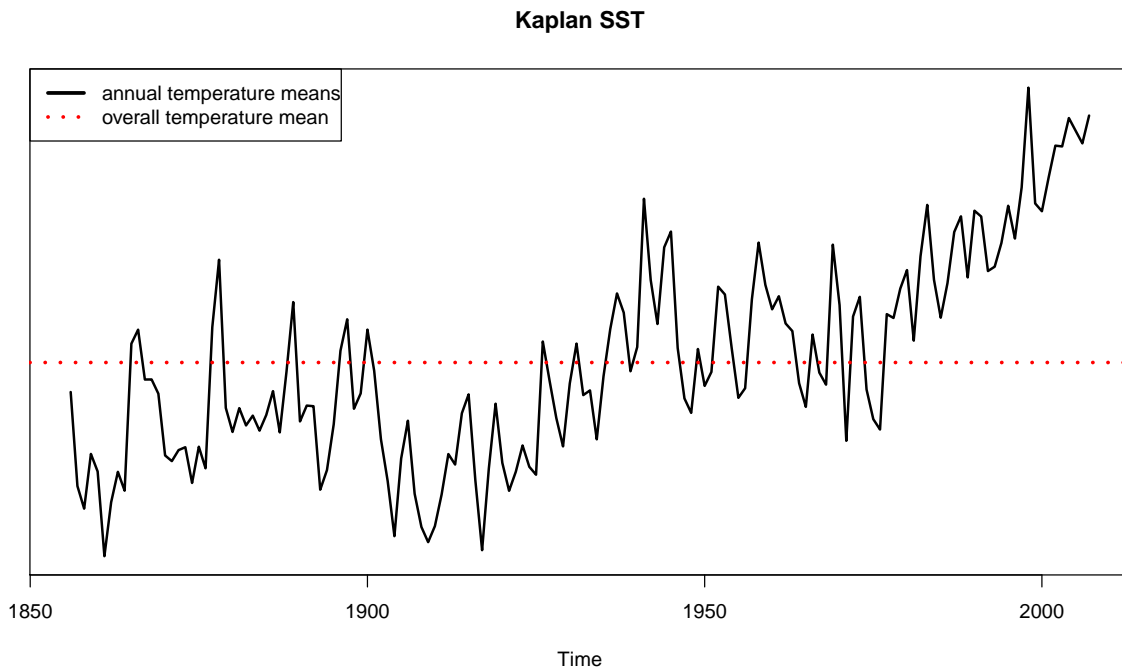
Kaplan SST

Figure 12.4: Kaplan SST data averaged over all grid points.

It might be helpful to consult the info.functions pdf stored in the doc folder to get detailed informa-
tion. Only one more example is given in this documentation file: Dealing with correlation analysis,
the `plot_pFieldcor` function is presented.

Examples   Correlation analysis is a powerful tool to evaluate the extent of climate variability.
Again referring to his work ("Climate signatures of grape harvest dates", 2010), Moritz Krieger
suggests that the grape harvest time is connected to the temperatures from April-August, which
would make the grape harvest data interesting for reconstructions of past climate conditions. Ev-
idence is provided by experiments in which climate data sets are correlated with a grape harvest
time series from the Burgundy region. These use the `plot_pFieldcor` routine, a function that
correlates a pField object with a pTs object and tests the obtained coefficients for significance.

Taking the Kaplan SST data and the grape harvest time series, this is as easy as

```
plot_pFieldcor(field=kaplan.sst_pField,ts=grape_pTs,
        area=c(lon1=260,lon2=100,lat1=20,lat2=80),plot.sig="contour").
```

The results are displayed in Figures 12.5, 12.6 and 12.7, where the `plot.sig` argument respectively takes on values "contour", "siglines" and "sigshade".
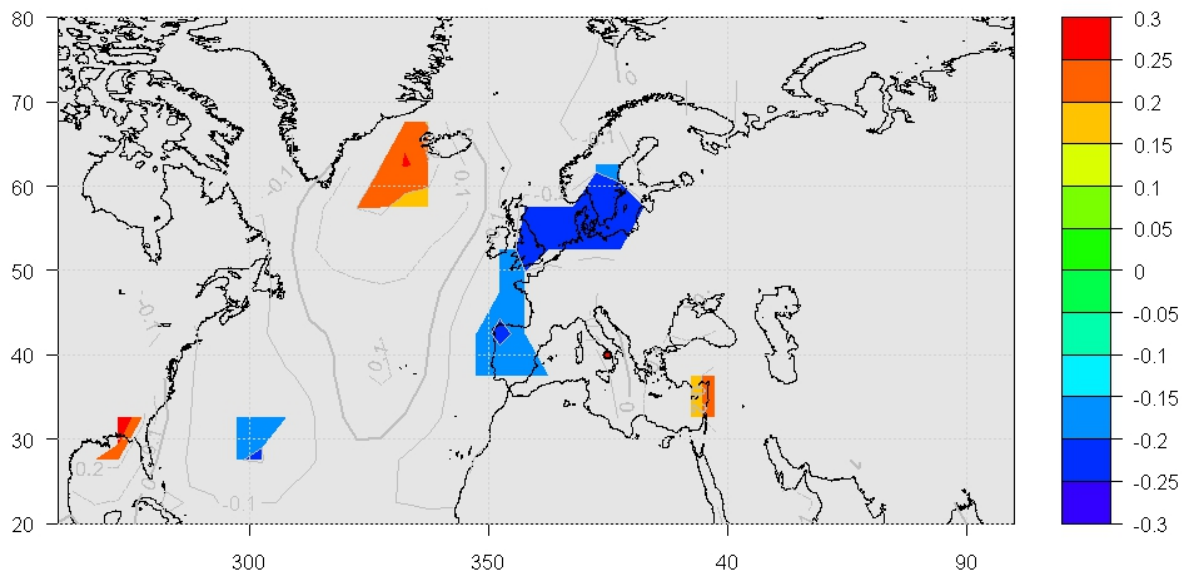
Figure 12.5: `plot_pFieldcor` output for Kaplan SST, grape harvest data and `plot.sig=contour` (only significant areas are colored).
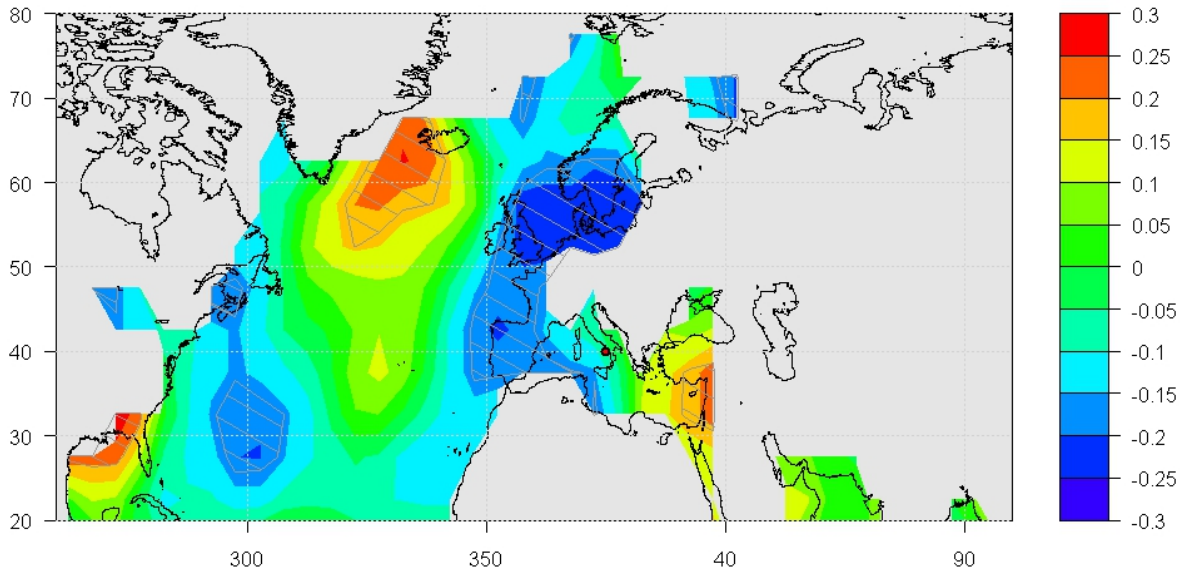
Figure 12.6: `plot_pFieldcor` output for Kaplan SST, grape harvest data and `plot.sig=siglines` (all correlations are colored, significant areas are hatched).
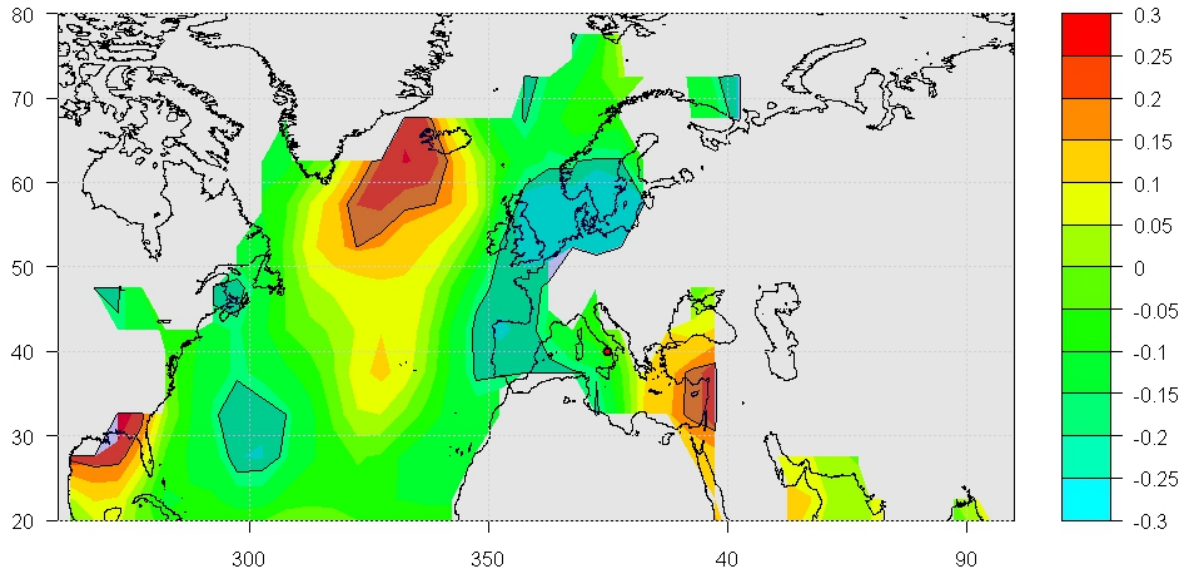
Figure 12.7: `plot_pFieldcor` output for Kaplan SST, grape harvest data and `plot.sig=sigshade` (all correlations are colored, significant areas are shaded in light blue).

Exercise 82   –   Correlation plots

1. Calculate the temperature and precipitation values for the closest point to your home town. Display the time series for the annual mean and the 4 seasons, respectively.

2. Calculate the correlation with large-scale sea level pressure and SST in a similar way as in Figures 12.5, 12.6 and 12.7. Discuss the seasonal correlation signal in the data.

Plot paleoLibrary data objects   As has been shown in one of the previous examples, plotting is particularly straightforward for pTs objects. The plot arguments are the same as for regular R time series ("ts" objects; see R help for plot.ts). Also, additional lines and points can be inserted into a graphics device after a pTs plot has been created. This is different from a pField plot, which uses the R function filled.contour (to get an overview of the graphic parameters, look up the R help or the description of `filled.contour.own` / `plotmap.pField` in the info.functions pdf). Here, additional input can be passed on through the `FUN` argument.

Example   One might want to plot the Kaplan SST field at time 1875 and highlight, for some reason, the capital of Australia, Canberra. It is a good start to put in a point at Canberra using its geographical coordinates:

```
canberra<-c(lon=149.12,lat=-35.27)

plot(kaplan.sst_pField[20,],FUN=points(x=canberra[1],
      y=canberra[2],pch=21,bg="green",cex=2))
```

Then again, this mark is not striking enough. It thus seems appropriate to label the point, leading to two input functions (R functions text and points). These are passed on to the field plot by entering

```
canberra.function<-function()
  {
    canberra<-c(lon=149.12,lat=-35.27)
    points(x=canberra[1],y=canberra[2],pch=21,bg="green",cex=2)
    text(x=canberra[1],y=canberra[2],labels="Canberra, Austr.",
    col="white",cex=1.5,pos=4)
  }


plot(kaplan.sst_pField[20,],FUN=canberra.function())
```

For a visualization, see Figures 12.8 and 12.9.
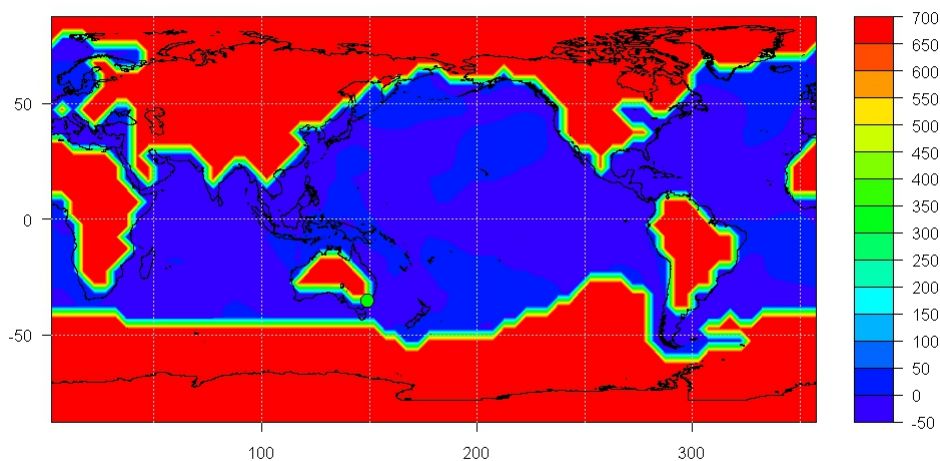
Figure 12.8: Kaplan SST data (year 1875) with the capital of Australia highlighted (green point).
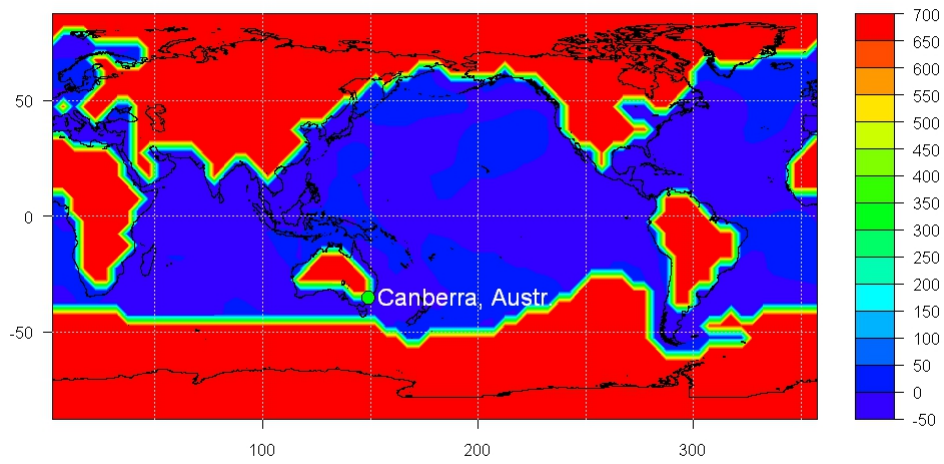
Figure 12.9: Kaplan SST data (year 1875) with the capital of Australia highlighted (green point plus text label).

Exercise 83   –   Correlation and Monte Carlo experiments

Tasks:

1. Just learn the correlation command: Create two vectors of equal length which have a linear relation, plot A against B

2. Use R to calculate the correlation between the two.

3. Get a feeling for the interpretation of the size of a correlation: Add noise to B using the rnorm function, plot the two, add more and more noise

4. How the correlation and the p-value in cor.test are changing? The $R^2 =$ coeffient of determination should be the same as the ratio of the variance of the signal B to the variance of A. Check this!

5. Sampling variability/correlation which appears by chance: Now create two vectors which are only containing noise; Calculate the correlation between the two.

6. Vary the number of your samples (length of your vectors), investigate how the correlations are changing. Now you can see why statistical tests are needed.

7. Monte Carlo experiment to investigate significance in correlations: Repeat the experiments very often (in a loop, this is called Monte Carlo experiment). Each time save the correlation between the two random vectors. Plot the histogram of this distribution.

8. You can now see how likely it is to get specific correlation value just by chance. This probability is strongly dependent on your sample size/degrees of freedom. Now determine the size of correlation which is needed if you want to have a probability of less/equal 5% to get the correlation by chance. Use the quantile command.

9. If you create two datasets which have this "limit" correlation you got by the Monte- Carlo experiment and the same degrees of freedom, cor.test should give you $p = 0.05$ (here this is calculated with a t-distribution).

10. The advantage of the Monte Carlo experiment is that you do not have to care about all the assumptions made in the analytic calculation of the p-value (normality, independence).
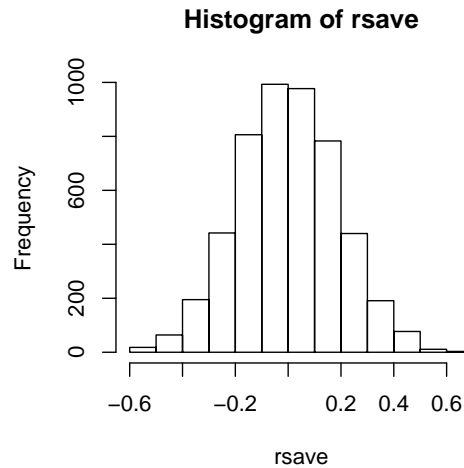
Figure 12.10: Histogram of the Monte Carlo experiment in exercise 83.

Solution

```
#Correlation, significance and Monte carlo experiments
#Correlation = normed regression

a<-1:1000
noise<-rnorm(1000)*500
b<-a + noise

plot(a,b)
cor(a,b)    #correlate a,b
cor.test(a,b)  #correlation + significance test

N<-30

a<-rnorm(N)
b<-rnorm(N)
cor(a,b)

#Now the same as a Monte Carlo experiment (some hundred times)

rsave<-vector()    #vector for saving
for (i in 1:5000)
{
  a<-rnorm(N)
  b<-rnorm(N)
  rsave[i]<-cor(a,b)
}
```

```
hist(rsave)

#abs is used to get the p-values for a two-sided test
quantile(abs(rsave),probs=0.95)
#dev.print(postscript, file="histogr.ps")
```

Exercise 84 – **Correlation and statistical testing**

Two independent time series covering the time from 1850 to 1970 with negligible autocorrelation are given. They are correlated with $\rho = -0.2$.

a) Describe a statistical test to obtain whether the correlation is significant with p=0.01 or not.

b) How is such a test called?

c) How can this test be realised in R? Write down the programming code. It should also plot a histogram and the quantiles.

Exercise 85 – **Statistical significance**

Fig. 12.11 shows the histogram of a statistical test with random time series. The thin red (green) lines show the p=0.05 (p=0.01) quantiles. Three pairs of time series with the same length has been correlated and yield the following correlation values:
a) $\rho = -0.32$, b) $\rho = 0.14$ and c) $\rho = 0.24$. Which of the correlation values are statistical significant with p=0.01 and p=0.05?

Exercise 86 – **Palaeolibrary**

With the palaeolibrary, climate data can be easily analysed. Describe what the following commands are doing and explain the meaning of the arguments.

a) `plot.final(field=sst.field,ts=nao.index,zlim=c(-0.6,0.6),`
   `plot_sig="siglines",area=c(lat1=30,lat2=75,lon1=340,lon2=60),`
   `point=c(lat=46,lon=10))`

b) `test1<-selspace(field.sst,lat1=30,lat2=45, lon1=20,lon2=60)`
   `test2<-applyspace(test1,mean)`
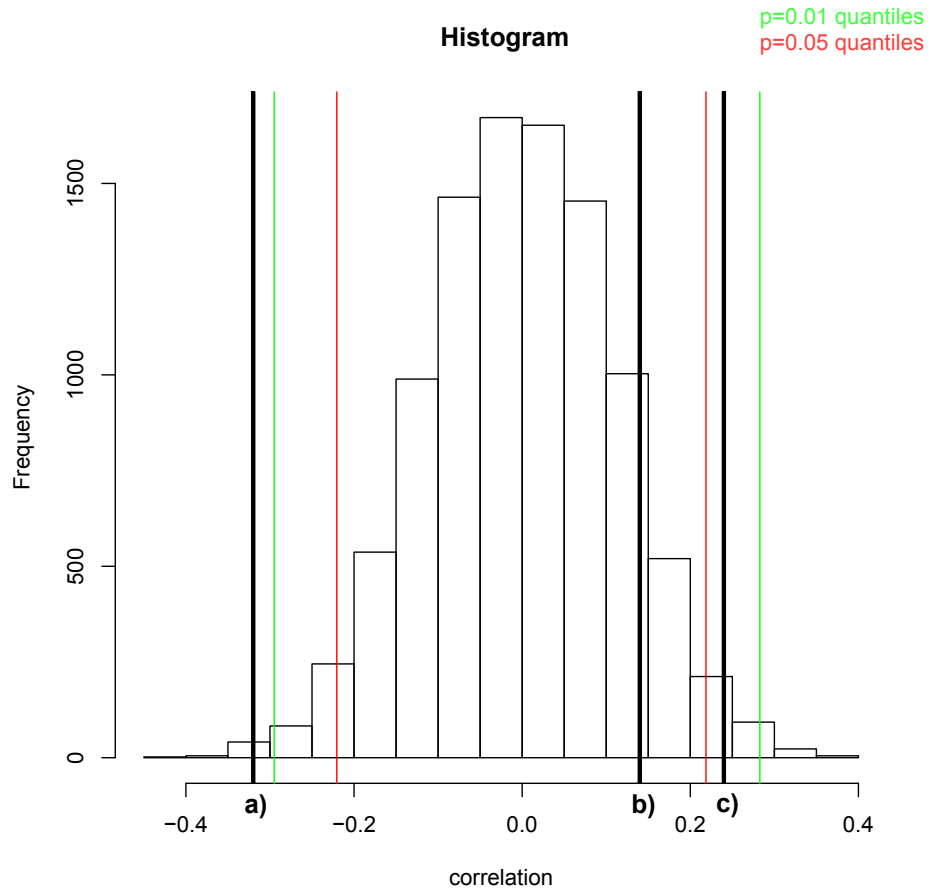
Figure 12.11: Histogram of a statistical test with random time series. The red (green) lines show the p=0.05 (p=0.01) quantiles. The bold black lines show correlation values of two time series, respectively.

c) ```
plot(roll.2(test2,nao.index,width=30,by=2,FUN=cor.pTs)
```

## 12.3  Examples in Python

This document describes some examples in the programming language python.  According to wikipedia and the development website:

> Python is a general-purpose, high-level programming language whose design philosophy emphasizes code readability. Python's syntax allows programmers to express concepts in fewer lines of code than would be possible in languages such as C, and the language provides constructs intended to enable clear programs on both a small and large scale.
>
> Python supports multiple programming paradigms, including object-oriented, imperative and functional programming styles.  It features a dynamic type system and automatic memory management and has a large and comprehensive standard library.
>
> Like other dynamic languages, Python is often used as a scripting language, but is also used in a wide range of non-scripting contexts.  Using third-party tools, Python code can be packaged into standalone executable programs.  Python interpreters are available for many operating systems.

A couple of thing should be noted. First and foremost, python relies on whitespace to organize the language syntax. `for`, `if`, `while`, and function definitions via `def` all need to be properly indented, either with tabs or spaces. This is a bit irritating if you are coming from a background of C or Matlab, where loops can easily be enclosed in braces or parentheses. Once you get used to it, though, proper use of whitespace makes for cleaner and legible code, without having to hunt and find where a certain block begins and ends.

Secondly, python is object orientated.  This means that anything that is defined (or declared) by the user gets a set of attributes depending on what sort of object/instance it is.  These can be accessed with a . behind the object's name.  Although a computer scientist would probably furiously disagree with this, I'll use object and instance interchangeably.[1]

---

[1]Strictly speaking, an object is location within the computer memory with a specific value referred to be an identifier. Objects are instances of classes

## 12.3.1 Plotting climate data

The code that comes with this document, `cosmos_plot.py`, is designed to plot 2 types of run-of-the-mill model outputs that you're bound to encounter sooner or later. First, we will plot a scalar variable from the ocean: the global salinity field for the surface layer. The function we'll use to do this is below.

```python
def plot_scalar(fin):
  fin = netcdf.netcdf_file(fin, 'r')
  global vars
  vars = fin.variables
  var = fin.variables['SAO'].data.squeeze()
  lon = filepath.variables['lon'].data.squeeze()
  lat = filepath.variables['lat'].data.squeeze()
  var, lon = shiftgrid(180.,var, lon, start=False)
  lons, lats = np.meshgrid(lon, lat)
  map = Basemap(projection='robin',lon_0=0,resolution='c')
  x, y = map(lons, lats)
  map.drawmapboundary(fill_color='0.6')
  map.drawcoastlines(linewidth=0.75)
  cs = map.contourf(x,y,var, cmap = matplotlib.cm.jet, levels=np.arange(20,40,1),
  cbar = map.colorbar(cs, location='right',pad="5%", format='%.2f', extend = 'bot
```

Let's go through this line by line and figure out what is happening. The first line should be fairly obvious, we define a function `plot_scalar()` which takes a single argument, `fin`. Notice that since `fin` is not assigned a default value, calling the function without any arguments would lead to an error.

Next, we reassign `fin` by calling the module netcdf which contains the function netcdf_file. Here, the `fin` we provided is read by the function with the option `'r'`. `fin` is now no longer a string but a netcdf_file object.

If we are running python interactively, it would be useful to see which variables are in the file we read even after the function has completed. Therefore, the variable vars is defined to be global, making it available to the user outside of the function environment.

Now we want to grab the variable we want to plot out of the dataset. Let's give it a sensible name, like `var`. It's taken from the netcdf object `fin`, it's data is extracted, and the squeeze command removes any dimensions which have a size of one. This is necessary for plotting, since

plotting in more than 2 dimensions gets a bit tricky. We similarly get the corresponding latitudes and longitutdes.

Line 8 might not make sense right away. The model gives us outputs on 0 to 360 degrees, so once around the entire planet. The map projection we will use later needs this to be -180 to 180 instead, otherwise half the plot will have missing values. The rearranged data in `var` and `lon` are overwritten with new values.

Next we set make an evenly spaced grid of the longitudes and latitudes with the meshgrid function, which is contained in the numpy module (usually abbreviated to np when it's imported). The map is set up with the Basemap command, which allows for various projections. We will use robin, centering the map at the Prime Meridian, and take a pretty coarse resolution, since this will already be finer than anything the model gives us.

The Basemap instance, now called `map`, can take the longitudes and latitudes and figure out how far they are in distances from the map origin, which will be needed for plotting, so we do this and assign the distances to `x` and `y`

A couple of cosmetic things follow. The ocean data has missing values, there's no such thing as a surface layer salinity in the Rockies! We'll get a nice grey color to fill these in using `drawmapboundary`. Coastlines are always nice, so we'll add those two. That'll give a feel for where the model coastlines are compared to the real ones.

Finally we can plot. This command is a bit more complicated, so part by part: The map.contourf calls a filled contourplot command to be drawn by the map instance. It takes the variables `x`, `y`, and `var`. `x` and `y` are `lon` and `lat` computed automatically by the map to be in distances `var` is the variable we read at the beginning, salt `cmap` tells contourf which type of colormap to use for the filled contour. We'll use jet, it's a nice span from blue to red over yellow and green `levels` tells contourf how many contours to draw and shade the `np.arange` command within levels gives the smallest, largest, and interval spacing `extend` is a 4 type switch with values of none, max, min, and both. It allows data that is outside the range allowed in levels to appear in the maximum (or minimum) color

Lastly, we draw a colorbar to see which values are where.

Next we'll look at a vector, surface ocean velocity. The plot commands are fairly similar.

```python
def plot_vector(fin):
    filepath = netcdf.netcdf_file(fin, 'r')
    us = filepath.variables['UKO'].data.squeeze()
    vs = filepath.variables['VKE'].data.squeeze()
    us = np.ma.masked_equal(us, -9e+33)
    vs = np.ma.masked_equal(vs, -9e+33)
    lon = filepath.variables['lon'].data
    lat = filepath.variables['lat'].data
    us, lon1 = shiftgrid(180.,us, lon, start = False)
    vs, lon = shiftgrid(180.,vs, lon, start = False)
    lons, lats = np.meshgrid(lon, lat)
    map = Basemap(projection='robin', lon_0 = 0, resolution='c')
    lons, lats = np.meshgrid(lon, lat)
    x, y = map(lons, lats)
    map.drawmapboundary(fill_color = '0.6')
    map.drawcoastlines(linewidth = 0.75)
    cs = map.contourf(x,y,(us**2 + vs**2)**0.5, cmap = matplotlib.cm.jet, levels =
    q = map.quiver(x[::5, ::5],y[::5, ::5],us[::5, ::5],vs[::5, ::5], scale = 10)
    qk = plt.quiverkey(q, 0.9, 1.05, 0.5, r'$0.5 \frac{m}{s}$', labelpos = 'E', for
    cbar = map.colorbar(cs, location = 'right',pad = "5%", format = '%.2f', extend
    print 'All done, call plt.show() or plt.savefig()'
```

The import commands are identical. The first new command is found in line 7 and 8; here, we mask out data that is missing from the ocean model and filled with a predefined number. This will eliminate some problems we would otherwise run into during the vector plotting step.

In line 9, we assign the rotated longitude to lon1, since we only want to rotate the globe once. Line 17 has a slight modification; the filled contour plot we want to have behind our vectors should show the magnitude, which is calculated from the individual vector components.

Line 18 will draw the vector arrows. Since we want to avoid clutter, we take every 5th data point. There's lots of customization you can do with the quiver command, but the default looks rather nice.

Finally, we make a quiver key to show how long a default arrow is. The position (0.9, 1.05) is specified relative to the axis frame, and the standard size to be shown is selected (0.5). The label requires a prefix r before the string to prevent escape characters from being evaluated. Notice also that python can evaluate LATEXstandard commands in it's labels.

Executing the Script and result

There are a couple of ways of executing the script, depending on how much you want to learn about python. The functions themselves can be given to an interpret "as is" in the following manner:

```
$ python
>>> execfile('cosmos_plots.py')
```

The functions will now be available to call, just remember to give the filename argument `fin` as a string in single quotes.

Alternatively, you could use the ipython interpreter (i for interactive, not to be confused with numerous apple products). This has several advantages, namely that is much more interactive and has a build in help. Any object followed by a ? will yield a quick help, and a double ?? will show the source code of the function. Start this one with:

```
$ ipython -i cosmos_plots.py
```

The -i flag allows the session to be interactive after the script is executed. ipython also has a very good traceback system for coding errors.

Finally you could enter the function codes line by line to see how the program evolves, either as a function using `def` or as is. ipython offers the ability to see optional keyword arguments I didn't discuss. If you like the quality and look-and-feel of the plots produced with this script and want to use python for additional graphing work, I'd recommend this to get a bit of practice. That being said, if you find yourself making the same sorts of graphs over and over, it's generally a good idea to pack the bulk of the coding into a function with some manipulatable options to set things like colors, font sizes, etc. This will save lots of work and looking through rather repetitive codes. The plots that are produced by the program are shown below (Figs. 12.12, 12.13), barring any implementation or porting errors.

12.3.2   Itation map

Calculate the map

$$z_{n+1} = z_n^2 + c \tag{12.1}$$
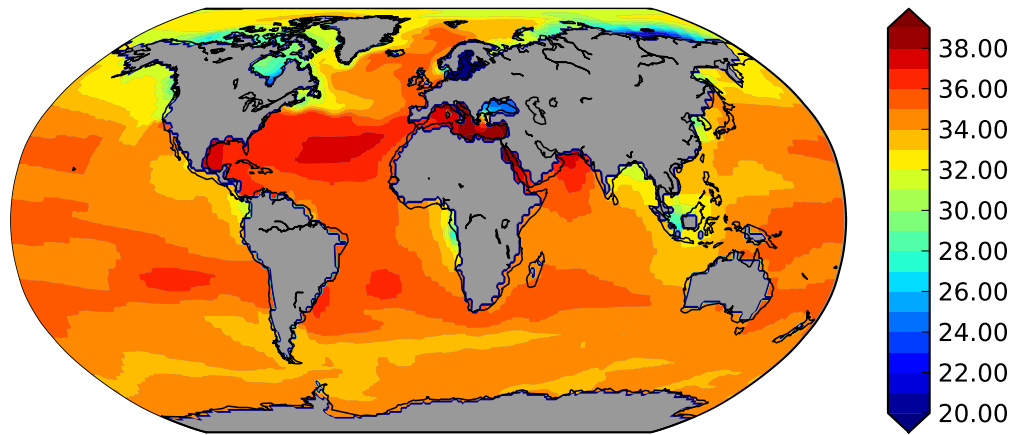
Figure 12.12: Plot of ocean surface salinity using the python script.
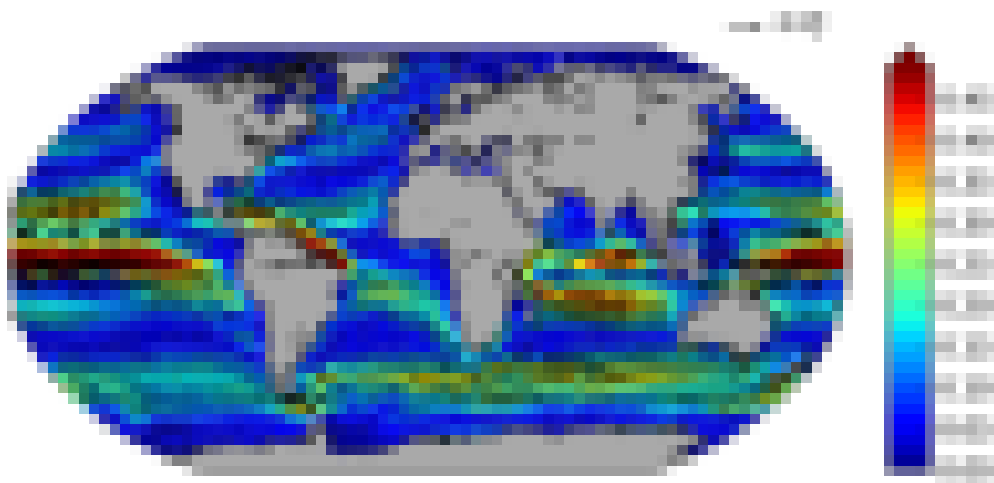
Figure 12.13: As in Fig. 12.12, but for ocean surface horizonal velocity.

in the complex plane with c being a complex number. This set is called Mandelbrot set. Here is the solution for mandel.py

```python
from numpy import array
## Parameters ##

gridSize=500
N=50
zInit=0+0j

## Actions ##

gridHalf=int(gridSize/2)
def mandelbrot(no, yes):
    for i in range(-gridHalf, gridHalf):
        for j in range(-gridHalf, gridHalf):
            c=(i+j*1j)/gridHalf*2
            Z=zInit
            for n in range(0, N):
                Z=Z*Z+c
                if abs(Z)>=2:
                    break
            if abs(Z)>=2:
                no.append((i, j))
            else:
                yes.append((i, j))

no=[]
yes=[]
mandelbrot(no, yes)

from matplotlib.pyplot import plot, axis, show, legend

xYes, yYes=zip(*yes)
xYes=list(array(xYes)/gridHalf)
yYes=list(array(yYes)/gridHalf)
plot(xYes, yYes, 'ob', markersize=0.5, label='The Mandelbrot Set')

legend()
axis([-1, 1, -1, 1])
show()
```

There are a couple of ways of executing the script, depending on how much you want to learn about python. The functions themselves can be given to an interpret "as is" in the following manner:

```
$ python
>>> execfile('mandel.py')
```

The functions will now be available to call, just remember to give the filename argument `fin` as a

string in single quotes.

Alternatively, you could use the ipython interpreter (i for interactive, not to be confused with numerous apple products). This has several advantages, namely that is much more interactive and has a build in help. Any object followed by a ? will yield a quick help, and a double ?? will show the source code of the function. Start this one with:

```
$ ipython -i mandel.py
```

The -i flag allows the session to be interactive after the script is executed. ipython also has a very good traceback system for coding errors.

Finally you could enter the function codes line by line to see how the program evolves, either as a function using `def` or as is. ipython offers the ability to see optional keyword arguments.[2]

12.3.3 The Linear Advection Equation

The linear advection equation is simply:

$$a_t + u a_x = 0 \tag{12.2}$$

where $a(x,t)$ is some scalar quantity and u is the velocity at which it is advected ($u > 0$ advects to the right). The solution to Eq. 12.2 is to simply take the initial data, $a(x, t = 0)$, and displace it to the right at a speed u. The shape of the initial data is preserved in the advection. Many hyperbolic systems of PDEs, e.g. the equations of hydrodynamics, can be written in a form that looks like a system of (nonlinear) advection equations, so the advection equation provides important insight into the methods used for these systems.

Exercise 87 – **Linear advection analytic solution**

---

[2]If you like the quality and look-and-feel of the plots produced with this script and want to use python for additional graphing work, it is recommend to do this to get a bit of practice. That being said, if you find yourself making the same sorts of graphs over and over, it's generally a good idea to pack the bulk of the coding into a function with some manipulatable options to set things like colors, font sizes, etc. This will save lots of work and looking through rather repetitive codes.

Show that $a(x - ut)$ is a solution to Eq. 12.2 for any choice of a. This means that the solution is constant along the lines $x = ut$ (the curves along which the solution is constant are called the characteristics).

First-order advection in 1-d and finite-differences

To get a flavor of the methods for advection, we will use a simple finite-difference discretization—here, the domain is divided into a sequence of points where we store the solution. We will solve Eq. 12.2 numerically by discretizing the solution at these points. The index i denotes the point's location, and a_i denotes the discrete value of $a(x)$ in zone i. The data in each zone can be initialized as $a_i = a(x_i)$. Figure 12.14 shows the grid.

We also need to discretize in time. We denote the time-level of the solution with a superscript, so $a_i^n = a(x_i, t^n)$. For a fixed Δt, time level n corresponds to a time of $t = n\Delta t$.

A simple first-order accurate discretization is:

$$\frac{a_i^{n+1} - a_i^n}{\Delta t} = -u \frac{a_i^n - a_{i-1}^n}{\Delta x} \tag{12.3}$$

This is an *explicit* method, since the new solution, a_i^{n+1}, depends only on information at the old time level, n.

Finally, we also need to specify a boundary condition for this. Our choice of spatial derivative is one-sided—it uses information from the zone to the left of the zone we are updating. This is because information is flowing from left to right in this problem ($u > 0$). This choice of the derivative is called *upwinding*—this choice of derivative results in a stable method. Notice that if we use Eq. 12.3 to update the data in the first zone inside the boundary, we need data to the left of this zone—outside of the domain. This means that we need a single ghost point to implement the boundary conditions for our method. The presence of the ghost points allow us to use the same update equation (e.g. Eq. 12.3) for all zones in the domain.

The last piece of information needed to update the solution is the timestep, Δt. It can be shown that for the solution to be *stable*, the timestep must be less than the time it takes information to
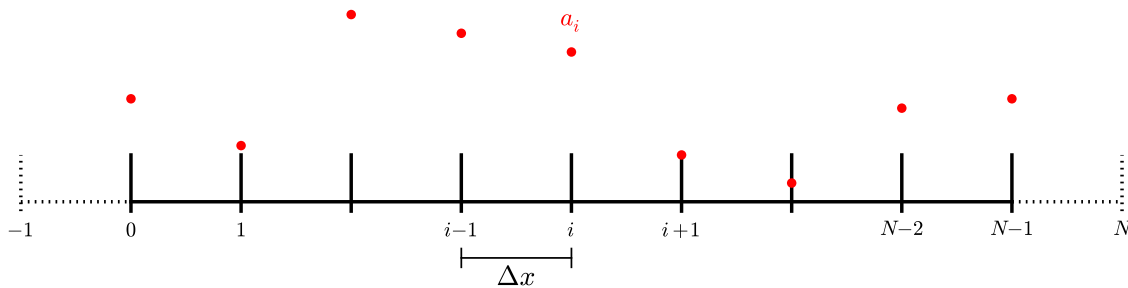
Figure 12.14: A simple finite-difference grid. The solution is stored at each of the labeled points. The dotted lines show the ghost points used to extend our grid past the physical boundaries to accommodate boundary conditions. Note that if we are periodic, then points 0 and $N - 1$ are at the same physical point in space, so we would only need to update one of them.

propagate across a single zone. That is:

$$\Delta t \leq \frac{\Delta x}{u} \ .$$ (12.4)

This is called the *Courant-Friedrichs-Lewy* or *CFL* condition. A dimensionless quantity called the *CFL number* is defined as

$$C = \frac{\Delta t u}{\Delta x}$$ (12.5)

Stability requires $C \leq 1$. We traditionally write the timestep as

$$\Delta t = C \frac{\Delta x}{u}$$ (12.6)

and specify C as part of the problem (a typical value may be $C = 0.7$).

Exercise 88   –   Perfect advection with a Courant number of 1

Show analytically that when you use $C = 1$ in the first-order differenced advection equation (Eq. 12.3) that you advect the profile exactly, without any numerical error.

Keep in mind that, in general, we will be solving a non-linear system of equations, so it is not possible to run with $C = 1$, since u (and therefore C) will change from zone to zone. Instead,

one looks at the most restrictive timestep over all the zones and uses that for the entire system.

Exercise 89   –   A 1-d finite-difference solver for linear advection

Write a code to solve the 1-d linear advection equation using the discretization of Eq. 12.3 on the domain $[0, 1]$ with $u = 1$ and periodic boundary conditions. For initial conditions, try both a Gaussian profile and a top-hat:

$$a = \begin{cases} 0 & if & & x & < 1/3 \\ 1 & if & 1/3 \le & x & < 2/3 \\ 0 & if & 2/3 \le & x \end{cases} \tag{12.7}$$

Note: For a general treatment of boundary conditions, you would initialize the ghost points to their corresponding periodic data and apply the difference equations to zones $0, \ldots, N - 1$. However, for periodic BCs on this grid, points 0 and $N - 1$ are identical, so you could do the update in this special case on points $1, \ldots, N - 1$ without the need for ghost points and then set $a_0 = a_{N-1}$ after the update.

Run you program for one or more periods (one period is $T = 1/u$) with several different CFL numbers and notice that there is substantial numerical dissipation (see Figure 12.15).

Exercise 90   –   FTCS and stability

You may think that using a centered-difference for the spatial derivative, $u_x \sim (u_{i+1} - u_{i-1})/(2\Delta x)$ would be more accurate. This method is called FTCS (forward-time, centered-space). Try this. You will find that the solution is unconditionally unstable.

The classic method for understanding stability is to consider the growth of a single Fourier mode in our discetization. That is, substitute in $a_i^n = A^n e^{ji\theta}$, where $j = \sqrt{-1}$, and θ represents a phase. A method is stable if $|A^{n+1}/A^n| \le 1$. Performing this with FTCS shows that no value of C can make the method stable. Doing the same analysis for Eq. 12.3 would show that the upwind method is stable for $0 \le C \le 1$. (Note that this stability analysis only works for linear
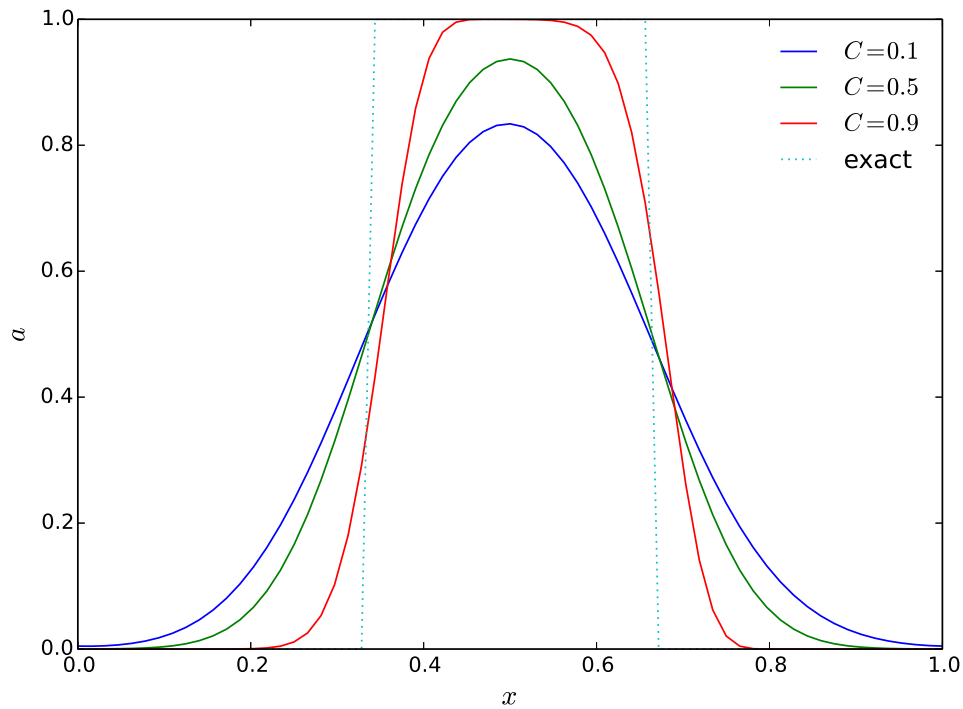
Figure 12.15: Finite-difference solution to the first-order finite-difference upwind method for advection, using 65 points and a variety of CFL numbers.

equations, where the difference Fourier modes are decoupled, nevertheless, we use its ideas for nonlinear advection problems as well).

Exercise 91 – Stability analysis

To get an alternate feel for stability, we can ask what the terms left out by truncation look like. That is, we can begin with the discretized equation:

$$a_i^{n+1} - a_i^n = -\frac{u\Delta t}{\Delta x}(a_i^n - a_{i-1}^n) \tag{12.8}$$

and replace a_i^{n+1} with a Taylor expansion in time, and replace a_{i-1}^n with a Taylor expansion in space, keeping terms to $O(\Delta t^3)$ and $O(\Delta x^3)$. Replacing $\partial a/\partial t$ with $-u\partial a/\partial x$ in the higher-

order terms, show that our difference equation more closely corresponds to

$$a_t + u a_x \; = \; \frac{u \Delta x}{2} \left(1 - \frac{\Delta t u}{\Delta x} \right) \frac{\partial^2 a}{\partial x^2} \tag{12.9}$$

$$= \; \frac{u \Delta x}{2} (1 - C) \frac{\partial^2 a}{\partial x^2} \tag{12.10}$$

Notice that the righthand side looks like a diffusion term, however, if $C > 1$, then the coefficient of the diffusion is negative—this is unphysical. This means that the diffusion would act to take smooth features and make them more strongly peaked—the opposite of physical diffusion.

An alternate approach to time-discretization is to do an implicit discretization. Here our upwind method would appear as:

$$\frac{a_i^{n+1} - a_i^n}{\Delta t} = -u \frac{a_i^{n+1} - a_{i-1}^{n+1}}{\Delta x} \tag{12.11}$$

We can write this as a linear system with coupled equations:

$$- C a_{i-1}^{n+1} + (1 + C) a_i^{n+1} = a_i^n \tag{12.12}$$

In matrix form, solving for the points $1, \ldots, N-1$, this is:

$$
\begin{pmatrix}
1+C & & & & & & -C \\
-C & 1+C & & & & & \\
 & -C & 1+C & & & & \\
 & & -C & 1+C & & & \\
 & & & \ddots & \ddots & & \\
 & & & & -C & 1+C & \\
 & & & & & -C & 1+C
\end{pmatrix}
\begin{pmatrix}
u_1^{n+1} \\
u_2^{n+1} \\
u_3^{n+1} \\
u_4^{n+1} \\
\vdots \\
u_{N-2}^{n+1} \\
u_{N-1}^{n+1}
\end{pmatrix}
=
\begin{pmatrix}
u_1^{n} \\
u_2^{n} \\
u_3^{n} \\
u_4^{n} \\
\vdots \\
u_{N-2}^{n} \\
u_{N-1}^{n}
\end{pmatrix}
$$

$$(12.13)$$

This requires a matrix solve—this makes implicit methods generally more expensive than explicit methods. However, stability analysis would show that this implicit discretization is stable for any choice of C. (But one must not confuse stability with accuracy—the most accurate solutions with this method will still have a small C). Also note that the form of the matrix will change depending on the choice of boundary conditions. Figure 12.16 shows the result of solving this implicit system.
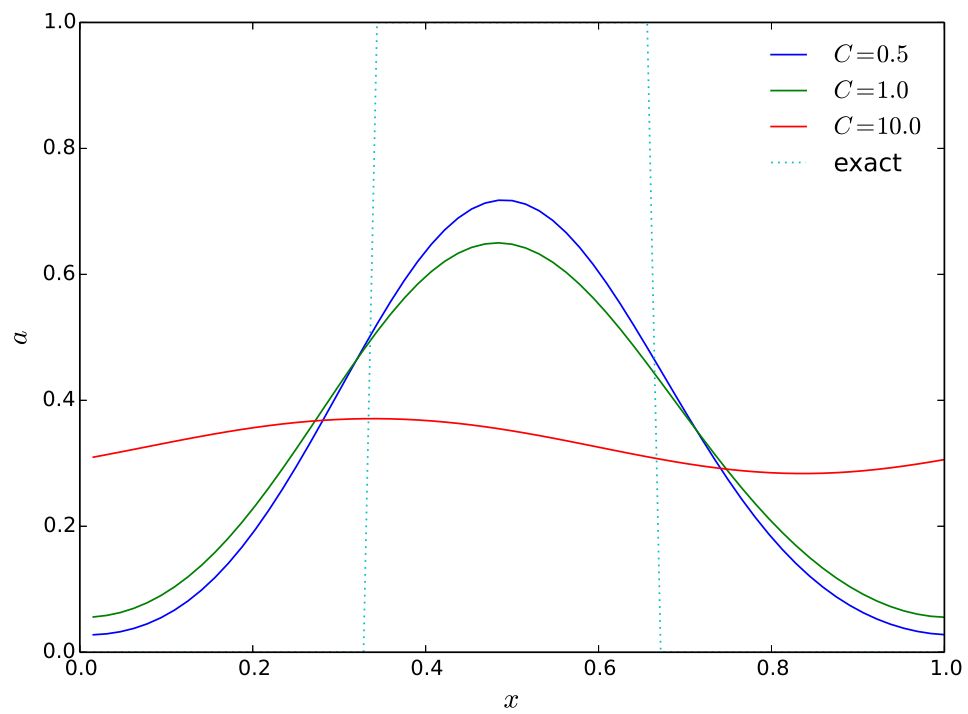
Figure 12.16: Finite-difference solution to the implicit first-order finite-difference upwind method for advection, using 65 points and a variety of CFL numbers.

Chapter 13

Appendix: Questions

13.1 Test for exam

1. **Several questions about the course** (*40 points*, **for each Q** *2 points*).

   Q1: Please clarify: On the Northern Hemisphere, particles tend to go to the right or left relative to the direction of motion due to the Coriolis force?

   Q2: The Coriolis parameter f is defined as
   a) $f = \Omega \cos \varphi$
   b) $f = 2\Omega \cos \varphi$
   c) $f = 2\Omega \sin \varphi$
   d) $f = \beta y$

   Q3: Please write down the equation of state for the ocean and atmosphere! (for the ocean: linearized version)

   Q4: What are the two dominant terms in the horizontal momentum balance for the large-scale dynamics at mid-latitudes?

   Q5: What is the hydrostatic approximation in the momentum equations?

Q6: a) What is the Fourier and Laplace transform of a function $x(t)$?

b) What is the Fourier transform of the $\delta(t)$-function?

Q7: Name three different dimensionless parameters which characterize the flow.

Q8: a) Please write down the Euler forward numerical scheme for $\frac{d}{dt}x = f(x)$!

b) Consider also the special case $f(x) = rx - x^2$!

Q9: Please write down the barotropic potential vorticity equation for large-scale motion!

Q10: Please write down the dispersion relation for Rossby and gravity waves!

Q11: What is the necessary condition for stability in a linear system

$$\frac{d}{dt}x = Ax$$

with real vector x and $n \times n$ matrix A?

... and for the analogous non-linear case $\frac{d}{dt}x = f(x)$?

Q12: a) What is the definition of auto-correlation and auto-covariance?

b) How is the Fourier transformation of the auto-covariance called?

Q13: Explain the stochastic climate model.

Q14: Draw a schematic figure of the Atlantic Ocean meridional overturning!

Q15: Explain the Taylor-Proudman Theorem! (remember $f = f_0$, barotropic circulation)

Q16: What is the dispersion relation for Kelvin waves? Make a sketch of the coastally trapped Kelvin wave on the Northern Hemisphere ocean basin.

Q17: Explain the difference between dispersive and non-dispersive waves! You could use the $\omega(k)$ formula for Rossby and Kelvin waves.

Q18: Please state: The dimensionless Reynolds number is $Re = U/(L\nu)$ or $Re = UL/\nu$ or $Re = U^2 L/\nu$? ν denotes the kinematic viscosity, L a length-scale L determined by the geometry of the flow, and U a characteristic velocity.

Q19: Describe in words the Rayleigh-Bénard instability. The basic state possesses a steady-state solution in which there is no motion, and the temperature varies linearly with depth:

$$u = w = 0 \tag{13.1}$$

$$T_{eq} = T_0 + \left(1 - \frac{z}{H}\right)\Delta T \tag{13.2}$$

When this solution becomes unstable, ... (please continue)
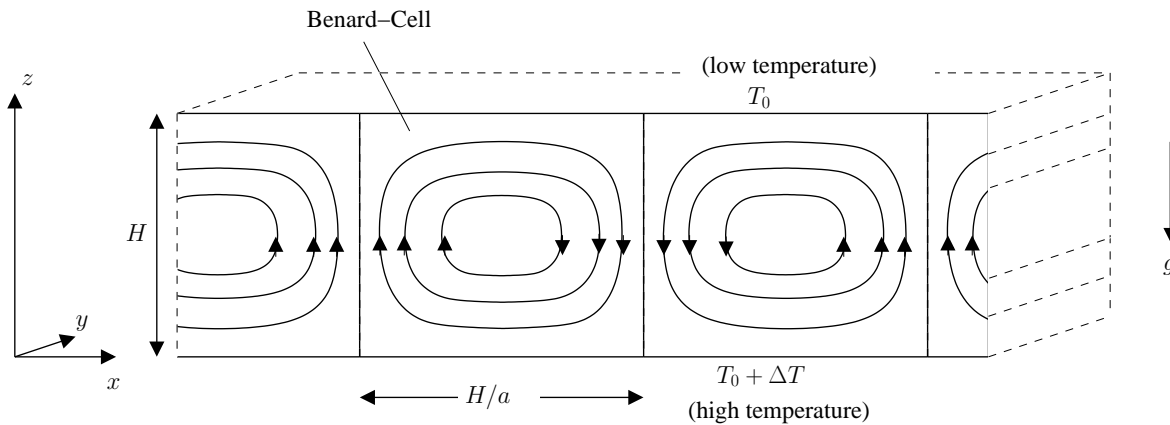
Figure 13.1: Geometry of the Rayleigh-Bénard system.

Q20: Make a sketch of the Foucault pendulum and explain the horizontal dynamics

$$\ddot{x} = 2\Omega \sin\varphi\, \dot{y} - \frac{g}{L}x \tag{13.3}$$

$$\ddot{y} = -2\Omega \sin\varphi\, \dot{x} - \frac{g}{L}y \tag{13.4}$$

2. **Earth's curvature** *(6 points)*

   I. The highest building on the campus of the University of Bremen is the so-called drop tower with a height of h=110 metres (Fig. 13.2 upper panel). How far one can look onto the horizon under good weather conditions?

   Hint: Denote this distance by d. Remember the Earth's radius $R = 6378km$ and apply Pythagoras.

   II. Show that the rule-of-thumb $d = \sqrt{2hR}$ is a good approximation? (For h=10m this means d=11 km.) Notice: When h is in m, d in km, the formula can be written as $d = 3.5\sqrt{\frac{h}{m}}$ km.

   III. The town Bremerhaven which is at the North Sea is about 60 km north of Bremen. How big must a tower in Bremen be in order to see the coast in Bremerhaven? (Fig. 13.2 lower panel).

Figure 13.2: Upper panel: Drop tower in Bremen. Lower panel: Harbor in Bremerhaven, ca. 60 km north of Bremen.

	Quantity	Atmosphere	Ocean
horizontal velocity	U	$10\ ms^{-1}$	$10^{-1}\,ms^{-1}$
horizontal length	L	$10^6\,m$	$10^6\,m$
horizonal Pressure changes	δP (horizontal)	$10^3\,Pa$	$10^4\,Pa$
time scale	T	$10^5\,s$	$10^7\,s$
Coriolis parameter at 45°N	$f_0 = 2\Omega \sin \varphi_0$	$10^{-4}\,s^{-1}$	$10^{-4}\,s^{-1}$
density	ρ	$1\ kgm^{-3}$	$10^3\,kgm^{-3}$
viscosity (turbulent)	ν	$10^{-5}\,kgm^{-3}$	$10^{-6}\,kgm^{-3}$

Table 13.1: Table shows the typical scales in the atmosphere and ocean system.

3. **Scaling of the dynamical equations** *(6 points)*

We work in the rotating frame of reference of the Earth. The equation can be scaled by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. We can estimate the relative contributions in units of m/s^2 in the horizontal momentum equations:

$$\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{U/T\sim10^{-8}} + \underbrace{\mathbf{v}\cdot\nabla\mathbf{v}}_{U^2/L\sim10^{-8}} = \underbrace{-\frac{1}{\rho}\nabla p}_{\delta P/(\rho L)\sim10^{-5}} + \underbrace{2\Omega\times\mathbf{v}}_{f_0 U\sim10^{-5}} + \underbrace{fric}_{\nu U/H^2\sim10^{-13}} \qquad (13.5)$$

where fric denotes the contributions of friction due to eddy stress divergence (usually $\sim \nu\nabla^2\mathbf{v}$). Typical values are given in Table 13.1. The values have been taken for the ocean.

a) Please repeat the estimate for the atmosphere using Table 13.1.

b) The Rossby number Ro is the ratio of inertial (the left hand side) to Coriolis (second term on the right hand side) in (13.5): terms

$$Ro = \frac{(U^2/L)}{(fU)} = \frac{U}{fL} \quad . \qquad (13.6)$$

Ro is small when the flow is in a so-called geostrophic balance. Please calculate Ro for the atmosphere and ocean using Table 13.1.

4. **Concept of dynamic similarity**   *(8 points)*

For the case of an incompressible flow, assuming the temperature effects are negligible and external forces are neglected, the Navier-Stokes equations consist of conservation of mass

$$\nabla \cdot \mathbf{u} = 0 \tag{13.7}$$

and conservation of momentum

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu \nabla^2 \mathbf{u} \tag{13.8}$$

where $\mathbf{u}$ is the velocity vector and p is the pressure, ν denotes the kinematic viscosity.

a) Show: The equations (13.7,13.8) can be made dimensionless by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. For example: $u = U \cdot u_d$.

Note: the units of $[\rho_0] = kg/m^3$, $[p] = kg/(ms^2)$, and $[p]/[\rho_0] = m^2/s^2$. Therefore the pressure gradient term in (4.21) has the scaling U^2/L.

b) Show: The scalings vanish completely in front of the terms except for the $\nabla^2 u_d$-term! The dimensionless parameter is the Reynolds number and the only parameter left!

Remark: For large Reynolds numbers, the flow is turbulent. In most practical flows Re is rather large $(10^4 - 10^8)$, large enough for the flow to be turbulent.

5. **Elimination of the pressure term** *(10 points)*

   Assume a 2D flow without non-linear terms, where the equations reduce to:

   $$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \tag{13.9}$$

   $$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad . \tag{13.10}$$

   a) Show: Taking the curl of (13.9,13.10) results in the elimination of pressure.

   b) Show: Defining the stream function ψ through

   $$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \tag{13.11}$$

   (mass continuity being unconditionally satisfied), the incompressible Newtonian 2D momentum and mass conservation degrade into one equation:

   $$\partial_t \left(\nabla^2 \psi \right) = \frac{\mu}{\rho} \, \nabla^4 \psi \tag{13.12}$$

   c) We now consider the rotating framwork and add the Coriolis terms $-\rho f v$ and $\rho f u$ to the left hand side of (13.9,13.10). Subtract $\partial/\partial y$ of (13.9) from $\partial/\partial x$ of (13.10) to eliminate the pressure terms to derive the vorticity equation! Show that (13.12) changed into

   $$\partial_t \left(\nabla^2 \psi \right) + D_t f = \frac{\mu}{\rho} \, \nabla^4 \psi \tag{13.13}$$

   or

   $$D_t \left(\nabla^2 \psi + f \right) = \frac{\mu}{\rho} \, \nabla^4 \psi \tag{13.14}$$

   when the non-linear terms are neglected which was the assumption in (13.9,13.10).
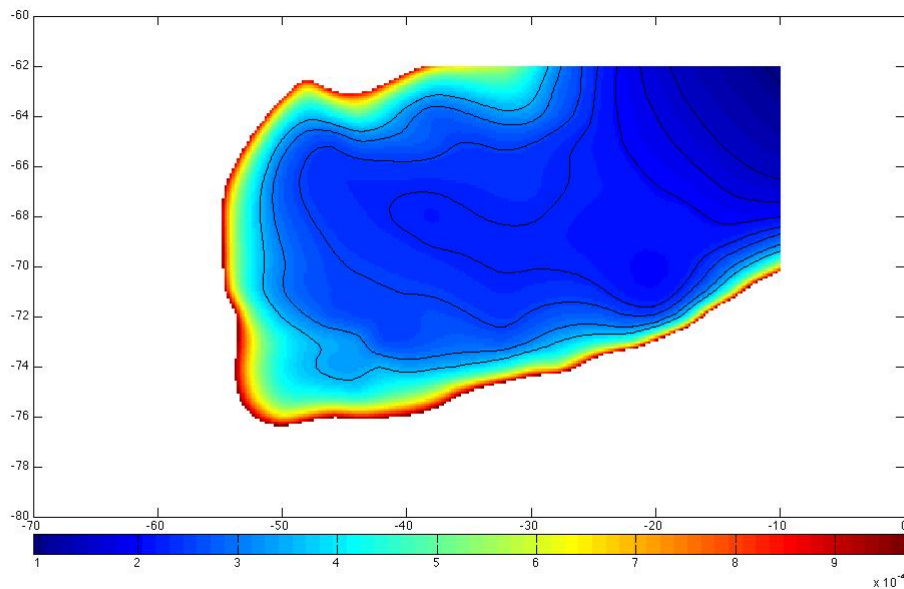
Figure 13.3: f/h countours in the Weddell Sea for $34 \cdot 10^6$ years before present.

6. **f/h contours** *(6 points)*

For the Oligocene (about 34 Million years ago), the topography was reconstructed. The f/h-contours are shown in Fig. 13.3. The length of the day was nearly as today.

a) Draw the barotropic streamlines in this figure?

b) How would the circulation look like if the Earth would rotate faster?

c) What will happen if we include a clockwise wind circulation on top of the Weddell Sea? Please specify: Horizontal variations in this wind stress will cause vertical Ekman pumping which presses down *or* Ekman suction which pulls up the water below the Ekman layer? Use equation (13.17) !
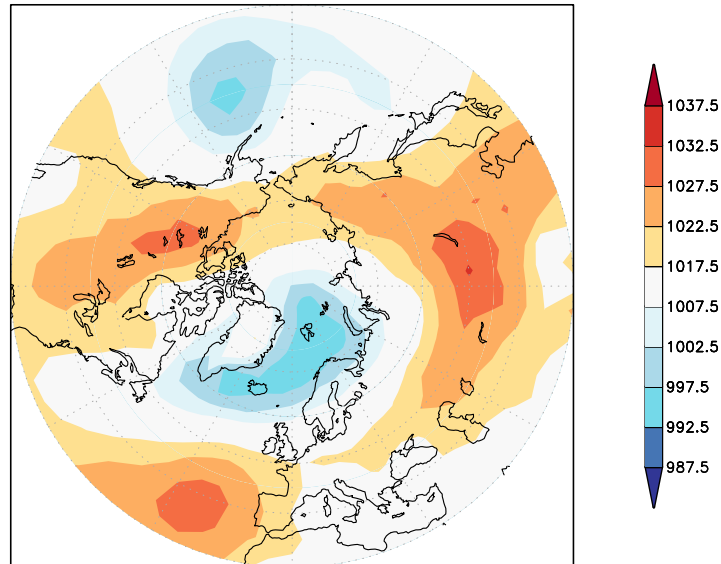
Figure 13.4: Sea level pressure (hPa) field for February 2015. In February, the circulation is characterized by a low pressure over the Greenland-Iceland-Norwegian Sea, and a surrounded high pressure. Data are from Trenberth and Paolino (1980).

7. **Geostrophy** *(4 points)*

    a) Write down the geostrophic balance.

    b) Draw the direction of large-scale motions in the atmosphere in Fig. 13.4 using the geostrophic balance.

8. **Wind-driven ocean circulation** *(10 points)*

the Sverdrup transport V for the depth-integrated flow is calculated by

$$\rho_0 \beta V = \frac{\partial}{\partial x}\tau_y - \frac{\partial}{\partial y}\tau_x \qquad (13.15)$$

where τ_x and τ_y are the components of the wind stress.

The Ekman transports V_E, U_E describe the dynamics in the upper mixed layer:

$$f V_E = -\tau_x/\rho_0 \qquad , \qquad f U_E = \tau_y/\rho_0 \qquad (13.16)$$

where $U_E = \int_{-E}^{0} u\,dz$ and $V_E = \int_{-E}^{0} v\,dz$ are the depth-integrated velocities in the thin friction-dominated Ekman layer at the sea surface. Denote w_E as the Ekman vertical velocity at the bottom of the Ekman layer. Using the continuity equation, the divergence of the Ekman transports leads to a vertical velocity w_E at the bottom of the Ekman layer:

$$-\int_{-E}^{0} \frac{\partial w}{\partial z}\,dz = w_E = \frac{\partial}{\partial x}U_E + \frac{\partial}{\partial y}V_E = \frac{\partial}{\partial x}\left(\frac{\tau_y}{\rho_0\, f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_x}{\rho_0\, f}\right) \quad (13.17)$$

a) Assume that the windstress is only zonal with

$$\tau_x = -\tau_0 \cos(\pi y/B) \qquad (13.18)$$

for an ocean basin $0 < x < L, \quad 0 < y < B$. Calculate the Sverdrup transport, Ekman transports, and Ekman pumping velocity for this special case. Make a schematic diagram of the windstress, Sverdrup transport, Ekman transports, and Ekman pumping velocity.

b) Using a), at what latitudes y are $|V|$ and $|V_E|$ maximum? Calculate their magnitudes. Take constant $f = 10^{-4}\,\text{s}^{-1}$ and $\beta = 1.8\cdot 10^{-11}\,\text{m}^{-1}\text{s}^{-1}$ and $B = 5000\,\text{km}, \tau_0/\rho_0 = 10^{-4}\,\text{m}^2\text{s}^{-2}$.

c) Using the values in b), calculate the maximum of w_E for constant f.

9. **Rossby, gravity, and Kelvin waves** *(10 points)*

Start with the shallow water equations

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{13.19}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{13.20}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{13.21}$$

with H=const. as mean depth and η as surface anomaly.

a) with the elimination of the fast gravity waves in equation (13.90)

$$\frac{\partial \eta}{\partial t} = 0$$

derive the dispersion relation for divergence-free Rossby waves! Ansatz: Introduce a stream-function for u,v: $\Psi \sim \exp(ikx + ily - i\omega t)$

b) with the assumption of $f = f_0 = 0$ derive the dispersion relation for gravity waves! The restoring force is related to gravity. Ansatz: take one of the equations (13.88,13.89,13.90) and derive the solution.

c) Kelvin wave. Assume a vertical wall at x=0 along the y-axis (an idealized coast) and u=0. Derive the solution for $v(x, y, t)$ and $\eta(x, y, t)$ using the equations (13.89,13.90)! Specify the x-dependence of the solutions using (13.88) and discuss the trapping distance from the coast!

Hint: For the x-dependence of η and v use the ansatz

$$\eta(x, y, t) = \tilde{\eta}(x) \exp(ily - i\omega t) \tag{13.22}$$

$$v(x, y, t) = \tilde{v}(x) \exp(ily - i\omega t) \ . \tag{13.23}$$

10. **Rossby wave formula (long waves in the westerlies)** *(10 points)*

    Consider the vorticity equation

    $$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0 \qquad\qquad (13.24)$$

    a) Assume a mean flow with constant zonal velocity $u = U = const > 0$ and a varying north-south component $v = v(x,t)$ which gives the total motion a wave-like form. Furthermore, h =const. Write down the vorticity equation for this specific flow!

    b) Use a) and the ansatz

    $$v(x,t) = A\cos[(kx - \omega t)] \qquad\qquad (13.25)$$

    to determine the disperion relation $\omega(k)$, group velocity $\frac{\partial \omega}{\partial k}$, and the phase velocity $c = \omega/k$.

    c) Derive the wavelength $L = 2\pi/k$ of the stationary wave given by $c = 0$.

11. **Potential vorticity:** *(5 points)*

    An air column at $53°$N with $\zeta = 0$ initially streches from the surface to a fixed tropopause at 10 km height. If the air column moves until it is over a mountain barrier of 2 km height at $30°$N, what is its absolute vorticity and relative vorticity as it passes the mountain top?

    Assume: $\sin 53° = 0.8$; $\sin 30° = 0.5$

    The angular velocity of the Earth $\Omega = 2\pi/(1 day)$.

12. **Angular momentum and Hadley cell** *(10 points)*

Consider a zonally symmetric circulation (i.e., one with no longitudinal variations) in the atmosphere. In the inviscid upper troposphere one expects such a flow to conserve absolute angular momentum, i.e.,

$$\frac{DA}{Dt} = 0, \tag{13.26}$$

where A is the absolute angular momentum per unit mass (parallel to the Earth's rotation axis)

$$A = r\left(u + \Omega r\right) = \Omega R^2 \cos^2 \varphi + uR \cos \varphi \quad . \tag{13.27}$$

Ω is the Earth rotation rate, u the eastward wind component, $r = R \cos \varphi$ is the distance from the rotation axis, R the Earth's radius, and φ latitude.

a) Show, for inviscid zonally symmetric flow, that the relation $\frac{DA}{Dt} = 0$ is consistent with the zonal component of the equation of motion

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{13.28}$$

in (x, y, z) coordinates, where $y = R\varphi$.

b) Use angular momentum conservation to describe in words how the existence of the Hadley circulation explains the existence of both the subtropical jet in the upper troposphere and the near-surface trade winds.

c) If the Hadley circulation is symmetric about the equator, and its edge is at $20°$ latitude, determine the strength of the subtropical jet. Use (13.26, 13.117).

d) Is the Hadley Cell geostrophically driven or not?

13. **Analytical EBM** *(5 points)*

The temperature is described as $T(y)$ and the heat transport (sensible, latent and ocean) is modelled as diffusion:

$$C_p\, \partial_t T \; + \; k\, \partial_y^2 T \;=\; (1-\alpha)Q_S^{top} - (A + B\, T) \tag{13.29}$$

a) Show the solution if the planetary albedo α is chosen as a constant parameter. Hint: Use the ansatz with a global component and a latitude component

$$T(y,t) \;=\; T_0(t) + T_1(t)\cdot\cos(ay) \tag{13.30}$$

$$Q_S^{top} \;=\; Q_0 + Q_1\cdot\cos(ay) \tag{13.31}$$

with $y = R\varphi$ and $a = 2/R$.

b) Determine the solution with the assumption of planetary albedo α deprending on the global temperature:

$$\alpha(T) = \alpha_0 + \alpha_1\cdot T_0 \tag{13.32}$$

Hint: Separate the dynamics for T_0 and T_1 and use

$$\int_{-90°}^{90°} \cos(2\varphi)\, d\varphi = 0 \tag{13.33}$$

14. **Bifurcation:** *(6 points)*

    Consider the differential equation

    $$\frac{d}{dt}x \;=\; rx\,(1-x)^2 \tag{13.34}$$

    a) Calculate the bifurcation with respect to the parameter r and draw the bifurcation diagram!

    b) Discuss the stability in terms of the potential V(x) ! (remember that the potential can be calculated from the right hand side of equation (13.34): -V' = rhs of (13.34))

    c) Discuss the stability in terms of the graphical method!

    d) Which types of bifurcations do you know?

15. **Graphical method for bifurcations** *(5 points)*

We introduce a graphical method to obtain stability or instability. Consider the '''saddle-node bifurcation''', one of the equilibrium points is unstable (the saddle), while the other is stable (the node). In Fig. 13.5, we can plot $\frac{dx}{dt} = f(x)$ dependent on x (left panel) for

$$\frac{dx}{dt} = b + x^2 \tag{13.35}$$

with $b < 0$ in this particular case (For $b > 0$ we would have no equilibrium, and we have no point x_e with $f(x_e) = 0$.). We just consister the slope $f'(x_e)$ and see that the filled circles with positive slope are unstable, the open circles with negative slopes are stable (right panel in Fig. 13.5).

   I. Draw the bifurcations as in Fig. 13.5 for the pitchfork bifurcation.
  II. Draw the bifurcations as in Fig. 13.5 for the transcritical bifurcation.
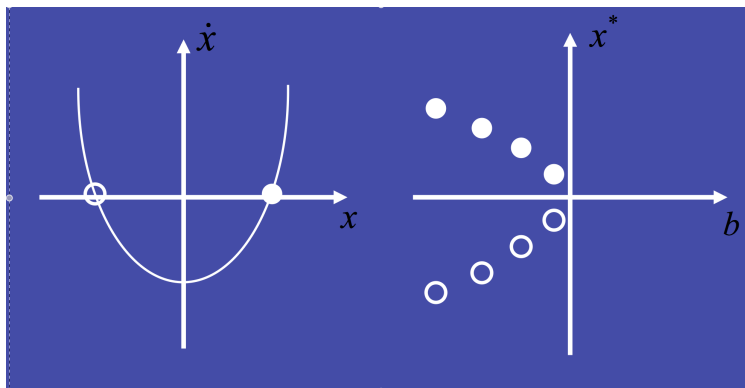
Figure 13.5: Saddle-node bifurcation diagram using the graphical method.

16. **Stochastic climate model** *(8 points)*

Imagine that the temperature of the ocean mixed layer of depth h is governed by

$$\frac{dT}{dt} = -\lambda T + \frac{Q_{net}}{\gamma_O} ,\qquad(13.36)$$

where coefficient γ_O is given by the heat capacity $c_p \rho h$, and λ is the typical damping rate of a temperature anomaly. The air-sea fluxes due to weather systems are represented by a white-noise process $Q_{net} = \hat{Q}_\omega e^{i\omega t}$ where $\hat{Q}_\omega$ is the amplitude of the random forcing at frequency ω. $\hat{Q}^*$ is the complex conjugate.

a) Solve Eq. 13.104 for the temperature response $T = \hat{T}_\omega e^{i\omega t}$ and hence show that:

$$\hat{T}_\omega = \frac{\hat{Q}_\omega}{\gamma_O \left(\lambda + i\omega\right)}\qquad(13.37)$$

b) Show that it has a spectral density $\hat{T}_\omega \hat{T}_\omega^*$ is given by:

$$\hat{T}\,\hat{T}^* = \frac{\hat{Q}\,\hat{Q}^*}{\gamma_O^2 \left(\lambda^2 + \omega^2\right)}\qquad(13.38)$$

and the spectrum

$$S(\omega) =< \hat{T}\,\hat{T}^* >= \frac{1}{\gamma_O^2 \left(\lambda^2 + \omega^2\right)}.\qquad(13.39)$$

The brackets $< \cdots >$ denote the ensemble mean. Make a sketch of the spectrum using a log-log plot and show that fluctuations with a frequency greater than λ are damped.

17. **Laplace transform** *(12 points)*

$$\text{is given by} \qquad \mathcal{L}\left\{x(t)\right\} = L(s) = \int_0^\infty e^{-st} x(t) dt \qquad (13.40)$$

a) Show that integration by parts leads to

$$\mathcal{L}\left\{\frac{d}{dt} x(t)\right\} = sL(s) - x(0) \qquad (13.41)$$

b) Show that

$$\mathcal{L}\left\{\exp(-at)\right\} = \frac{1}{s+a} \qquad (13.42)$$

c) Laplace transformation of mixed layer model: Imagine that the temperature of the ocean mixed layer is governed by

$$\frac{dT}{dt} = -\lambda T + Q(t), \qquad (13.43)$$

where λ is the typical damping rate of a temperature anomaly and $Q(t)$ a forcing.

I. Use the Laplace transformation to show

$$L(s) = \frac{Q(s) + T(0)}{s + \lambda} \quad . \qquad (13.44)$$

where $Q(s) = \mathcal{L}\left\{Q(t)\right\}$

II. Consider the special case $Q(t) = \exp(i\omega_0 t)$, then $Q(s) = \frac{1}{s - i\omega_0}$. The forcing and the temperature is of course a real number, but by representing it as a complex number we can simultaneously keep track of both phase components. Show that

$$L(s) = \frac{T(0) + Q(s)}{s + \lambda} = \frac{T(0)}{s + \lambda} + \frac{1}{(s + \lambda)} \frac{1}{(s - i\omega_0)} \qquad (13.45)$$

and via the Laplace back-transformation and (13.42, 13.47) that

$$T(t) = \exp(-\lambda t)T(0) + \frac{[\exp(i\omega_0 t) - \exp(-\lambda t)]}{\lambda + i\omega_0} \quad . \tag{13.46}$$

III. Calculate the real and imaginary part of (13.46).

IV. Take the real part. Show: At low frequencies, the output $T(t)$ is similar to the forcing $Q(t)$. At high frequencies it rolls off as $1/\omega$ (it is a low-pass filter) and is out of phase by $90°$.

Hint:

$$\mathcal{L}\{-\exp(-at) + \exp(-bt)\} \;=\; \frac{-1}{s+a} + \frac{1}{s+b} = \frac{a-b}{(s+a)(s+b)} \tag{13.47}$$

18. **Climate sensitivity and variability in the Stochastic Climate Model** *(8 points)*

Imagine that the temperature of the ocean mixed layer of depth h is governed by

$$\frac{dT}{dt} = -\lambda T + Q_{net} + f(t), \tag{13.48}$$

where the air-sea fluxes due to weather systems are represented by a white-noise process with zero average $< Q_{net} >= 0$ and δ-correlated in time $< Q_{net}(t)Q_{net}(t+\tau) >= \delta(\tau)$. The function $f(t)$ is a time dependent deterministic forcing. Assume furthermore that $f(t) = c \cdot u(t)$ with $u(t)$ as unit step or the so-called Heaviside step function.

a) Show via Laplace transform that

$$< T(t) > = \mathcal{L}^{-1}\{L(s)\}(t) = \mathcal{L}^{-1}\left\{ \frac{< T(0) >}{s+\lambda} + \frac{c}{s} \cdot \frac{1}{s+\lambda} \right\} \tag{13.49}$$

$$= T(0) \cdot \exp(-\lambda t) + \frac{c}{\lambda}(1 - \exp(-\lambda t)) \tag{13.50}$$

b) Show that the equilibrium response is

$$\Delta T = \lim_{t\to\infty} < T(t) >= \frac{c}{\lambda}. \tag{13.51}$$

c) Calculate the spectrum of (13.48) for $f(t) = 0$! What is the relationship of the dissipation (through λ) and the fluctuations (through the spectrum $S(\omega)$) ?

19. **Ocean thermohaline circulation:** *(10 points)*

In the lecture, we introduced an interhemispheric box model of the deep ocean circulation to study the feedbacks in the climate system. The box model consists of four oceanic and three atmospheric boxes, as indicated in Fig. 13.6. The ocean boxes represent the Atlantic Ocean from $80°N$ to $60°S$. The indices of the temperatures T, the salinities S, the surface heat fluxes H, the atmospheric heat fluxes F, the radiation terms R as well as later on the volumes bear on the different boxes (N for the northern, M for the tropical, D for the deep and S for the southern box).
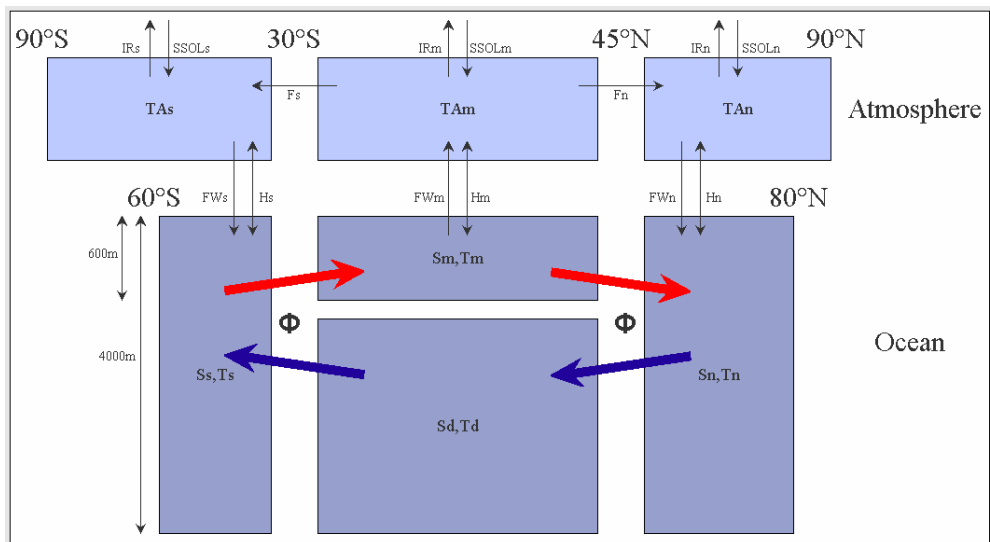
Figure 13.6: Schematic illustration of the Climate-Box-Model

The prognostic equation for the temperatures of the northern ocean box reads

$$\frac{d}{dt}T_N = - (T_N - T_M)\, \frac{\Phi}{V_N} + \frac{H_N}{\rho_0 c_p dz_2},$$

(13.52)

where ρ_0 denotes a reference density for saltwater and c_p the specific heat capacity of water. The factors dz and V_N indicate the depths and volumes of the discrete ocean box. The overturning flow Φ is assumed to be proportional to the density gradients of the the northern

and the southern box:

$$\Phi = c\left[-\alpha\left(T_N - T_S\right) + \beta\left(S_N - S_S\right)\right] \tag{13.53}$$

a) Explain the terms in equation (13.52).

b) Write down the associated equations for salinities! (only for S_N).

c) Explain in words what will happen if the high-latitude salinity S_N is reduced (e.g., due to melting of Greenland).

20. **Short programming questions.** *(6 points)*

    Write down the output for the following R-commands:

    a) `0:10`

    b) `a<-c(0,5,3,4); mean(a)`

    c) `max(a)-min(a)`

    d) `paste("The mean value of a is",mean(a),"for sure",sep="_")`

    e) `a*2+c(1,1,1,0)`

    f) `my.fun<-function(n){return(n*n+1)}`
       `my.fun(10)-my.fun(1)`

21. **Interpretation of SST correlation maps and modes of variability** *(5 points)*

    The two maps show correlations with local climate indices (Fig. 13.8).

    a) Which climate modes correspond to the sea surface temperature correlation-maps?

    b) Describe and draw schematically the associated sea level pressure patterns!

    c) What are the dominant time scales and dominant seasons for these two modes in the climate system?
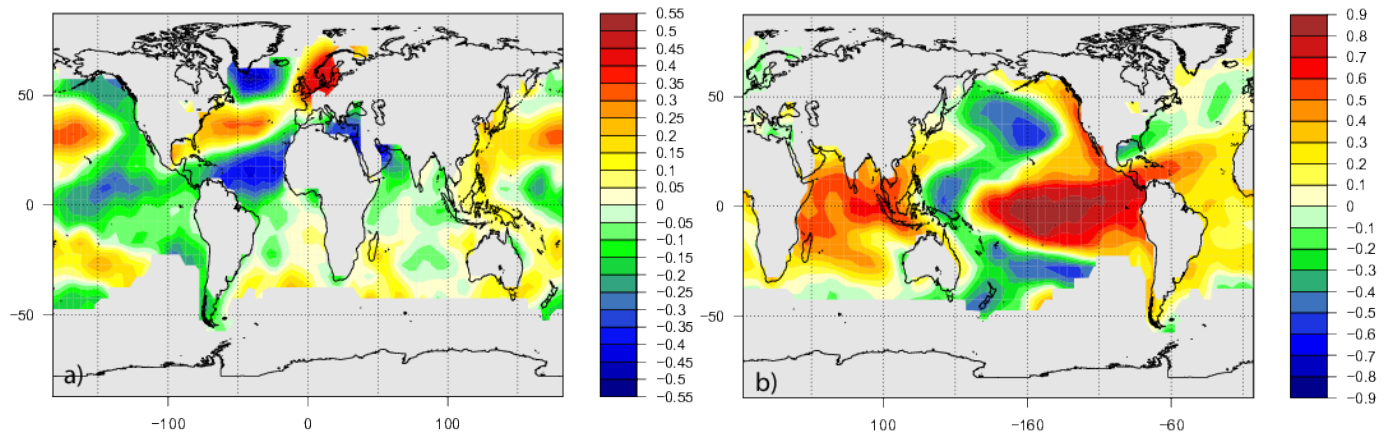
Figure 13.7: Correlation map of climate indices with global sea surface temperature.

22. **cdo** *(12 points)*

In the following tasks we will process NetCDF files using the CDO. For convenience, some useful CDO commands are summarized in the following listing.

```
1  #extract a variable named "varname" from file input.nc
2  cdo selvar,varname input.nc output.nc
3
4  #extract the first month of all years in file input.nc
5  cdo selmon,1 input.nc output.nc
6
7  #calculate a time average over a time series input.nc
8  cdo timmean input.nc output.nc
9
10 #generate a seasonal mean from input.nc
11 cdo seasmean input.nc output.nc
12
13 #generate a year mean from input.nc
14 cdo yearmean input.nc output.nc
15
16 #calculate an average annual cycle from file input.nc
17 cdo ymonmean input.nc output.nc
18
19 #select a region from input.nc, from longitude "a" to "b", from latitude "c" to "d"
20 cdo sellonlatbox,a,b,c,d input.nc output.nc
21
22 #calculate a spatial average of field input.nc
23 cdo fldmean input.nc output.nc
24
25 #write the output of a CDO operator "a" to the screen (omits file output.nc)
26 cdo output -a input.nc
27
28 #calculate the difference between two NetCDF files input1.nc and input2.nc
29 cdo sub input1.nc input2.nc output.nc
30
31 #multiply two fields input1.nc and input2.nc
32 cdo mul input1.nc input2.nc output.nc
33
34 #add a constant "a" to field input.nc
35 cdo addc,a input.nc output.nc
36
37 #select only regions of input2.nc, for which mask input1.nc is true (i.e. 1)
38 #represents an if-then programming construct
39 cdo ifthen input1.nc input2.nc output.nc
40
41 #use input2.nc, where mask input1.nc is true - otherwise use input3.nc
42 #represents an if-then-else programming construct
43 cdo ifthenelse input1.nc input2.nc input3.nc output.nc
44
45 #reduce a data range (a,b) in input.nc to the constant value "c"
46 cdo setrtoc,a,b,c input.nc output.nc
47
48 #replace a data range (a,b) in input.nc by the missing value ("NaN")
49 cdo setrtomiss,a,b input.nc output.nc
50
51 #calculate the trend of a time series in input.nc;
52 #the trend is defined by offset "a" and slope "b" of the regression line;
53 #"a" is stored in a.nc, "b" is stored in b.nc
54 cdo trend input.nc a.nc b.nc
55
56 #calculate the horizontal area covered by each grid cell of input.nc
57 cdo gridarea input.nc output.nc
```

The file INIOM_PD_3901-4000_tsurf_seaice.nc contains two time series of climatological fields. Reduce the NetCDF file by performing the following tasks using the CDO. Please write down the necessary commands.

- Task 1: Split the data set in two separate data sets, one for variable tsurf, one for variable seaice.

- Task 2: Calculate a time average over the full time period available in each of the separate data sets created in Task 1.

- Task 3: For each of the data sets created in Task 1 calculate an average annual cycle that is representative for the full time period of the time series.

  Hint: In order to create an average annual cycle (multi-year monthly mean), you have to generate a new data set that contains twelve months. The data stored in each time step (month) n of this new data set o must represent the average over all corresponding months contained in the full time series of the initial data set i, i.e. (see CDO documentation):

$$o\left(n, x\right) = \text{mean}\left(i\left(t, x\right), \text{month}(i(t)) == n\right); \qquad n \in (1, 12)$$

- Task 4: Calculate a seasonal mean from the average annual cycle of both variables retrieved in Task 3.

- Task 5: Select only the Northern Hemisphere of the time average retrieved in Task 2.

- Task 6: Calculate the global average temperature from the data set retrieved in Task 2.

13.2   Other questions

1. Ocean thermohaline circulation: Consider a geostrophic flow (u, v)

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{13.54}$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{13.55}$$

with pressure $p(x, y, z, t)$.

a) Use the hydrostatic approximation $\frac{\partial p}{\partial z}$ and equation (13.110) in order to derive the meridional overturning stream function $\Phi(y, z)$. Φ is defined via

$$\Phi(y, z) = \int_0^z \frac{\partial \Phi}{\partial \tilde{z}} \, d\tilde{z} \qquad (13.56)$$

$$\frac{\partial \Phi}{\partial \tilde{z}} = \int_{x_e}^{x_w} v(x, y, \tilde{z}) \, dx \quad \text{(zonally integrated transport)}, \qquad (13.57)$$

where x_e and x_w are the eastward and westward boundaries in the ocean basin (think e.g. of the Atlantic Ocean). Units of Φ are $m^3 s^{-1}$. At the surface $\Phi(y, 0) = 0$.

b) Another definition of $\Phi(y, z)$ is via

$$-\frac{\partial \Phi}{\partial y} = \int_{x_e}^{x_w} w \, dx \qquad (13.58)$$

with vertical velocity w. Why is the definition of a streamfunction useful? What is the physical law and equation behind that?

c) Consider now a water planet with flat bottom (unlike the Earth). Provide the meridional overturning stream function $\Phi(y, z)$ in this ocean! Is there a meridional transport due to geostrophy in the atmosphere?

2. Ekman layer: Consider a geostrophic flow $(\bar{u}, \bar{v})$

$$-f\bar{v} = -\frac{1}{\rho_0} \frac{\partial \bar{p}}{\partial x} \qquad (13.59)$$

$$f\bar{u} = -\frac{1}{\rho_0} \frac{\partial \bar{p}}{\partial y} \qquad (13.60)$$

with pressure $\bar{p}(x, y, t)$. The boundary-layer equations are then

$$-f(v - \bar{v}) = \nu \frac{\partial^2 u}{\partial z^2} \tag{13.61}$$

$$f(u - \bar{u}) = \nu \frac{\partial^2 v}{\partial z^2} \quad . \tag{13.62}$$

The boundary conditions are specified to be at the surface

$$\rho_0 \nu \frac{\partial u}{\partial z} = \tau^x \tag{13.63}$$

$$\rho_0 \nu \frac{\partial v}{\partial z} = \tau^y \tag{13.64}$$

and for $z \to -\infty$: $u = \bar{u}, \quad v = \bar{v}$.

a) Calculate the flow (u,v) as the departure from the interior flow $(\bar{u}, \bar{v})$!

b) Calculate the net wind-driven horizontal transport through integration

$$V = \int_{-\infty}^{0} dz(v - \bar{v}) \tag{13.65}$$

$$U = \int_{-\infty}^{0} dz(u - \bar{u}) \quad . \tag{13.66}$$

What is the direction of U and V in terms of the surface wind stress τ?

c) For the case $f = f_0$ of constant Coriolis parameter, determine the divergence of the flow

$$\int_{-\infty}^{0} dz\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \tag{13.67}$$

which is identical to the vertical velocity across the Ekman layer (since w(0)=0).

3. Energy

Start with the shallow water equations

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{13.68}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{13.69}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{13.70}$$

with H=const. as mean depth and η as surface anomaly.

a) Show that the total energy throughout the shallow water column satisfies

$$\frac{\partial (PE + KE)}{\partial t} + \nabla \cdot F = 0, \tag{13.71}$$

where

$$KE = \frac{1}{2}\rho H(u^2 + v^2), \quad PE = \frac{1}{2}\rho g\eta^2, \text{and vector } F = \rho gH \quad (u\eta, v\eta). \tag{13.72}$$

b) Determine the dispersion relation for plane wave solutions to this problem of the form

$$(u_0, v_0, \eta_0) \exp i(kx + ly - \omega t) \tag{13.73}$$

and express u_0, v_0 in terms of η_0.

c) Let

$$\exp i(kx + ly - \omega t) = e^{i\Psi} \tag{13.74}$$

and let an overbar denote the average of the quantity over a full phase of the wave, i.e., over the interval $0 \leq \Psi \leq 2\pi$. Show that

$$\frac{\bar{KE}}{\bar{PE}} = \frac{\omega^2 + f^2}{\omega^2 - f^2}. \tag{13.75}$$

Under what circumstances is there an equipartion of energy $\bar{K}E = \bar{P}E$?

30 points

4. Barotropic wave:

A jet stream of speed 50m/s meanders with 6000 km wavelength and 1500 km amplitude, centred at $45°N$. Compute the path (the meridional displacement) of the barotropic wave and find the phase speed. The path can be approximated:

$$y = A \, \cos[2\pi(x - ct)/\lambda] \tag{13.76}$$

$$2\Omega/R_{earth} = 2.29 \cdot 10^{-11} m/s \tag{13.77}$$

$$c_0 = -\beta(\frac{\lambda}{2\pi})^2 \tag{13.78}$$

Write:

$\beta =?$

$c =?$

$y(x) =?.$

30 points

5) Ice Dynamics:

a) The forces, acting on a block of ice are given in terms of stresses. What are the two stress components and how is the stress tensor defined.

b) In glaciology the deviatoric stress is responsible for ice deformation. Give its definition.

c) Ice reacts like on external stresses by internal deformation. How is the general flow law after Glen defined.

d) Consider a column of ice with height H and unit cross-section perpendicular to an inclined plane of angle alpha. The weight of the ice column has a component parallel to the plane which is compensated by the basal shear stress. In general, ice becomes afloat at basal shear of 100 kPa. Estimate the thickness of a perfect plastic glacier, which surface slope is measured to be 10 deg? Use: $\rho = 917 kg/m^3, g = 9.81 m/s$

Answers:

a) normal stress σ_{ii} and shear stress $\tau_{ij}, i, j = x, y, z$

$$\sigma = \begin{pmatrix} \sigma_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{zz} \end{pmatrix} \tag{13.79}$$

b)

$$\tau_{ij} = \sigma_{ii} - \delta_{ij} 1/3(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

c)

$$\dot{\epsilon}_{xy} = A\tau^{n-1}\tau_{xy}, \qquad n = \text{flow-parameter} = 3$$

d)

$$\tau_b = \rho g H sin(\alpha)$$

$$\tau_b = 100 kPa, H = 64m$$

30 points

6) Ocean thermohaline circulation: a) Consider a geostrophic flow (u, v)

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{13.80}$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{13.81}$$

with pressure $p(x, y, z, t)$.

Use the hydrostatic approximation

$$\frac{\partial p}{\partial z} = -g\rho \tag{13.82}$$

and equation (13.110) in order to derive the meridional overturning stream function $\Phi(y, z)$ as a fuction of density ρ at the basin boundaries! Φ is defined via

$$\Phi(y, z) = \int_0^z \frac{\partial \Phi}{\partial \tilde{z}}\, d\tilde{z} \tag{13.83}$$

$$\frac{\partial \Phi}{\partial \tilde{z}} = \int_{x_e}^{x_w} v(x, y, \tilde{z})\, dx \quad \text{(zonally integrated transport),} \tag{13.84}$$

where x_e and x_w are the eastward and westward boundaries in the ocean basin (think e.g. of the Atlantic Ocean). Units of Φ are $m^3 s^{-1}$. At the surface $\Phi(y, 0) = 0$.

b) Draw a figure of the Atlantic overturning!

10 points

7) Methods:

a) Consider the differential equation

$$\frac{d}{dt}x = ax - \epsilon x^2 \tag{13.85}$$

Explain the asymptotic method in case of $\epsilon << 0$. Expand x variables in a power series.

b) Provide an example from atmosphere-ocean dynamics!

10 points

8) Cells:

What are the names of the 3 meridional cells in the atmosphere? Draw a picture!

How is the equatorial cell driven? Are these cells geostrophically driven or not?

10 points

20 points

9) The Stommel model: of the wind-driven circulation in a homogeneous ocean of constant depth h is described by

$$R\nabla^2\psi + \beta\partial_x\psi = \text{curl}(\tau_0/\rho_0) \tag{13.86}$$
$$= \partial_x\tau_0^y/\rho_0 - \partial_y\tau_0^x/\rho_0 \tag{13.87}$$

where R is a coefficient of bottom friction, β the derivative of the Coriolis frequency at a central latitude, and the τ_0 the windstress vector. Finally, ψ is the streamfunction of the depth integrated velocity

$$U = (U, V) = \int_{-h}^{0} u\,dz$$

i.e.

$$U = -\partial_y\psi, V = \partial_x\psi$$

a) Derive this equation from the conservation of momentum (linearized) and mass (volume!) assuming $w = 0$ at the mean surface $z = 0$ and at the bottom $z = -h$. For simplicity take Cartesian coordinates for the horizontal, $\beta = df/dy$. Boundary condition for the flux of momentum are $\tau(z = 0) = \tau_0$ and $\tau(z = -h) = R(-V, U)$.

b) in the boundary layer the terms on the left hand side of (13.86) get large. Show by scaling that the width of the layer is $W = R/\beta$.

c) how large must R be to get a width $W = 100$ km? ($\beta = 2 \times 10^{-11} \text{ m}^{-1}\text{s}^{-1}$).

d) can you imagine a planet form or rotation conditions where the other circulations are possible for the Stommel gyre?

20 points

13.3 Exam 3

Instructions before you start: *The perfect score for this exam is* 100 *points, although the sum of the problems is* 140. *Therefore, you can choose among the problems to solve.* 50 *points are necessary for the course.*

You are allowed to use a calculator & pen and only the paper we provide. Collaboration or use of alternative sources of information is <u>not</u> allowed. Good luck!

1. **Several questions about the course** (*32 points*, **for each Q** *2 points*).

    Q1: The Coriolis parameter f is defined as

    a) $f = \Omega \cos \varphi$

    b) $f = 2\Omega \cos \varphi$

    c) $f = 2\Omega \sin \varphi$

    d) $f = \beta y$

    Q2: Please clarify: On the Northern Hemisphere, particles tend to go to the right or left relative to the direction of motion due to the Coriolis force?

    Q3: Please write down the equation of state for the ocean and atmosphere!

    Q4: What are the dominant terms in the momentum balance for the large-scale dynamics at mid-latitudes?

    Q5: What is the hydrostatic approximation in the momentum equations?

Q6: Which are the characteristics of baroclinic and barotropic atmosphere (baroclinic vs. barotropic)?

Q7: Name the at least five monsoon systems and enumerate the time scales of the monsoon variability .

Q8: Please write down the Euler forward numerical scheme for $\frac{d}{dt}x = f(x)$!
Consider also the special case $f(x) = rx - x^2$!

Q9: Please write down the barotropic potential vorticity equation for large-scale motion!

Q10: Please write down the dispersion relation for Rossby and gravity waves!

Q11: What is the necessary condition for stability in a linear system

$$\frac{d}{dt}x = Ax$$

with real vector x and $N \times N$ matrix A?

... and for the analogous non-linear case $\frac{d}{dt}x = f(x)$?

Q12: What is the definition of correlation and covariance?
How is the Fourier transformation of the covariance called?

Q13: Explain the δ notation for stable oxygen isotopes! Provide an example of an stable and

an unstable isotope in nature! Why are they important?

Q14: Draw a schematic figure of the Atlantic Ocean meridional overturning!

Q15: What are the names of the 3 meridional cells in the atmosphere? Draw a picture with the direction! How is the equatorial cell driven? Are these cells geostrophically driven or not?

Q16: Explain dispersive and non-dispersive waves!

2. **Rossby, gravity, and Kelvin waves** *(12 points)*

Start with the shallow water equations

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x} \tag{13.88}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y} \tag{13.89}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \tag{13.90}$$

with H=const. as mean depth and η as surface anomaly.

a) with the elimination of the fast gravity waves in equation (13.90)

$$\frac{\partial \eta}{\partial t} = 0$$

derive the dispersion relation for divergence-free Rossby waves! Ansatz: Introduce a streamfunction for u,v:

$$\Psi \sim \exp(ikx + ily - i\omega t)$$

b) with the assumption of $f = f_0 = 0$ derive the dispersion relation for gravity waves! The restoring force is related to gravity. Ansatz: take one of the equations (13.88,13.89,13.90) and derive the solution.

c) Kelvin wave. Assume a vertical wall at x=0 along the y-axis (an idealized coast) and u=0. Derive the solution for $v(x, y, t)$ and $\eta(x, y, t)$ using the equations (13.89,13.90)! Specify the x-dependence of the solutions using (13.88) and discuss the trapping distance from the coast!

3. **Rossby wave formula (long waves in the westerlies)** *(11 points)*

   Consider the vorticity equation

$$\frac{D}{Dt}[(\zeta + f)/h] = 0 \tag{13.91}$$

   with h =const., u and v are the velocity components.

   a) Assume a mean flow with constant zonal velocity U

$$u = U = const > 0 \tag{13.92}$$

   and a varying north-south component

$$v = v(x, t) \tag{13.93}$$

   which gives the total motion a wave-like form. Derive the vorticity equation!

   b) With the ansatz

$$v(x, t) = A \cos[(kx - \omega t)] \tag{13.94}$$

   determine the disperion relation $\omega(k)$, group velocity $\frac{\partial \omega}{\partial k}$, and the phase velocity $c = \omega/k$.

   c) Derive the wavelength $L = 2\pi/k$ of the stationary wave given by $c = 0$.

4. **Wind-driven ocean circulation** *(8 points)*

   When the windstress is only zonal, the Sverdrup transport is

$$\rho_0 \beta V = \text{curl } \tau = -\frac{\partial}{\partial y}\tau^x \tag{13.95}$$

and Ekman transports and Ekman pumping velocity are

$$\rho_0 f V_E \;=\; -\tau^x \tag{13.96}$$

$$\rho_0 w_E \;=\; \operatorname{curl}\tau = -\frac{\partial}{\partial y}\tau^x. \tag{13.97}$$

Assume furthermore

$$\tau^x = -\tau_0 \cos(\pi y/B) \tag{13.98}$$

for an ocean basin $0 < x < L, 0 < y < B$.

a) at what latitudes y are $|V|$ and $|V_E|$ maximum? Calculate their magnitudes. Take constant $f = 10^{-4}$ s^{-1} and $\beta = 1.8 \cdot 10^{-11}$ m^{-1}s^{-1} and $B = 5000$ km, $\tau_0/\rho_0 = 10^{-4}$ m^{2}s^{-2}.

b) calculate the maximum of w_E for constant f (value see above).

5. **Potential vorticity:** *(6 points)*

   An air column at $53°$N with $\zeta = 0$ initially streches from the surface to a fixed tropopause at 10 km height. If the air column moves until it is over a mountain barrier 2.5 km hight at $30°$N, what is its absolute vorticity and relative vorticity as it passes the mountain top?

   Assume: $\sin 53° = 0.8$; $\sin 30° = 0.5$

   The angular velocity of the Earth $\Omega = 2\pi/(1day)$.

6. **Bifurcation:** *(6 points)*

   Consider the differential equation

$$\frac{d}{dt}x \;=\; rx\,(1-x) \tag{13.99}$$

Similar dynamics can be derived from the logistic growth or Stommel's box model of the ocean circulation (see lecture).

a) Calculate the bifurcation with respect to the parameter r and draw the bifurcation diagram!

b) Is this bifurcation a transcritical or Hopf bifurcation?

c) which types of bifurcations do you know?

7. **Lorenz equations** *(12 points)*

Consider the Lorenz equations (which were derived from the Rayleigh-Bernard system)

$$\dot{x} = \sigma(y - x) \tag{13.100}$$

$$\dot{y} = rx - xz - y \tag{13.101}$$

$$\dot{z} = xy - bz \tag{13.102}$$

with $\sigma, r, b > 0$. σ is the Prandtl number. Rayleigh number $R_a \sim \Delta T$, critical Rayleigh number R_c, and $r = R_a / R_c$.

a) Evaluate the equilibrium points and determine their stability through linearization!

b) Show the symmetry: The Lorenz equation has the following symmetry $(x, y, z) \rightarrow (-x, -y, z)$ independent on the parameters σ, r, b.

c) Lorenz system has bounded solutions: Show that all solutions of the Lorenz equation will enter an ellipsoid centered at $(0, 0, 2r)$ in finite time, and the solution will remain inside the ellipsoid once it has entered. To observe this, define a Lyapunov function

$$V(x, y, z) = rx^2 + \sigma y^2 + \sigma(z - 2r)^2 \tag{13.103}$$

8. **Stochastic climate model** *(9 points)*

Imagine that the temperature of the ocean mixed layer of depth h is governed by

$$\frac{dT}{dt} = -\lambda T + \frac{Q_{net}}{\gamma_O},$$

(13.104)

where coefficient γ_O is given by the heat capacity $c_p \rho h$, and λ is the typical damping rate of a temperature anomaly. The air-sea fluxes due to weather systems are represented by a white-noise process $Q_{net} = \hat{Q}_\omega e^{i\omega t}$ where $\hat{Q}_\omega$ is the amplitude of the random forcing at frequency ω. $\hat{Q}^*$ is the complex conjugate.

a) What is a white-noise process? Remember that

$$\int_R \exp(i\omega t)\delta(t - 0)dt = 1$$

(13.105)

and use the Fourier transformation.

b) Solve Eq. 13.104 for the temperature response $T = \hat{T}_\omega e^{i\omega t}$ and hence show that:

$$\hat{T}_\omega = \frac{\hat{Q}_\omega}{\gamma_O (\lambda + i\omega)}$$

(13.106)

c) Show that it has a spectral density $\hat{T}_\omega \hat{T}_\omega^*$ is given by:

$$\hat{T}\hat{T}^* = \frac{\hat{Q}\hat{Q}^*}{\gamma_O^2 (\lambda^2 + \omega^2)}$$

(13.107)

and the spectrum

$$S(\omega) = < \hat{T}\hat{T}^* > = \frac{1}{\gamma_O^2 (\lambda^2 + \omega^2)}.$$

(13.108)

The brackets $< \cdots >$ denote the ensemble mean. Make a sketch of the spectrum using a

log-log plot and show that fluctuations with a frequency greater than λ are damped.

9. **Carbon-14 ages** *(10 points)*

The number of decays per time is proportional to the current number of radioactive atoms. This is expressed by the following differential equation, where N is the number of radioactive atoms and λ is a positive number called the decay constant:

$$\frac{dN}{dt} = -\lambda N \tag{13.109}$$

$\frac{1}{\lambda}$ is the radiocarbon mean- or average-life = 8033 years (Libby value)

a) Calculate the radiocarbon half-life based on Libby's value of $\frac{1}{\lambda}$! The correct half-life is $5,730 \pm 40$ years.

b) Plants take up atmospheric carbon dioxide by photosynthesis, and are ingested by animals, so every living thing is constantly exchanging carbon-14 with its environment as long as it lives. Once it dies, however, this exchange stops, and the amount of carbon-14 gradually decreases through radioactive beta decay. Calculate a raw radiocarbon date from N and an initial condition of N_0 for radioactive atoms at $t = 0$! Why is it difficult to report C-14 ages greater than 60,000 years?

c) Typical values for δ^{14}C are $-50\%_o$ in the mixed layer ocean with depth 50m and $-150\%_o$ in deep water (layer depth 4 km). What are the ^{14}C ages of surface and deep water?

10. **Short programming questions.** *(6 points)*

Write down the output for the following R-commands:

   a) `0:10`

   b) `a<-c(0,5,3,4); mean(a)`

c) `max(a)-min(a)`

d) `paste("The mean value of a is",mean(a),"for sure",sep="_")`

e) `a*2+c(1,1,1,0)`

f) `my.fun<-function(n){return(n*n+1)}`

   `my.fun(10)-my.fun(1)`

11. **Ocean thermohaline circulation:** *(10 points)*

a) Consider a geostrophic flow (u, v)

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{13.110}$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{13.111}$$

with pressure $p(x, y, z, t)$.

Use the hydrostatic approximation

$$\frac{\partial p}{\partial z} = -g\rho \tag{13.112}$$

and equation (13.110) in order to derive the meridional overturning stream function $\Phi(y, z)$ as a fuction of density ρ at the basin boundaries! Φ is defined via

$$\Phi(y, z) = \int_0^z \frac{\partial \Phi}{\partial \tilde{z}}\, d\tilde{z} \tag{13.113}$$

$$\frac{\partial \Phi}{\partial \tilde{z}} = \int_{x_e}^{x_w} v(x, y, \tilde{z})\, dx \quad \text{(zonally integrated transport)}, \tag{13.114}$$

where x_e and x_w are the eastward and westward boundaries in the ocean basin (think e.g. of the Atlantic Ocean). Units of Φ are $m^3 s^{-1}$. At the surface $\Phi(y, 0) = 0$.

b) Draw a figure of the Atlantic overturning!

12. **Nabla***(4 points)*

Calculate the following operations for the function

$$f(x, y, z) = x^3 + 3x - 4xz + z^4 \quad : \tag{13.115}$$

∇f, calculate the divergence of the result!

Calculate the rotation of ∇f!

13. **Angular momentum and Hadley cell** *(8 points)*

Consider a zonally symmetric circulation (i.e., one with no longitudinal variations) in the atmosphere. In the inviscid upper troposphere one expects such a flow to conserve absolute angular momentum, i.e.,

$$\frac{DA}{Dt} = 0, \tag{13.116}$$

where A is the absolute angular momentum per unit mass (parallel to the Earth's rotation axis)

$$A = r\left(u + \Omega r\right) = \Omega R^2 \cos^2 \varphi + uR\cos\varphi \quad . \tag{13.117}$$

Ω is the Earth rotation rate, u the eastward wind component, $r = R\cos\varphi$ is the distance from the rotation axis, R the Earth's radius, and φ latitude.

a) Show, for inviscid zonally symmetric flow, that the relation $\frac{DA}{Dt} = 0$ is consistent with the zonal component of the equation of motion

$$\frac{Du}{Dt} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{13.118}$$

in (x, y, z) coordinates, where $y = R\varphi$.

b) Use angular momentum conservation to describe in words how the existence of the Hadley circulation explains the existence of both the subtropical jet in the upper troposphere and the near-surface trade winds.

c) If the Hadley circulation is symmetric about the equator, and its edge is at $20°$ latitude,

determine the strength of the subtropical jet. Use (13.117, 13.118).

d) Is the Hadley cells geostrophically driven or not?

14. **Interpretation of SST correlation maps and modes of variability** *(5 points)*

The two maps show correlations with local climate indices (Fig. 13.8).

a) Which climate modes correspond to the sea surface temperature correlation-maps?

b) Describe and draw schematically the associated sea level pressure patterns!

c) What are the dominant time scales and dominant seasons for these two modes in the climate system?
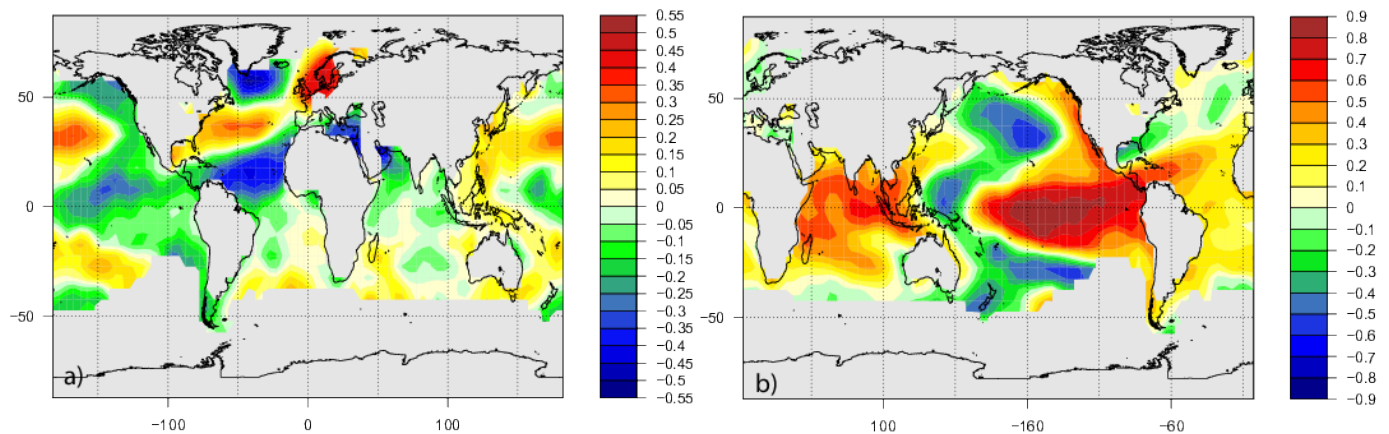
Figure 13.8: Correlation map of climate indices with global sea surface temperature.

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