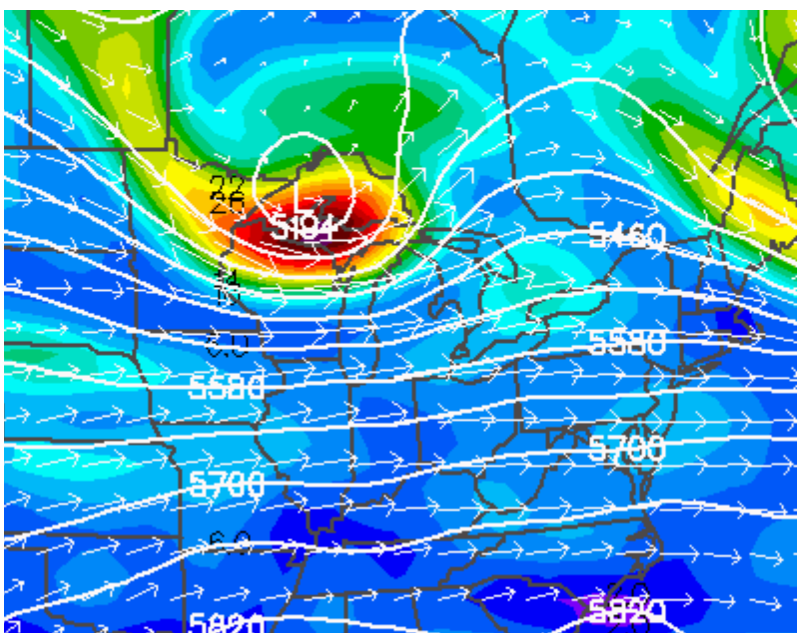
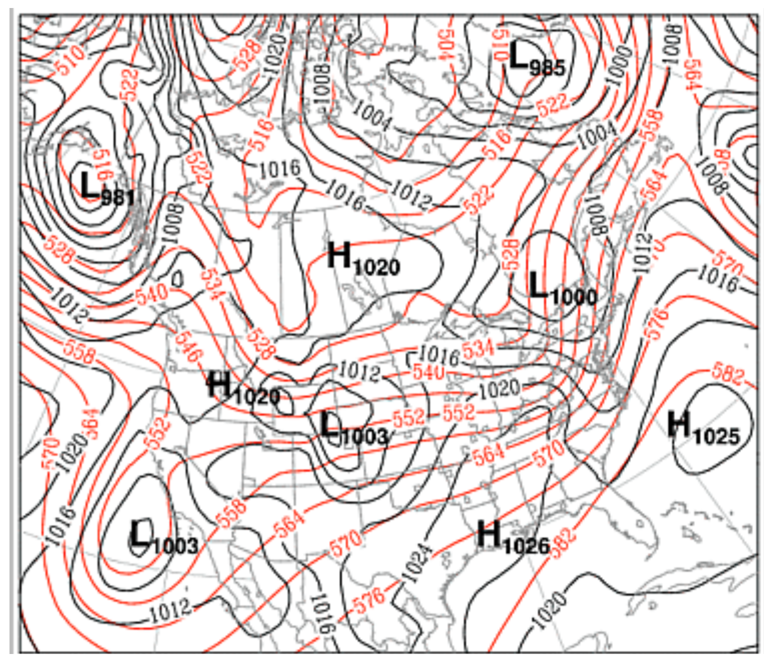
What Are Dynamics?

- Definition: The study of atmospheric and oceanic motions, with emphasis on the physical laws that govern such motions.





Course Objectives

- To lay a mathematical and theoretical foundation to be used in later applications.
- To apply the laws governing fluid motion (laws of hydrodynamics and thermodynamics) to the ocean and atmosphere

Basic Laws

- Conservation of mass (continuity equation)
- Conservation of energy (1st law of thermodynamics)
- Newton's 1st and 2nd Law (no resultant force → no change in motion, rate of change of motion of a body is proportional to resultant force acting on it)
- Conservation of angular momentum
- Newton's Law of Gravitation
- equation of state, e.g. Ideal Gas Law

Fundamental Physical Quantities

Quantity	**Symbol**	**Units**
length	L	meters (m)
time	t	seconds (s)
mass	M	kilograms (kg)
temperature	T	Kelvins (K)

Some Derived Quantities

Quantity	Symbol	Dimensions	Units
Velocity	**V**	L/t	m/s
Acceleration	**A**	L/t^2	m/s^2
Force	**F**	ML/t^2	$N = kg\ m/s^2$
Energy	E	ML^2/t^2	$J = kg\ m^2/s^2$
Pressure	P	M/Lt^2	$Pa = kg/m\ s^2$
Density	ρ	M/L^3	kg/m^3
Specific Volume	α	L^3/M	m^3/kg

Bold symbols indicate vector quantities.

Non-dimensional parameters

Basic concept and applications

$$\boldsymbol{u} = \boldsymbol{U} \cdot \boldsymbol{u_d} \tag{4.22}$$

$$\boldsymbol{t} = \boldsymbol{T} \cdot \boldsymbol{t_d} \tag{4.23}$$

$$\boldsymbol{x} = \boldsymbol{L} \cdot \boldsymbol{x_d} \tag{4.24}$$

with $\boldsymbol{U} = \boldsymbol{L}/\boldsymbol{T}$. From these scalings, we can also derive

$$\boldsymbol{\partial_t} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial t}} = \frac{\mathbf{1}}{\boldsymbol{T}} \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial t_d}} \tag{4.25}$$

$$\boldsymbol{\partial_x} = \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x}} = \frac{\mathbf{1}}{\boldsymbol{L}} \cdot \frac{\boldsymbol{\partial}}{\boldsymbol{\partial x_d}} \tag{4.26}$$

Note furthermore the units of $[\boldsymbol{\rho_0}] = \boldsymbol{kg/m^3}$, $[\boldsymbol{p}] = \boldsymbol{kg/(ms^2)}$, and $[\boldsymbol{p}]/[\boldsymbol{\rho_0}] = \boldsymbol{m^2/s^2}$. Therefore the pressure gradient term in (4.8) has the scaling $\boldsymbol{U^2/L}$. Furthermore, devide the equation (4.8) by $\boldsymbol{U^2/L}$ and the scalings vanish completely in front of the terms except for the $\boldsymbol{\nabla_d^2}\mathbf{u_d}$-term! This procedure yields therefore for (4.20,4.21):

$$\boldsymbol{\nabla_d} \cdot \mathbf{u_d} = \mathbf{0} \tag{4.27}$$

$$\frac{\boldsymbol{\partial}}{\boldsymbol{\partial t_d}}\mathbf{u_d} + (\mathbf{u_d} \cdot \boldsymbol{\nabla_d})\mathbf{u_d} = -\boldsymbol{\nabla_d p_d} + \frac{\mathbf{1}}{\boldsymbol{Re}}\boldsymbol{\nabla_d^2}\mathbf{u_d}$$

Expansion of Total Derivative

If $f = f(x, y, z, t)$ then

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

But $u \equiv \dfrac{dx}{dt}, \quad v \equiv \dfrac{dy}{dt}, \quad w \equiv \dfrac{dz}{dt}$

u = west-east component of fluid velocity

v = south-north component of fluid velocity

w = vertical component of fluid velocity

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\overbrace{\frac{dx}{dt}}^{u} + \frac{\partial f}{\partial y}\overbrace{\frac{dy}{dt}}^{v} + \frac{\partial f}{\partial z}\overbrace{\frac{dz}{dt}}^{w}$$

Euler's relation (expansion of total derivative):

$$\underset{\text{A}}{\frac{df}{dt}} = \underset{\text{B}}{\frac{\partial f}{\partial t}} + \underset{\text{C}}{u\frac{\partial f}{\partial x}} + \underset{\text{D}}{v\frac{\partial f}{\partial y}} + \underset{\text{E}}{w\frac{\partial f}{\partial z}}$$

Term A: Total rate of change of *f* following the fluid motion

Term B: Local rate of change of *f* at a fixed location

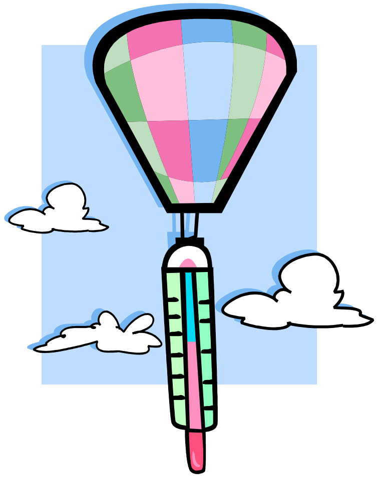
Term C: Advection of *f* in *x* direction by the *x*-component flow

Term D: Advection of *f* in *y* direction by the *y*-component flow

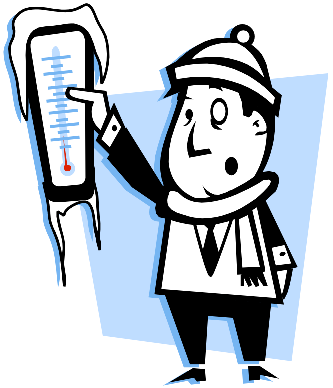
Term E: Advection of *f* in *z* direction by the *z*-component flow

Total Derivative vs. Local Derivative

Total derivative is the temporal rate of change following the fluid motion. Example: A thermometer measuring changes as a balloon floats through the atmosphere.

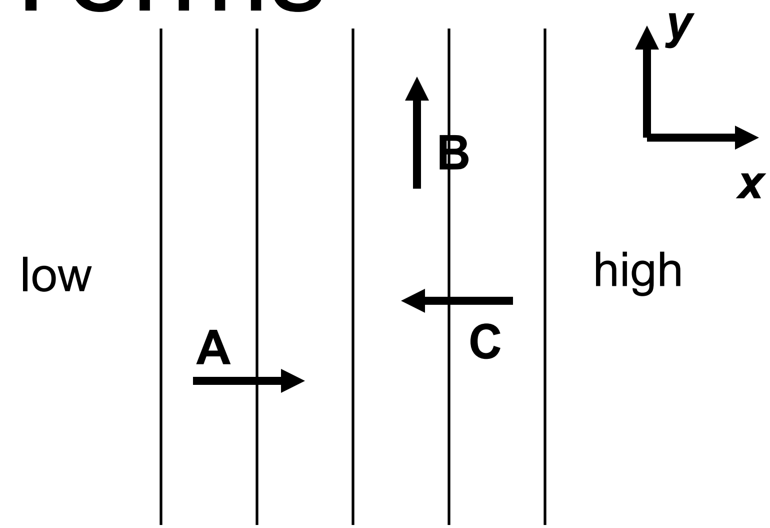


Local derivative is the temporal rate of change at a fixed point. Example: An observer measures changes in temperature at a weather station.



Advection Terms

Assume that thin lines are contours of a scalar quantity *f* and thick arrows indicate the fluid motion. We wish to evaluate the advection term $u\dfrac{\partial f}{\partial x}$



At point A: $u > 0, \dfrac{\partial f}{\partial x} > 0 \rightarrow u\dfrac{\partial f}{\partial x} > 0$ → **Transport from low to high: "negative advection of *f*"**

At point B: $u = 0, \dfrac{\partial f}{\partial x} > 0 \rightarrow u\dfrac{\partial f}{\partial x} = 0$ → **"neutral advection of *f*"**

At point C: $u < 0, \dfrac{\partial f}{\partial x} > 0 \rightarrow u\dfrac{\partial f}{\partial x} < 0$ → **Transport from high to low: "positive advection of *f*"**

Taylor Series

A function *f(x)* can be computed by Taylor expansion given the values of the function and its derivatives at a point x_0:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \ldots$$

$$f(x) = f(x_0) + \sum_{n=1}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

A truncated Taylor series can be used to approximate *f(x).*

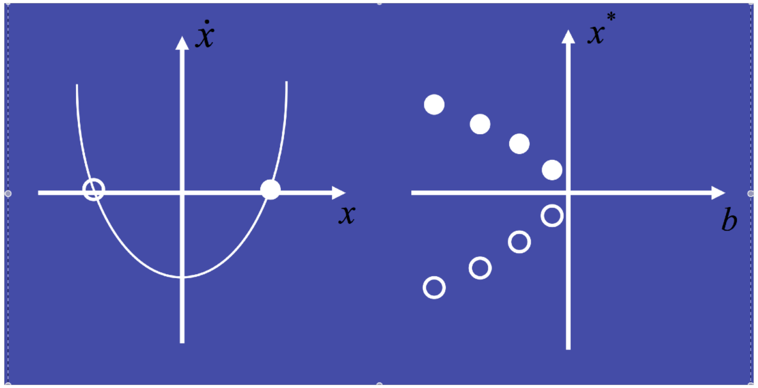
Stability theory

- basics

2.3.1 Linear stability analysis

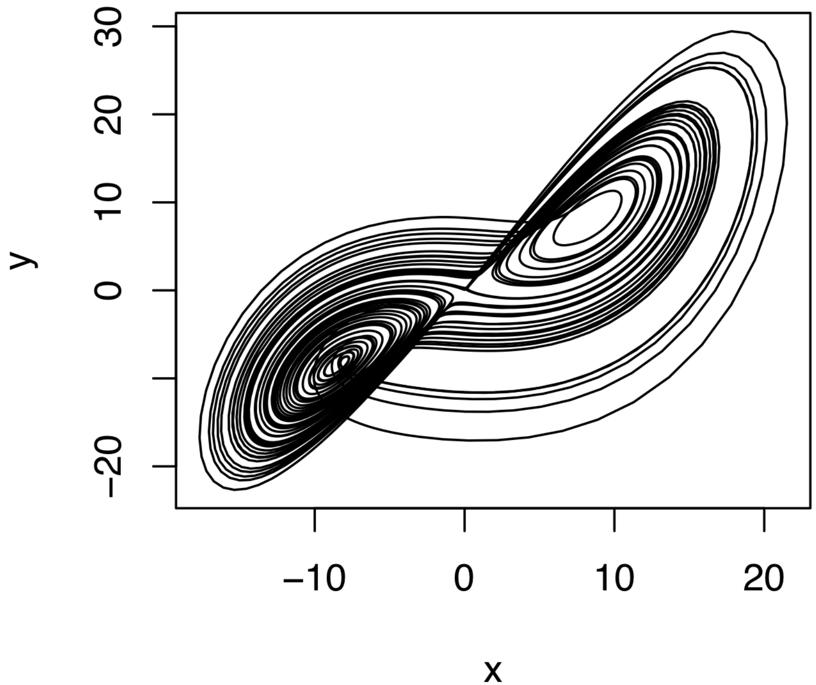
Consider the continuous dynamical system described by the ODE

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\lambda}) \quad \boldsymbol{f} : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n.$$



re 2.7: Saddle-node bifurcation diagram using the graphical method.

Lorenz model



$$\dot{X} = -\sigma X + \sigma Y$$
$$\dot{Y} = rX - Y - XZ$$
$$\dot{Z} = -bZ + XY$$

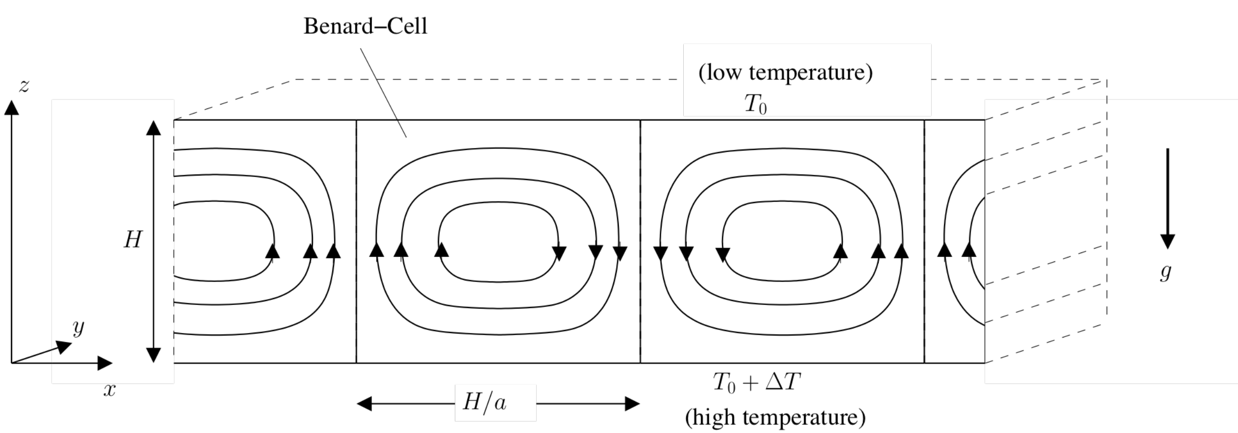


Figure 5.5: Geometry of the Rayleigh-Bénard system (see text for details).

The Atmospheric Continuum

- In atmospheric dynamics (ocean dynamics also) we do not treat the fluid as a collection of individual molecules.
- Instead we treat the fluid as a continuous medium (or continuum) in which a "point" is a volume element that is *very small compared to the total fluid volume* but still contains a *very large number of molecules*.
- These volume elements are commonly called *"air parcels"* or *"air particles."*
- The properties of these volume elements describe the state of the atmosphere.

Lattice Boltzmann

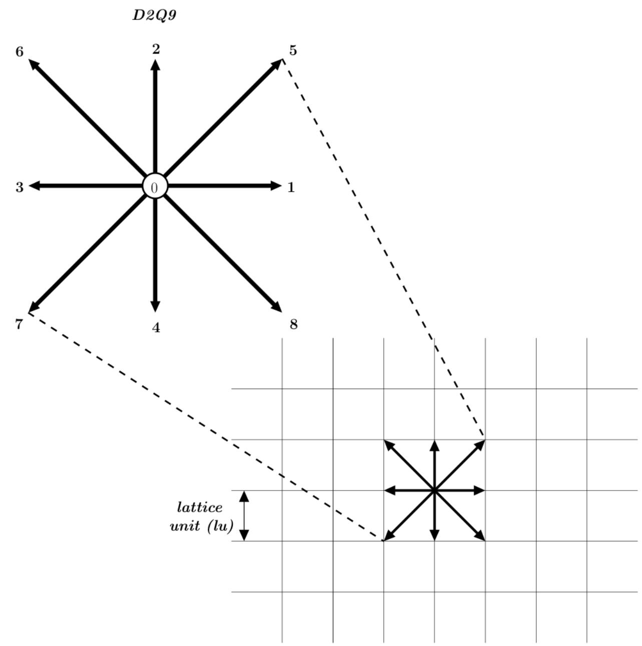


Figure 3.1: Discrete lattice velocities for the $\boldsymbol{D2Q9}$ model.

particle distribution functions (hereafter DFs) according to:

$$\rho = \sum_{a=0}^{\beta-1} f_a \qquad \text{(macroscopic fluid density)} \tag{3.52}$$

and

$$\vec{u} = \frac{1}{\rho}\sum_{a=0}^{\beta-1} f_a \vec{e}_a \qquad \text{(macroscopic velocity).} \tag{3.53}$$

The DFs at each lattice point are updated using the equation:

$$\underbrace{f_a(\vec{x} + \vec{e}_a\delta_t, t + \delta_t) = f_a(\vec{x}, t)}_{\text{Streaming}} - \underbrace{\frac{[f_a(\vec{x}, t) - f_a^{eq}(\vec{x}, t)]}{\tau}}_{\text{Collision}}, \tag{3.54}$$

mass and momentum are invariant. The equilibrium DFs can be obtained from the local Maxwell-

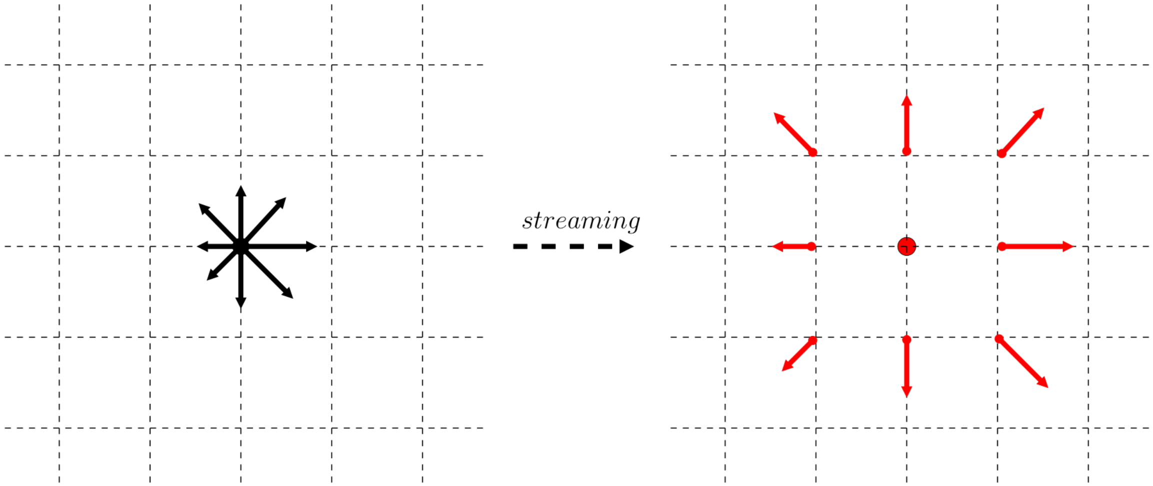


Figure 3.2: Illustration of the streaming process on a $\boldsymbol{D2Q9}$ lattice. Note that the magnitude of the DFs remain unchanged, but they move to a neighbouring node according to their direction.

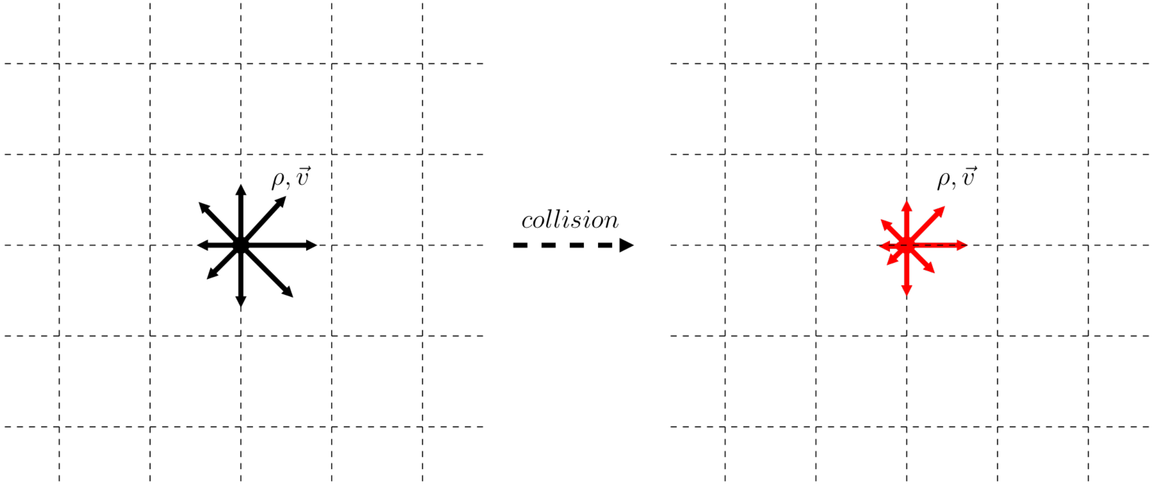


Figure 3.3: Illustration of the collision process on a $\boldsymbol{D2Q9}$ lattice. Note that the local density $\boldsymbol{\rho}$ and velocity $\vec{\boldsymbol{v}}$ are conserved, but the DFs change according to the relaxation-to-local-Maxwellian rule.

Coarse graining

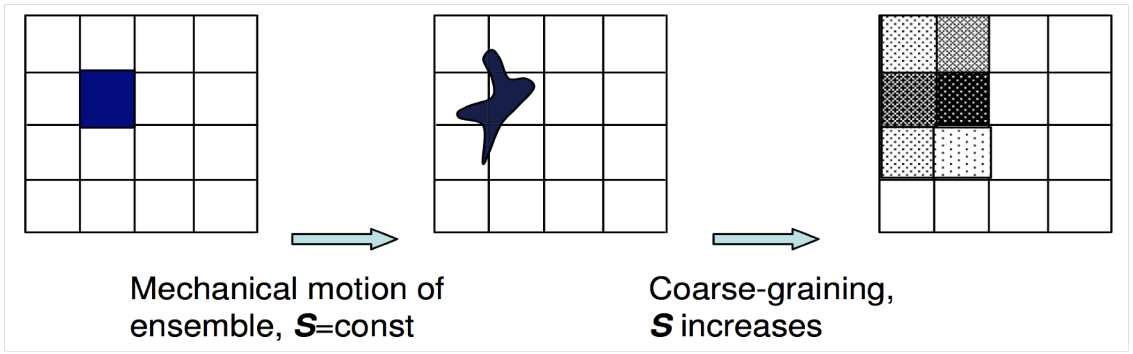


Figure 3.6: The Ehrenfests coarse-graining: two motion - coarse-graining cycles in 2D (values of probability density are presented by hatching density).

Navier-Stokes

$$\overbrace{\rho\Big(\underbrace{\frac{\partial \mathbf{u}}{\partial t}}_{\substack{\text{Unsteady}\\ \text{acceleration}}} + \underbrace{\mathbf{u}\cdot\nabla\mathbf{u}}_{\substack{\text{Advective}\\ \text{acceleration}}}\Big)}^{\text{Inertia (per volume)}} = \overbrace{\underbrace{-\nabla p}_{\substack{\text{Pressure}\\ \text{gradient}}} + \underbrace{\mu\nabla^2\mathbf{u}}_{\text{Viscosity}}}^{\text{Divergence of stress}} + \underbrace{\mathbf{F}}_{\substack{\text{Other}\\ \text{body}\\ \text{forces}}}.$$

4.4 Elimination of the pressure term

Taking the curl of the Navier-Stokes equation results in the elimination of pressure. This is especially easy to see if 2D Cartesian flow is assumed ($w = 0$ and no dependence of anything on z), where the equations reduce to:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4.14}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad . \tag{4.15}$$

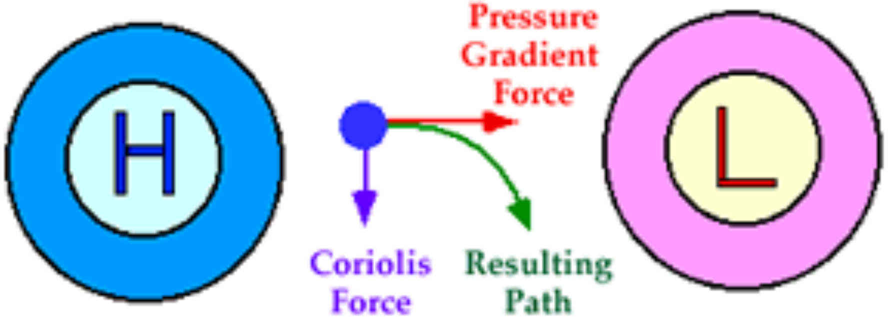
Differentiating the first with respect to y, the second with respect to x and subtracting the resulting equations will eliminate pressure and any potential force. Defining the stream function ψ through

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \tag{4.16}$$

$$D_t\zeta = \nu\nabla^2\zeta \quad .$$

Surface winds

An air parcel initially at rest will move from high pressure to low pressure (**pressure gradient force**)



Geostrophic wind blows parallel to the isobars because the **Coriolis force** and **pressure gradient force** are in balance.

Dynamics

Equation of motion with rotation
Considering all forces:

Pressure Gradient Force

Coriolis Force

Friction

Gravity

$$\frac{d\vec{v}}{dt} = \boxed{\vec{F}_p} + \boxed{\vec{F}_c} + \boxed{\vec{F}_R} + \boxed{\vec{F}_G}$$

$$\frac{d\vec{v}}{dt} = \boxed{-\frac{1}{\rho}\vec{\nabla}p} - \boxed{2\vec{\Omega} \times \vec{v}} - \boxed{a\vec{v}} + \boxed{\vec{g}}$$

Will later appear as f

Dynamics: Geostrophy

Equation of motion with rotation

x,y,z components:

Neglect friction in free atmosphere

Horizontal part

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} - \boxed{fv + f^* w} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x}}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\boxed{+ fu} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial y}}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} - f^* u = -\frac{1}{\rho}\frac{\partial p}{\partial z}\boxed{- g}$$

Vertical part

Dynamics: Geostrophy

In horizontal form:

Neglect friction in free atmosphere

Total derivative for u

$$\boxed{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

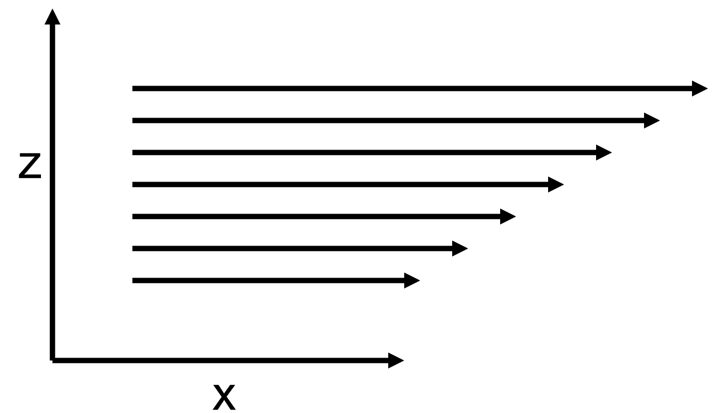
$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial z} - g$$

This is the hydrostatic equilibrium

Viscous Force

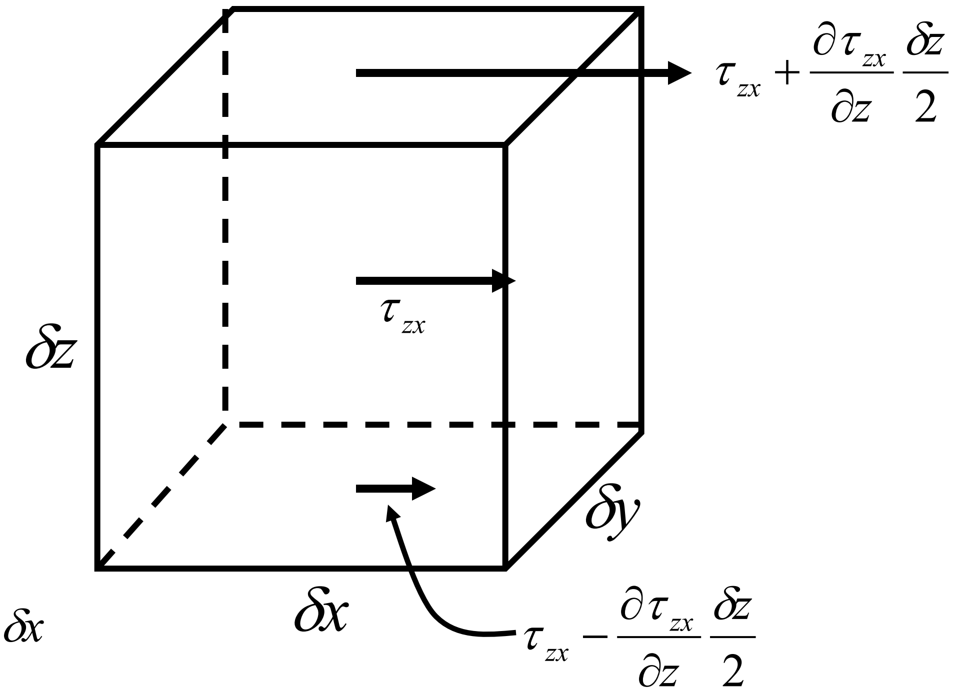
- If the wind velocity varies with height, random molecular motions will cause momentum to be transferred vertically.
- In other words, there is a drag exerted by the layers above and below the level of interest.



The stress due to the velocity shear is given by

$$\tau_{zx} = \mu \frac{\partial u}{\partial z}$$

where μ is the dynamic viscosity coefficient.



Using Taylor series expansion to express the net viscous force:

$$\left(\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta y\,\delta x - \left(\tau_{zx} - \frac{\partial \tau_{zx}}{\partial z}\frac{\delta z}{2}\right)\delta y\,\delta x$$

$$= \frac{\partial \tau_{zx}}{\partial z}\delta z\,\delta y\,\delta x$$

Substituting into Newton's second law:

$$\left(\frac{d\vec{V}}{dt}\right)_{inertial} = \frac{\sum \vec{F}}{m}$$

$$\frac{d\vec{V}}{dt} + 2\vec{\Omega}\times\vec{V} - \Omega^2\vec{R} = \frac{\sum \vec{F}}{m}$$

If the real forces acting on a fluid parcel are the pressure gradient force, gravitation and friction, then

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Rate of change of relative velocity following the relative motion in a rotating reference frame.

Coriolis acceleration

Pressure gradient force (per unit mass)

Gravity term (gravitation + centrifugal)

Friction

Vector momentum equation in rotating coordinates

Momentum Equations in Spherical Coordinates

- For a variety of reasons, it is useful to express the vector momentum equation for a rotating earth as a set of scalar component equations.
- The use of latitude-longitude coordinates to describe positions on earth's surface makes it convenient to write the momentum equations in spherical coordinates.
- The coordinate axes are (λ, ϕ, z) where λ is longitude, ϕ is latitude, and z is height.

Orientation of Coordinate Axes

The x- and y-axes are customarily defined to point east and north, respectively, such that

$$dx = a\cos\phi \, d\lambda$$

and

$$dy = a \, d\phi$$

Thus the horizontal velocity components are

$$u = \frac{dx}{dt}, \quad v = \frac{dy}{dt}$$

A Complication of Spherical Coordinates

When the x and y coordinates are defined in this way, the coordinate system is not strictly Cartesian, because the directions of the unit vectors depend on their position on the earth's surface.

This dependence on position can be accounted for mathematically (see Holton 2.3) by adding terms to each component of the total derivative:

$$\frac{d\vec{V}}{dt} = \left(\frac{du}{dt} - \frac{uv\,tan\,\phi}{a} + \frac{uw}{a}\right)\hat{i} + \left(\frac{dv}{dt} + \frac{u^2\,tan\,\phi}{a} + \frac{vw}{a}\right)\hat{j} + \left(\frac{dw}{dt} - \frac{u^2+v^2}{a}\right)\hat{k}$$

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Total derivative

Vector momentum equation in rotating coordinates

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = \ldots$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = \ldots$$

Vector momentum equation in rotating coordinates

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Coriolis acceleration

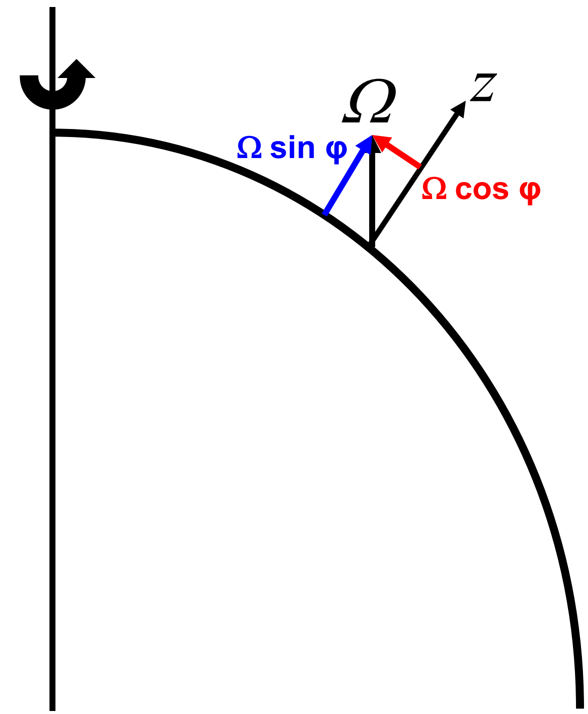
$$-2\vec{\Omega}\times\vec{V} = -2\Omega\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \cos\phi & \sin\phi \\ u & v & w \end{vmatrix}$$

$$= \left(2\Omega v\sin\phi - 2\Omega w\cos\phi\right)\hat{i} - 2\Omega u\sin\phi\,\hat{j} + 2\Omega\cos\phi\,\hat{k}$$

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = \underline{2\Omega v\sin\phi - 2\Omega w\cos\phi} + \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = \underline{-2\Omega u\sin\phi} + \ldots$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = \underline{2\Omega u\cos\phi} + \ldots$$



$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Vector momentum equation in rotating coordinates

Pressure gradient term

$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = 2\Omega v\sin\phi - 2\Omega w\cos\phi - \frac{1}{\rho}\frac{\partial p}{\partial x} + \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -2\Omega u\sin\phi - \frac{1}{\rho}\frac{\partial p}{\partial y} + \ldots$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = 2\Omega u\cos\phi - \frac{1}{\rho}\frac{\partial p}{\partial z} + \ldots$$

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Vector momentum equation in rotating coordinates

Gravity

$$\vec{g} = -g\hat{k}$$

g is a positive scalar = 9.8 m s^{-2} at earth's surface

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = 2\Omega v\sin\phi - 2\Omega w\cos\phi - \frac{1}{\rho}\frac{\partial p}{\partial x} + \ldots$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -2\Omega u\sin\phi - \frac{1}{\rho}\frac{\partial p}{\partial y} + \ldots$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = 2\Omega u\cos\phi - \frac{1}{\rho}\frac{\partial p}{\partial z} \underline{- g} + \ldots$$

$$\frac{d\vec{V}}{dt} = -2\vec{\Omega}\times\vec{V} - \frac{1}{\rho}\nabla p + \vec{g} + \vec{F}_r$$

Vector momentum equation in rotating coordinates

Friction

$$\vec{F}_r = F_{rx}\hat{i} + F_{ry}\hat{j} + F_{rz}\hat{k}$$

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = 2\Omega v\sin\phi - 2\Omega w\cos\phi - \frac{1}{\rho}\frac{\partial p}{\partial x} + \underline{F_{rx}}$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -2\Omega u\sin\phi - \frac{1}{\rho}\frac{\partial p}{\partial y} + \underline{F_{ry}}$$

$$\frac{dw}{dt} - \frac{u^2+v^2}{a} = 2\Omega u\cos\phi - \frac{1}{\rho}\frac{\partial p}{\partial z} - g + \underline{F_{rz}}$$

Momentum Equations in Spherical Coordinates

$$\boxed{\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x}} \boxed{+ 2\Omega v \sin\phi - 2\Omega w \cos\phi} \boxed{+ F_{rx}}$$

$$\boxed{\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial y}} \boxed{- 2\Omega u \sin\phi} \boxed{+ F_{ry}}$$

$$\boxed{\frac{dw}{dt} - \frac{u^2 + v^2}{a}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial z}} \boxed{+ 2\Omega u \cos\phi} \boxed{- g} \boxed{+ F_{rz}}$$

Momentum Equations in Spherical Coordinates

$$\boxed{\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial x}} \boxed{+ 2\Omega v \sin\phi - 2\Omega w \cos\phi} \boxed{+ F_{rx}}$$

$$\boxed{\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial y}} \boxed{- 2\Omega u \sin\phi} \boxed{+ F_{ry}}$$

$$\boxed{\frac{dw}{dt} - \frac{u^2 + v^2}{a}} = \boxed{-\frac{1}{\rho}\frac{\partial p}{\partial z}} \boxed{+ 2\Omega u \cos\phi} \boxed{- g} \boxed{+ F_{rz}}$$

total derivative

pressure gradient

Coriolis

gravity friction

Are All Of These Terms Important?

$$\frac{du}{dt} - \frac{uv\tan\phi}{a} + \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - 2\Omega w\cos\phi + F_{rx}$$

$$\frac{dv}{dt} + \frac{u^2\tan\phi}{a} + \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + F_{ry}$$

$$\frac{dw}{dt} - \frac{u^2 + v^2}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u\cos\phi - g + F_{rz}$$

Scale Analysis

- Goal: To determine relative importance of the terms in the basic equations for particular scales of motion.
- Approach: Estimate the following quantities
  1) The magnitude of the field variables.
  2) The amplitudes of fluctuations in the field variables. (Used to estimated derivatives
  3) The characteristic length, depth and time scales on which these flucutations occur.

Scaling Quantities

U = horizontal velocity scale

W = vertical velocity scale

L = length scale

H = depth scale

δP = horizontal pressure fluctuation

T = time scale (advective) = L/U

P_0 = surface pressure scale

Values of Scaling Quantities (midlatitude large-scale motions)

Quantity	Atmosphere	Ocean
U	10 m s^{-1}	10^{-1} m s^{-1}
W	10^{-2} m s^{-1}	10^{-4} m s^{-1}
L	10^{6} m	10^{6} m
H	10^{4} m	10^{3} m
δP (horizontal)	10^{3} Pa	10^{4} Pa
P_0	10^{5} Pa	10^{7} Pa
T	10^{5} s	10^{7} s

Physical Constants

$$g \approx 10\, m\, s^{-2}$$ gravity

$$a \approx 10^7\, m$$ radius of earth

$$\phi_0 = 45^\circ$$

$$f_0 = 2\Omega \sin\phi_0 = 2\Omega \cos\phi_0 = 10^{-4}\, s^{-1}$$

$$\rho_a = 1\, kg\, m^{-3}$$

density

$$\rho_a = 10^3\, kg\, m^{-3}$$

$$\nu_a = 10^{-5}\, m^2\, s^{-1}$$

viscosity

$$\nu_o = 10^{-6}\, m^2\, s^{-1}$$

Scaling The Horizontal Momentum Equations

$$\frac{du}{dt} - \cancel{\frac{uv\tan\phi}{a}} + \cancel{\frac{uw}{a}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v\sin\phi - \cancel{2\Omega w\cos\phi} + \cancel{F_{rx}}$$

$$\frac{dv}{dt} + \cancel{\frac{u^2\tan\phi}{a}} + \cancel{\frac{vw}{a}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u\sin\phi + \cancel{F_{ry}}$$

$\frac{U^2}{L}$	$\frac{U^2}{a}$	$\frac{UW}{a}$	$\frac{\delta P}{\rho L}$	$f_0 U$	$f_0 W$	$\frac{\nu U}{H^2}$
10^{-4}	10^{-5}	10^{-8}	10^{-3}	10^{-3}	10^{-6}	10^{-12}

Scaled Horizontal Momentum Equations (a.k.a. Equations of Motion)

$$\boxed{\begin{aligned} \frac{du}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin\phi \\ \frac{dv}{dt} &= -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\phi \end{aligned}}$$

Scaling The Vertical Momentum Equation

$$\cancel{\frac{dw}{dt}} - \cancel{\frac{u^2 + v^2}{a}} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \cancel{2\Omega u}\cos\phi - g + \cancel{F_z}$$

$$\frac{UW}{L} \qquad \frac{U^2}{a} \qquad \frac{P_0}{\rho H} \qquad f_0 U \qquad g \qquad \frac{\nu W}{H^2}$$

10^{-7} $\quad 10^{-5}$ $\quad 10$ $\quad 10^{-3}$ $\quad 10$ $\quad 10^{-15}$

The Hydrostatic Approximation

$$\boxed{\frac{\partial p}{\partial z} = -\rho g}$$

- For midlatitude synoptic-scale motions, vertical accelerations are very small compared to the vertical pressure gradient and gravity terms.
- This implies that vertical velocity cannot be determined from the vertical component of the momentum equation.

Geostrophic Balance

$$\cancel{\frac{du}{dt}} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin\phi$$

$$\cancel{\frac{dv}{dt}} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\phi$$

10^{-4} 10^{-3} 10^{-3}

- There is an approximate balance between the pressure gradient and Coriolis terms.
- Retaining only these two terms leads to the geostrophic approximation.
- The geostrophic approximation is a diagnostic relationship that cannot be used to predict the evolution of the velocity field.

The Geostrophic Approximation

$$\boxed{\begin{aligned} -fv &= -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ fu &= -\frac{1}{\rho}\frac{\partial p}{\partial y} \end{aligned}}$$

$$f \equiv 2\Omega \sin\phi$$

Coriolis parameter

The Geostrophic Wind

- The horizontal velocity field that satisfies the geostrophic approximation is known as the geostrophic wind.

$$v_g \equiv \frac{1}{\rho f}\frac{\partial p}{\partial x}$$

$$u_g \equiv -\frac{1}{\rho f}\frac{\partial p}{\partial y}$$

$$\vec{V}_g \equiv \hat{k} \times \frac{1}{\rho f}\nabla p$$

Hydrostatic Balance

$$\boxed{\frac{dp}{dz} = -\rho g}$$

- In the absence of atmospheric motions the gravity force must be exactly balanced by the vertical component of the pressure gradient force.
- Because vertical accelerations are very small for large-scale atmospheric motions, this is an excellent approximation for the vertical dependence of pressure in the real atmosphere.

$$\frac{dp}{dz} = -\rho g$$

$$dp = -\rho g\, dz$$

$$p(z) = \int_z^\infty \rho g\, dz$$

Pressure at any point is the weight per square meter of the atmospheric column overlying that point.

For average conditions,

$$p(0) = \int_0^\infty \rho g\, dz = 101.325\ kPa$$

This is the mean sea-level pressure.

We can define a quantity called the geopotential, which is related to gravity. Gravity can be represented as the gradient of the geopotential.

$$\nabla \Phi = -\vec{g}$$

Because $\vec{g} = -g\hat{k},$ then $\Phi = \Phi(z), \quad \dfrac{d\Phi}{dz} = g$

If the value of the geopotential is set to zero at mean sea level, the geopotential $\Phi(z)$ at height z is the work required to raise a unit mass to height z from mean sea level:

$$\Phi = \int_0^z g\ dz$$

Units of geopotential are J kg^{-1}, which are equivalent to $m^2\ s^{-2}$.

$$\Phi = \int_0^z g\, dz$$ implies that $$d\Phi = g\, dz$$

Since $$g\, dz = -\frac{1}{\rho} dp = -\alpha\, dp$$

then $$d\Phi = -\alpha\, dp = -\frac{RT}{p} dp = -RT\, d(\ln p)$$

The variation of geopotential with pressure depends on temperature.
Integrating in the vertical:

$$\boxed{\Phi(z_2) - \Phi(z_1) = R\int_{p_2}^{p_1} T\, d(\ln p)}$$

This is the hypsometric equation, which relates the difference in geopotential to the layer mean temperature.

Rather than express the hypsometric equation in terms of geopotential, meteorologists often rewrite it in terms of a quantity called geopotential height, which is defined as

$$Z \equiv \Phi(z)/g,$$

Units of geopotential are $m^2\ s^{-2}$, so units of geopotential height are m.

where $g = 9.8\ m\ s^{-2}$ is the global average gravity at sea level. The geopotential height is almost identical to the geometric height in the troposphere and lower stratosphere.

Thus the hypsometric equation

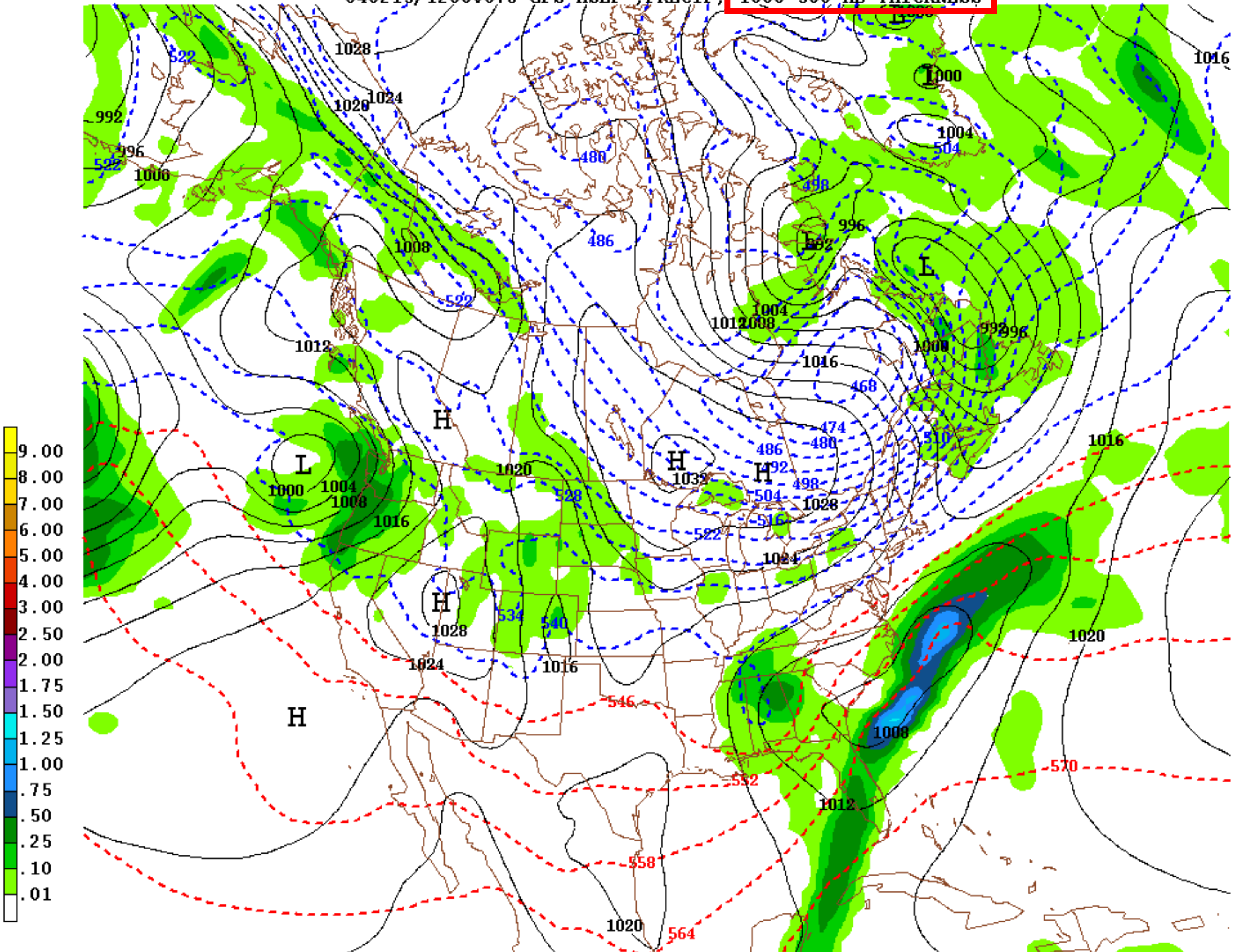
$$\Phi(z_2) - \Phi(z_1) = R\int_{p_2}^{p_1} T\, d(\ln p)$$

becomes

$$Z_T \equiv Z_2 - Z_1 = \frac{R}{g}\int_{p_2}^{p_1} T\, d(\ln p)$$

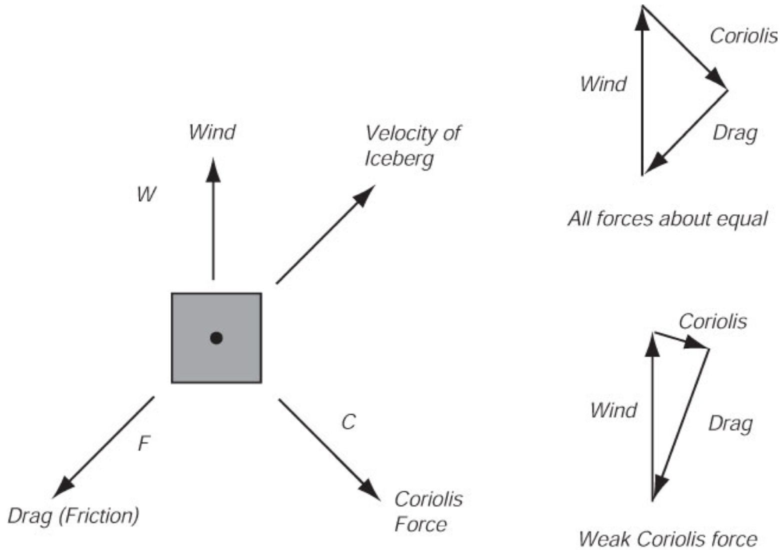
where Z_T is the thickness of the atmospheric layer between p_1 and p_2.

040215/1200V078 GFS MSLP ,PRECIP, 1000-500 MB THICKNESS

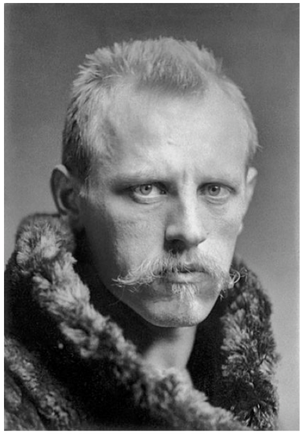


02/12/2004 06UTC 078HR FCST VALID SUN 02/15/2004 12UTC NCEP/NWS/NOAA

Nansen's Qualitative Arguments



Fridtjof Nansen noticed that wind tended to blow icebergs 20° – 40° to the right of the wind in the Arctic.



Nansen argued that three forces must be important:

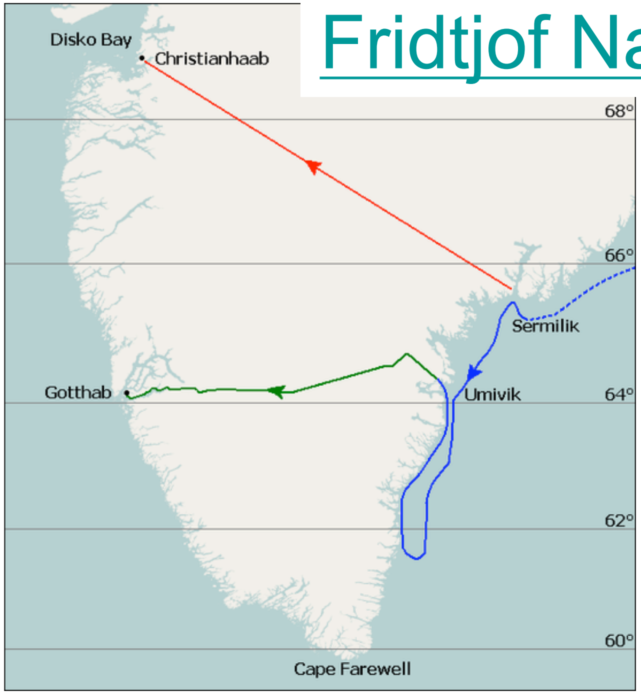
1) Wind Stress W

2) Friction F (otherwise the iceberg would move as fast as the wind)
Drag must be opposite the direction of the ice's velocity

3) Coriolis Force C.
Coriolis force must be perpendicular to the velocity

The forces must balance for steady flow: **W + F + C =** 0

Fridtjof Nansen's 1888 route across Greenland



Blue: Dotted line is the ship *Jason*'s journey from Iceland to near Sermilik fjord continuous blue: two small boats trying to reach the coast.

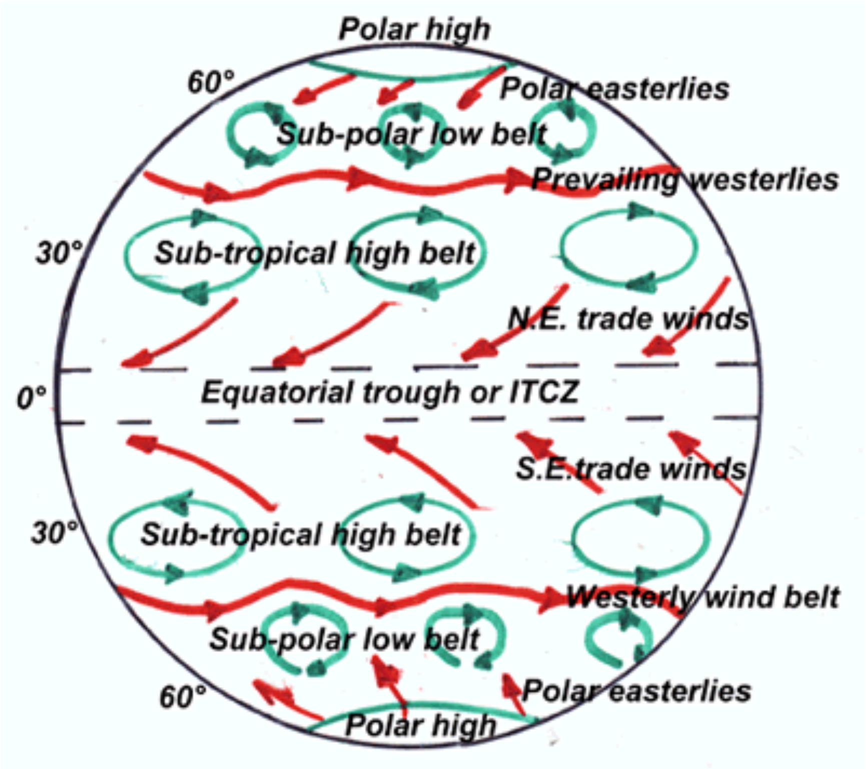
Red: Planned journey from Sermilik northwest to Christianhaab (today known as Qasigiannguit).

Green: Nansen's actual journey across Greenland from Umivik fjord to Gothaab (Nuuk).

Fridtjof Wedel-Jarlsberg Nansen (1861 – 1930)

In the final decade of his life Nansen devoted himself primarily to the League of Nations, in 1921 as the League's High Commissioner for Refugees.
In 1922 he was awarded the **Nobel Peace Prize** for his work on behalf of the displaced victims of the First World War and related conflicts.

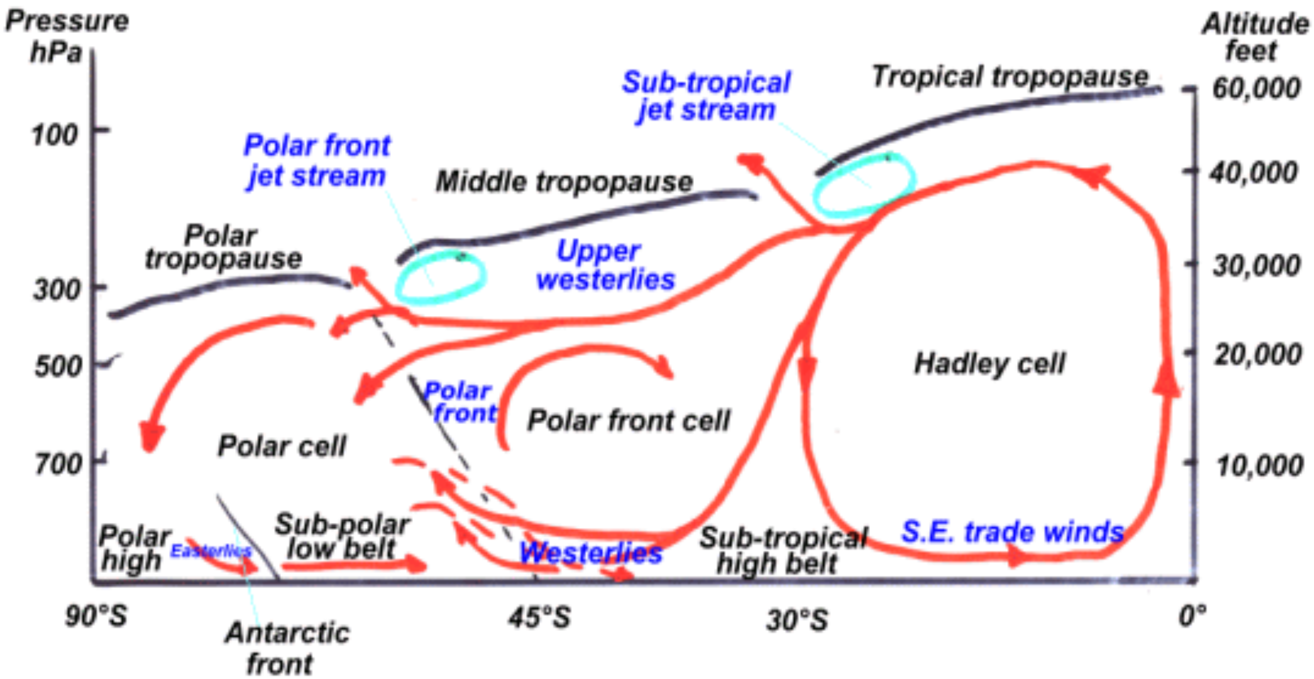
General global circulation



The atmosphere is rotating in the same direction as the Earth:

westerly winds move faster and easterly winds move slower than the Earth's surface.

Tropospheric Circulation

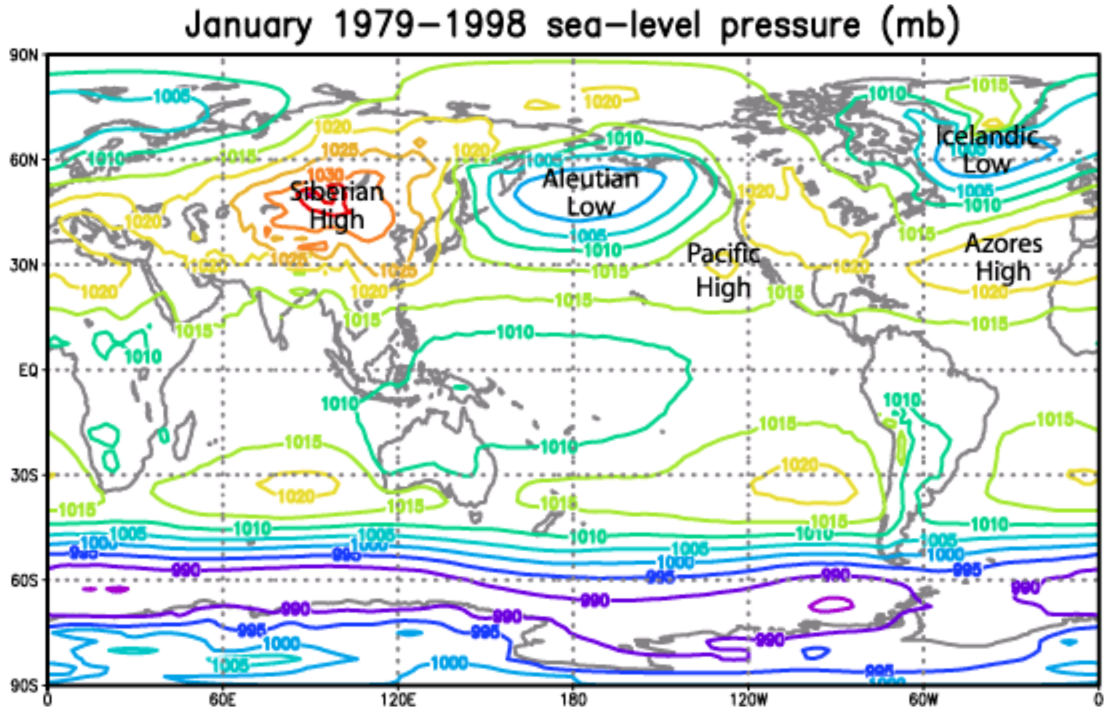


Intertropical Convergence Zone (ITCZ) and the Hadley cell

George Hadley [1685-1768], a British meteorologist, fromulated trade wind theory

50% of the Earth's surface 30° N - 30° S: Hadley cells affect half the globe

Climatology



Rossby waves and the westerly wind belt

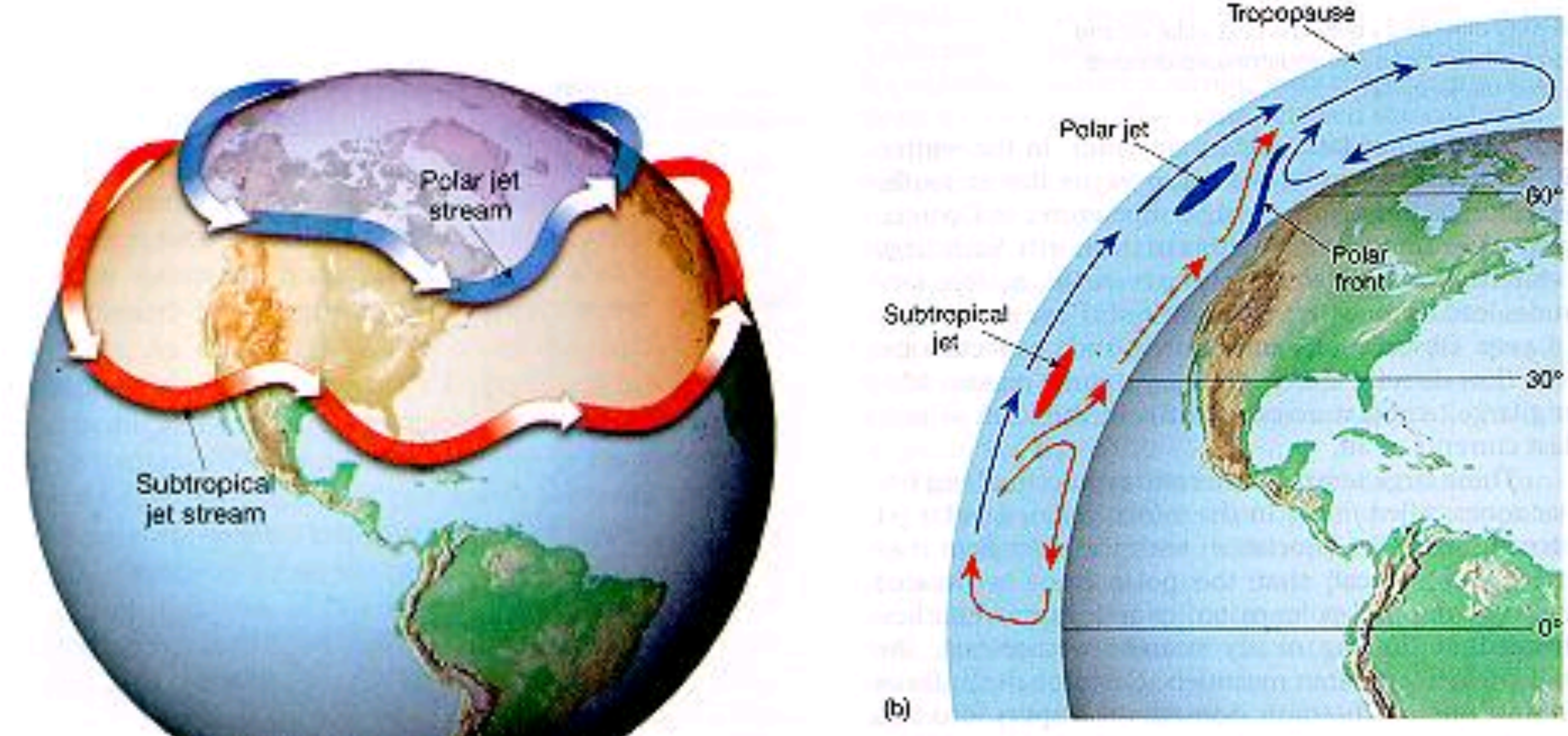
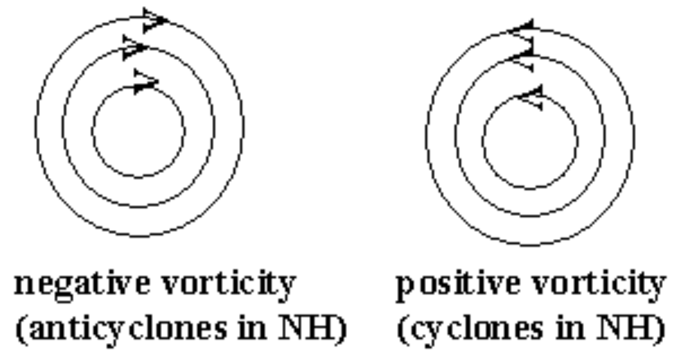


Figure 7–10 Jet streams. (a) Approximate positions of the polar and subtropical jet streams. Note that these fast-moving currents are generally not continuous around the entire globe. (b) A cross-sectional view of the polar and subtropical jets.

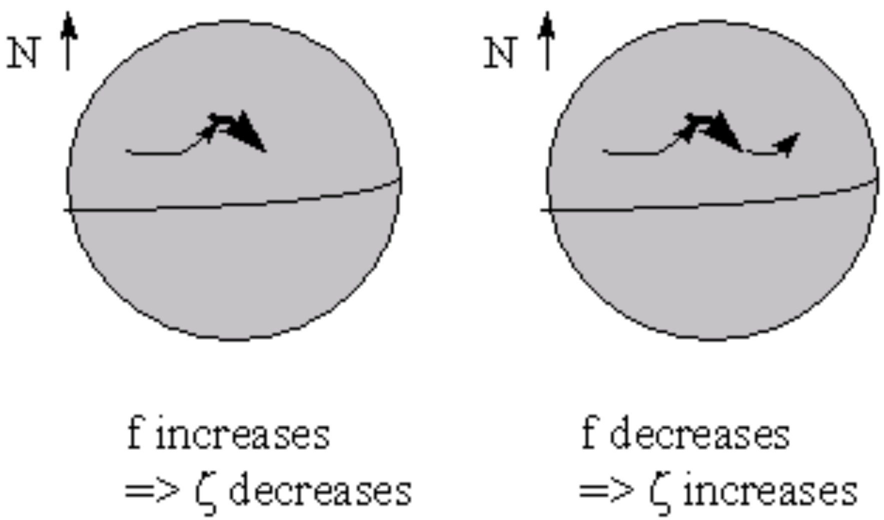
The jet stream is closely linked to the position of **Rossby waves**.

Rossby waves

Vorticity - the tendency to spin about an axis



On the spinnng Earth there is vorticity from the Earth's spin (**planetary vorticity**) and local vorticity due to cyclonic/anticyclonic behaviour (**relative vorticity**)
The absolute vorticity is conserved: *zeta + f = constant* *(f = 2 \omega sin \phi)*

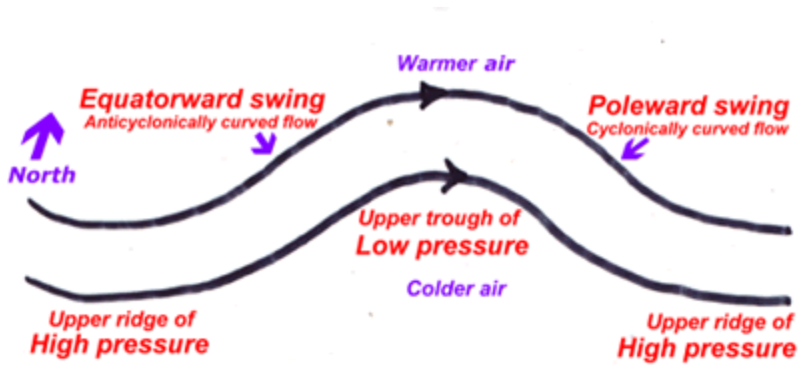


Oscillations: Rossby waves

Topographic Rossby waves: standing wave fixed to a permanent forcing location

Rossby waves and the westerly wind belt

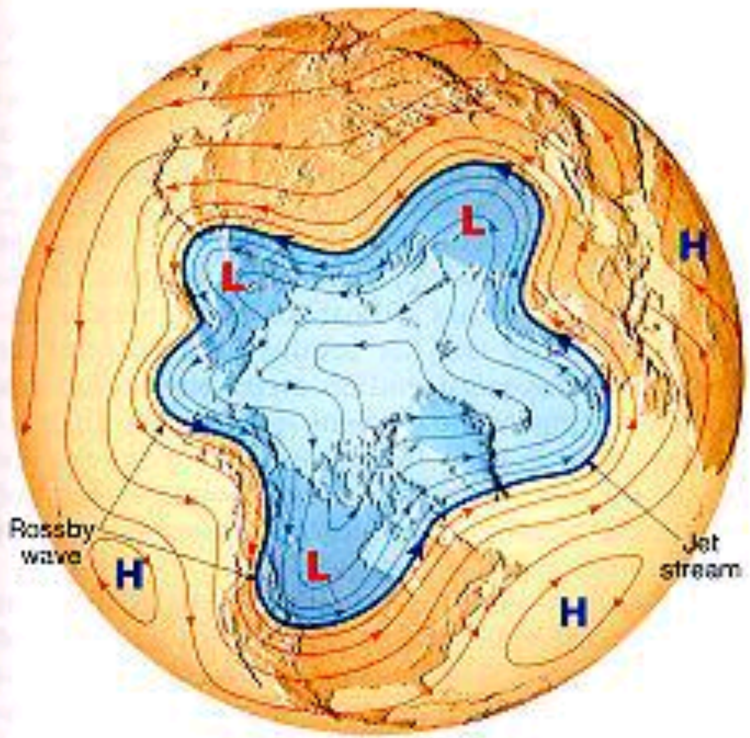
large-scale meanders of the mid-latitude jet stream.
Here Southern Hemisphere



Extend: 30° of latitude

Zeta + f = const.

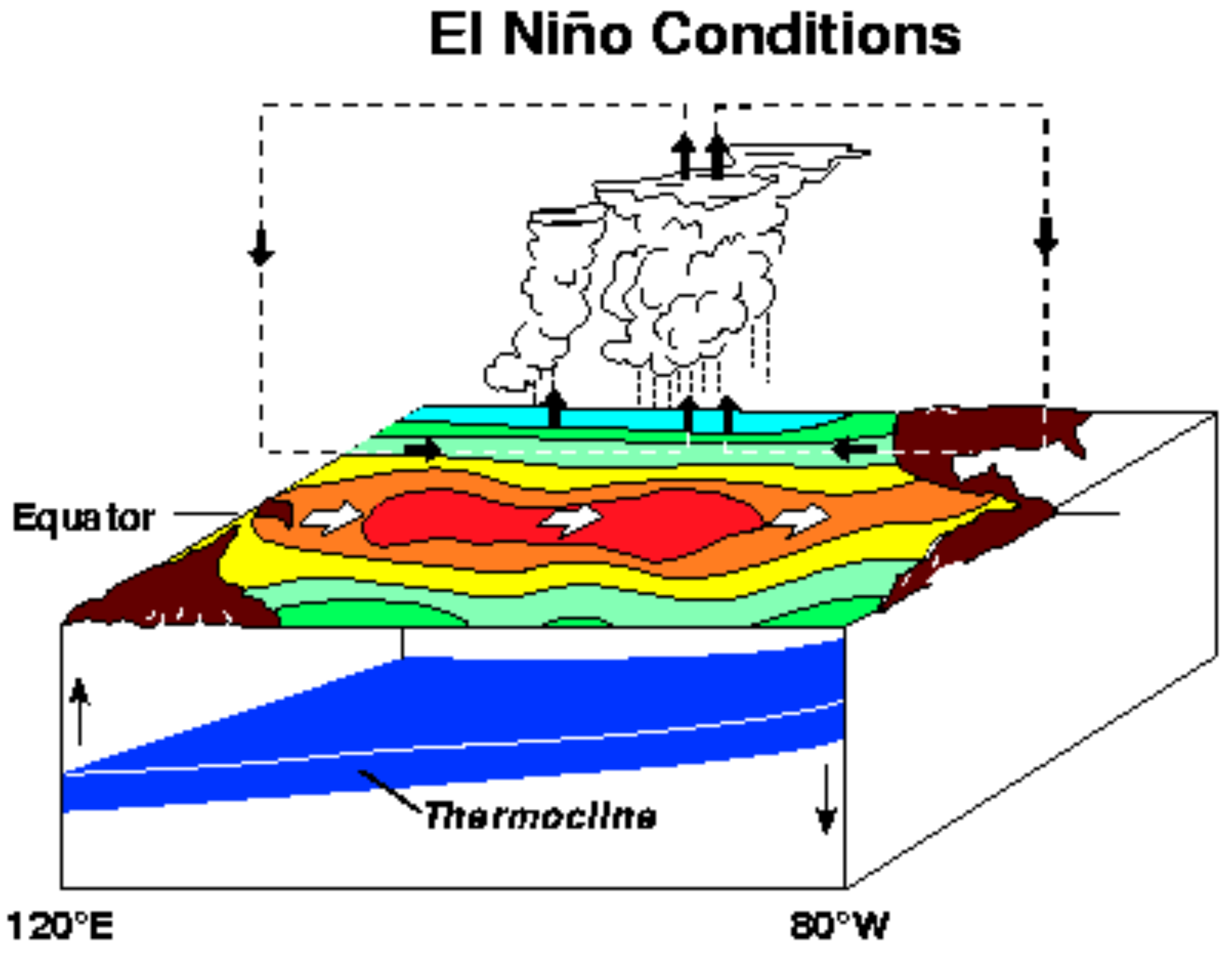
Atmosphere: Large-scale meanders



Idealized air flow of the westerlies at the 500-millibar level. The five long-wavelength undulations, called *Rossby waves*, compose this flow. The jet stream is the fast core of this wavy flow.

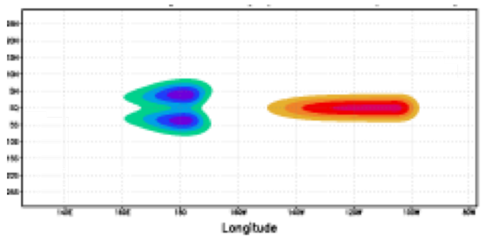
- Coriolis effect
  Scaling of the dynamical equations
  Geostrophy
  Vorticity
  Wind-driven ocean circulation

Upwelling and climate variability

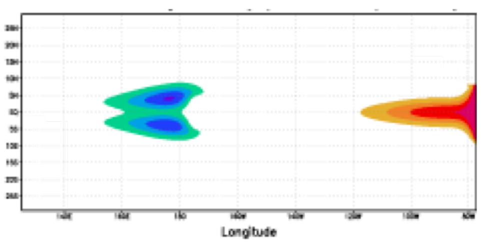


Kelvin wave (3 m/s, 70 days)

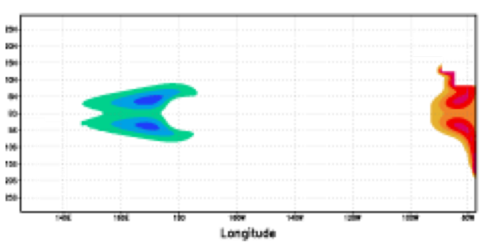
25 days



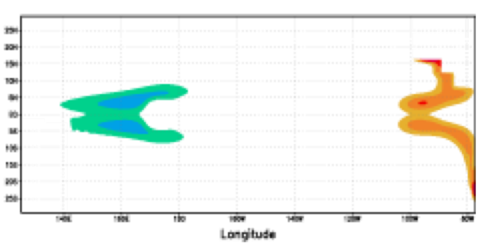
50 days



75 days

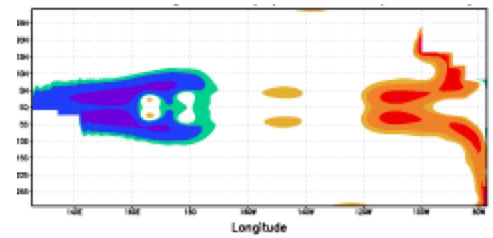


100 days

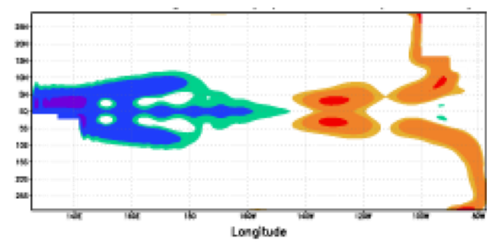


Rossby wave (1m/s, 210 days)

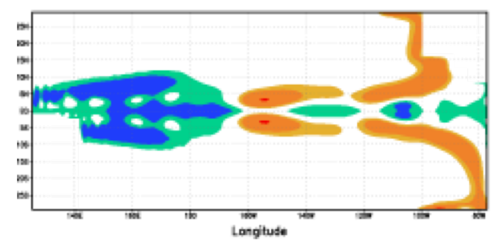
125 days



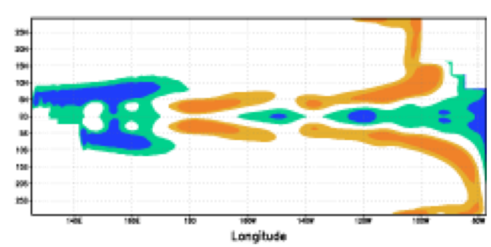
175 days



225 days



275 days



sea surface height anomalies

Ekman Mass Transport

- Integral of the Ekman Velocities down to a depth d:

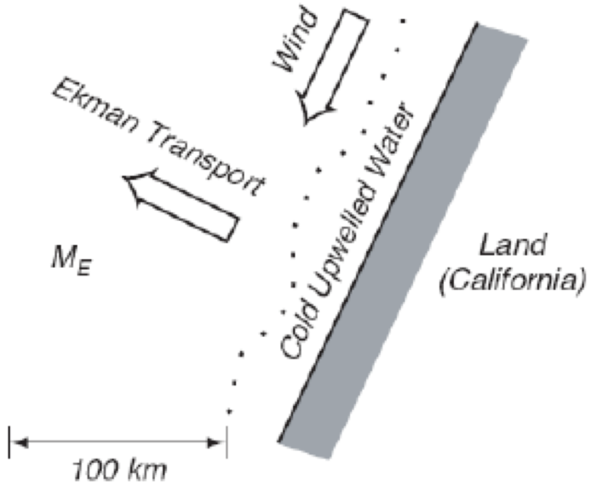
$$M_{Ex} = \int_{-d}^{0} \rho U_E \, dz, \qquad M_{Ey} = \int_{-d}^{0} \rho V_E \, dz$$

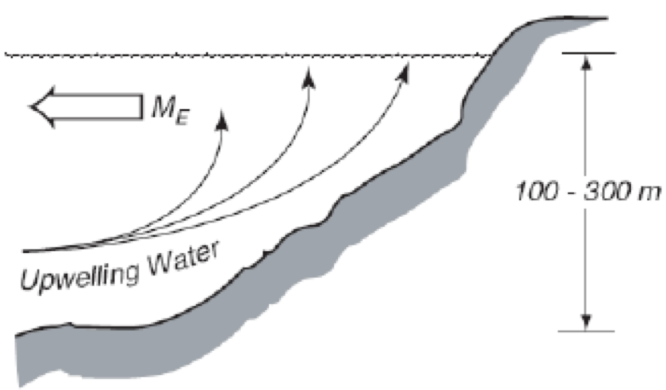
- Ekman transport relates the surface wind stress:

$$f \, M_{Ey} = -T_{xz}(0)$$
$$f \, M_{Ex} = \quad T_{yz}(0)$$

- Mass transport is perpendicular to wind stress
- In the northern hemisphere, *f* is positive, and the mass transport is in the *x* direction, to the east.

Coastal Upwelling





- Upwelling enhances biological productivity, which feeds fisheries.
- Cold upwelled water alters local weather. Weather onshore of regions of upwelling tend to have fog, low stratus clouds, a stable stratified atmosphere, little convection, and little rain.
- Spatial variability of transports in the open ocean leads to upwelling and downwelling, which leads to redistribution of mass in the ocean, which leads to wind-driven geostrophic currents via *Ekman pumping*.

Source: Introduction to Physical Oceanography, http://oceanworld.tamu.edu/home/course_book.htm

Ekman Pumping

- The horizontal variability of the wind blowing on the sea surface leads to horizontal variability of the Ekman transports.
- Because mass must be conserved, the spatial variability of the transports must lead to vertical velocities at the top of the Ekman layer.
- To calculate this velocity, we first integrate the continuity equation in the vertical direction:

$$\rho \int_{-d}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = 0$$

$$\frac{\partial}{\partial x} \int_{-d}^{0} \rho\, u\, dz + \frac{\partial}{\partial y} \int_{-d}^{0} \rho\, v\, dz = -\rho \int_{-d}^{0} \frac{\partial w}{\partial z}\, dz$$

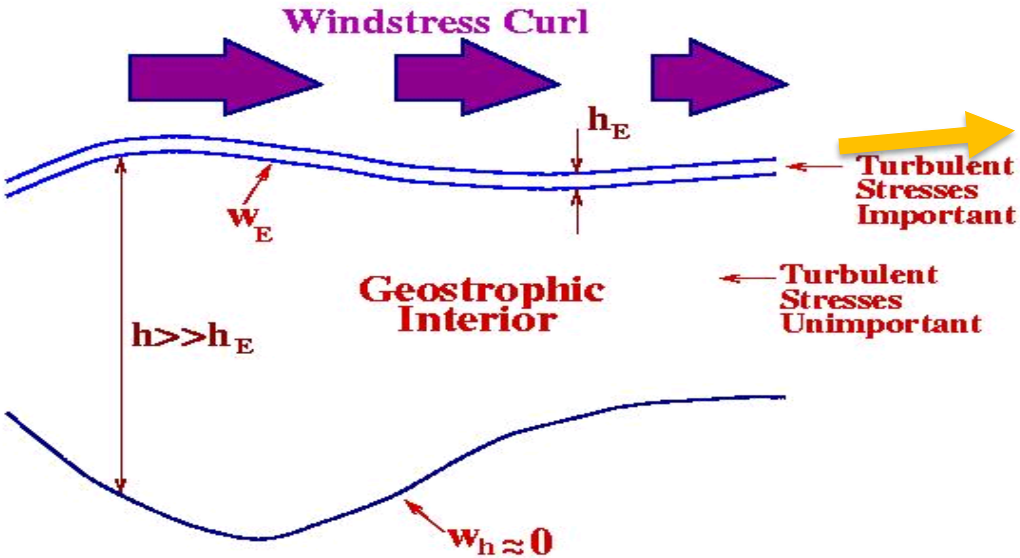
$$\frac{\partial M_{Ex}}{\partial x} + \frac{\partial M_{Ey}}{\partial y} = -\rho \left[w(0) - w(-d) \right]$$

Ekman v. Reality

- Inertial currents dominate
- Flow is nearly independent of depth within the mixed layer on time periods on the order of the inertial period (i.e. the mixed layer moves like a slab)
- Current shear is strongest at the top of the thermocline
- Flow averaged over many inertial periods is almost exactly that calculated by Ekman
- Ekman depth is typically on target with experiments, but velocities are often as much as half the calculated value
- Angle between wind and flow at surface depends on latitude and is near 45 degrees at mid-latitudes

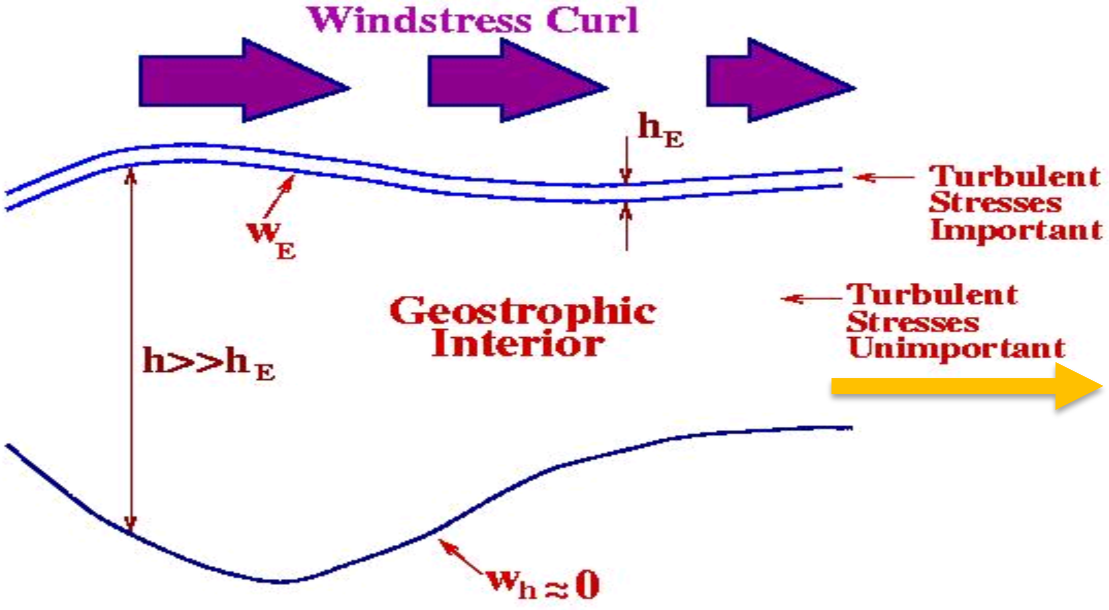
Ekman

- the vertical flow from the surface Ekman layer into the geostrophic interior is



$$w_E = \frac{1}{\rho_0 f}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)$$

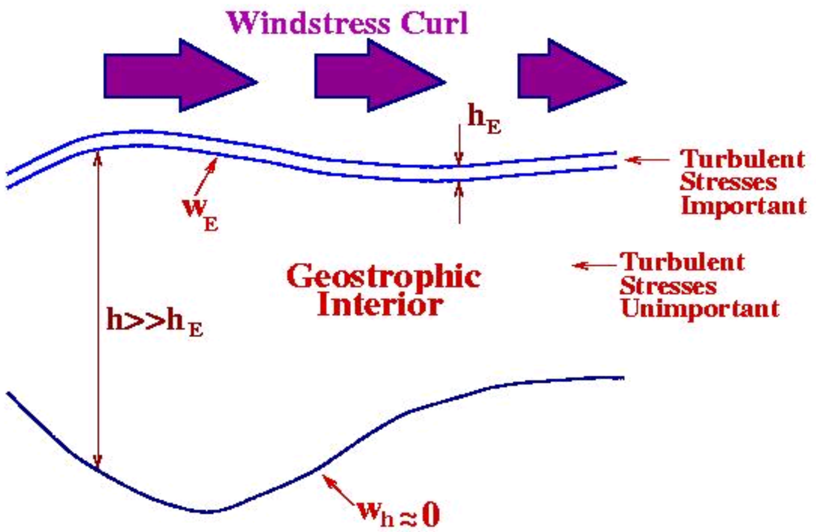
Sverdrup Balance



$$\beta \int_{\text{bot}}^{\text{top}} v \, dz = f \, w\big|_{\text{bot}}^{\text{top}}$$

Geostrophic balance

Sverdrup Balance



$$V \equiv \int_{\text{bot}}^{\text{top}} v \, dz$$

$$w_{\text{top}} = w_E = \frac{1}{\rho_0 f}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)$$

$$\beta V = \frac{1}{\rho_0}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)$$

… relates the integral meridional flow *throughout the vertical extent* of the treated layer to the local windstress curl.

Sverdrup Balance

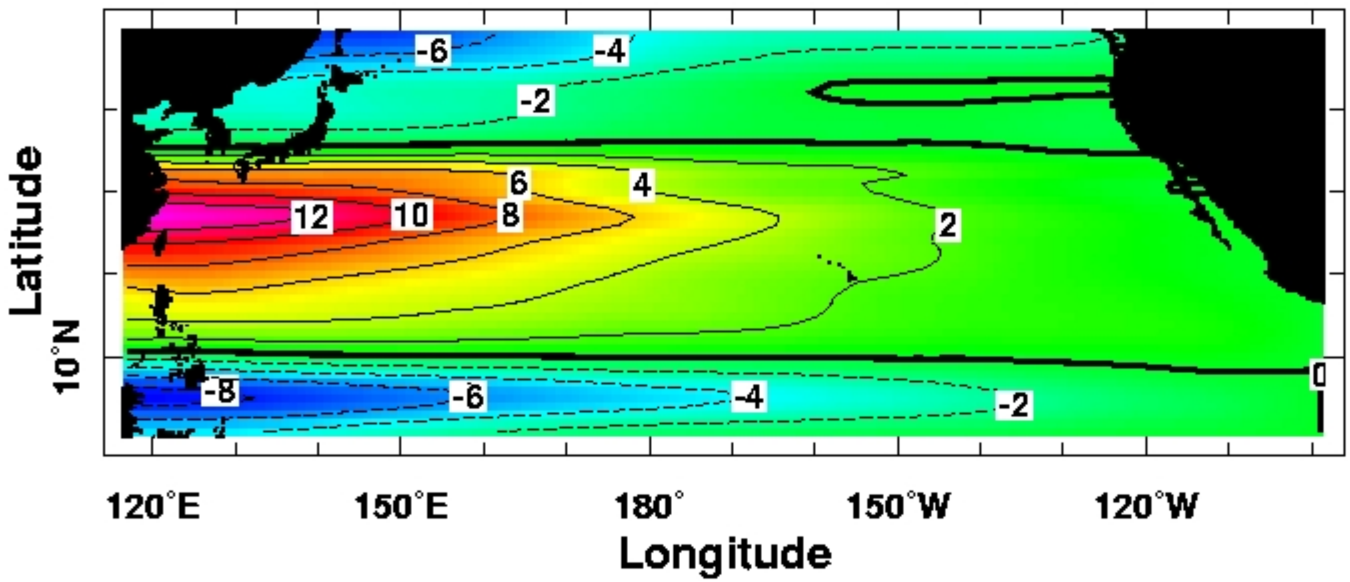
$$\beta V = \frac{1}{\rho_0}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)$$

Sverdrup Stream function

$$\frac{\partial \psi}{\partial x} = \frac{1}{\beta \rho_0}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)$$

$$\psi(x) = \frac{1}{\beta \rho_0}\int_{x_E}^{x}\left(\frac{\partial \tau_y}{\partial x} - \frac{\partial \tau_x}{\partial y}\right)\, dx$$

$$\psi(x,y) = \frac{1}{\beta\rho_0}\int_{x_E}^{x}\left(\frac{\partial\tau_y}{\partial x} - \frac{\partial\tau_x}{\partial y}\right)\, dx$$



Being that the curl is negative throughout the subtropics, it follows that the meridional flux must be everywhere equatorward. But such a situation, if sustained, will progressively empty the midlatitude oceans, while piling-up more and more water along the Equator; a clear physical impossibility! There must be somewhere a return poleward flow that `drains' the Equatorial region while replenishing the midlatitude missing volume.

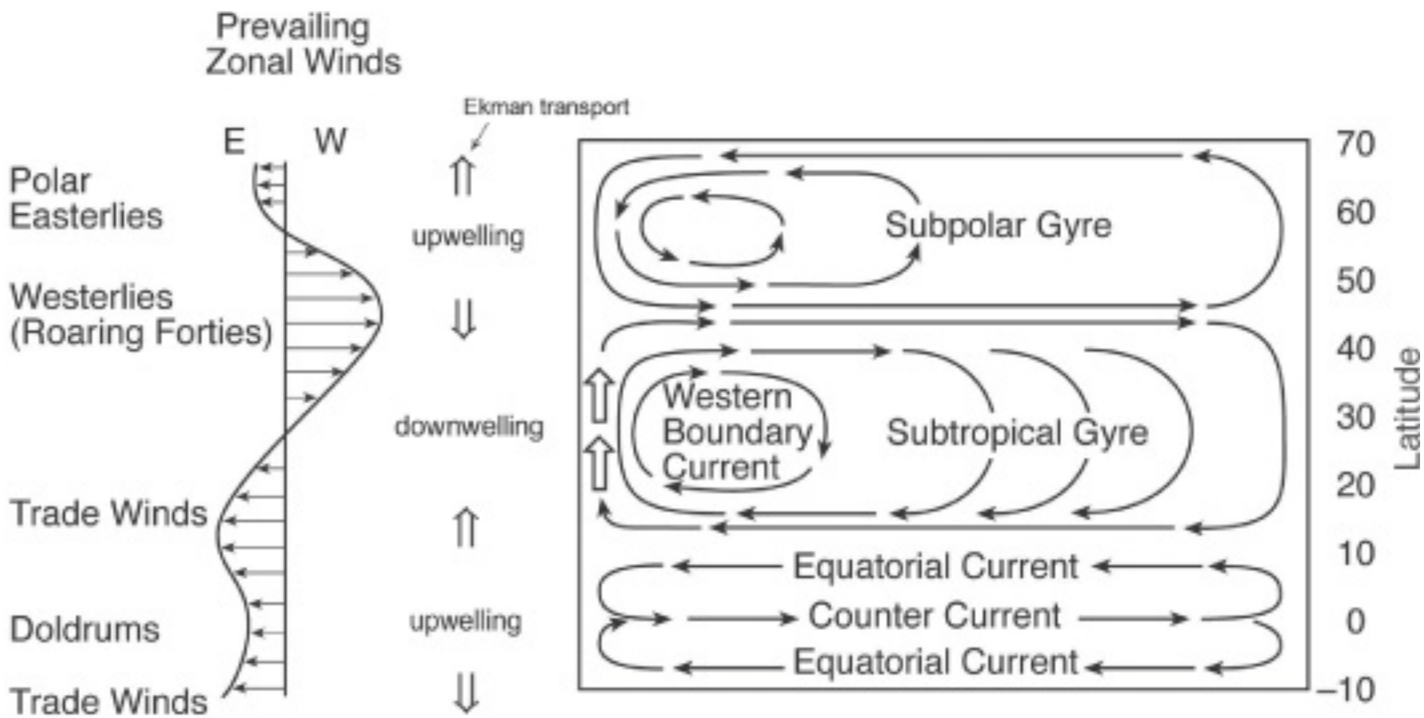


Figure 10.20: Schematic diagram showing the classification of ocean gyres and major ocean current systems and their relation to the prevailing zonal winds. The pattern of Ekman transport and regions of upwelling and downwelling are also marked.

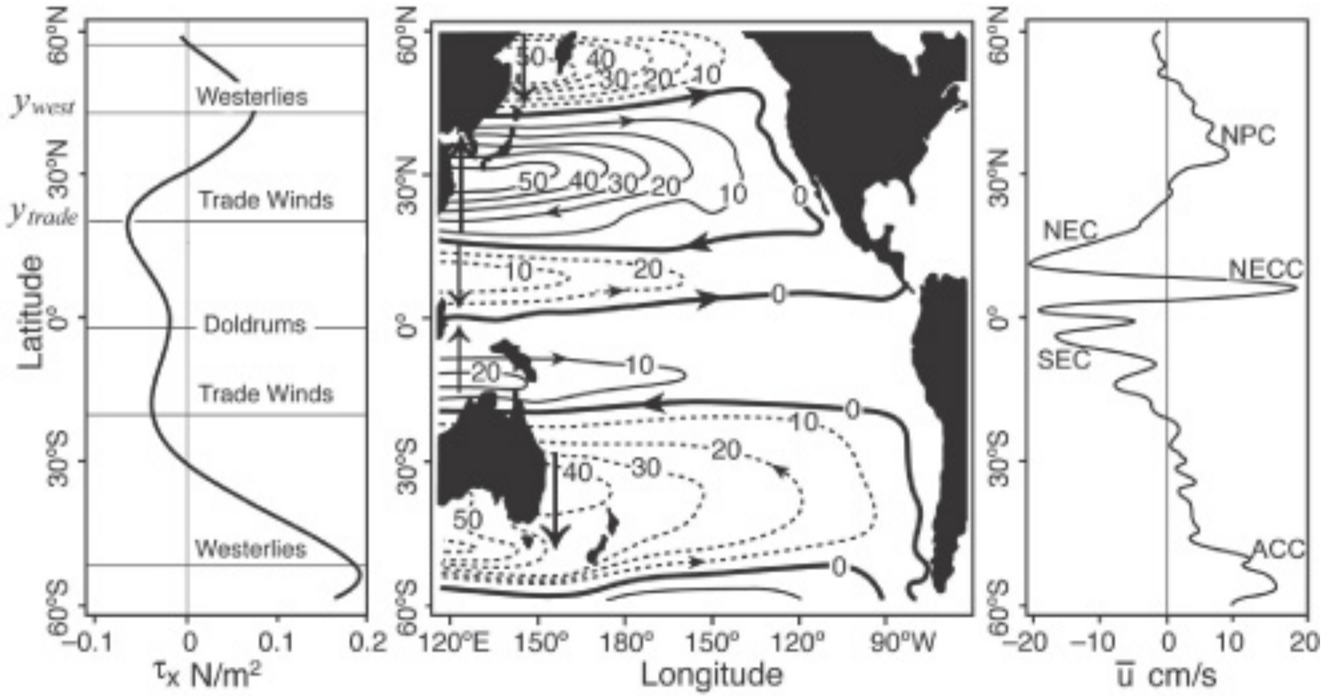
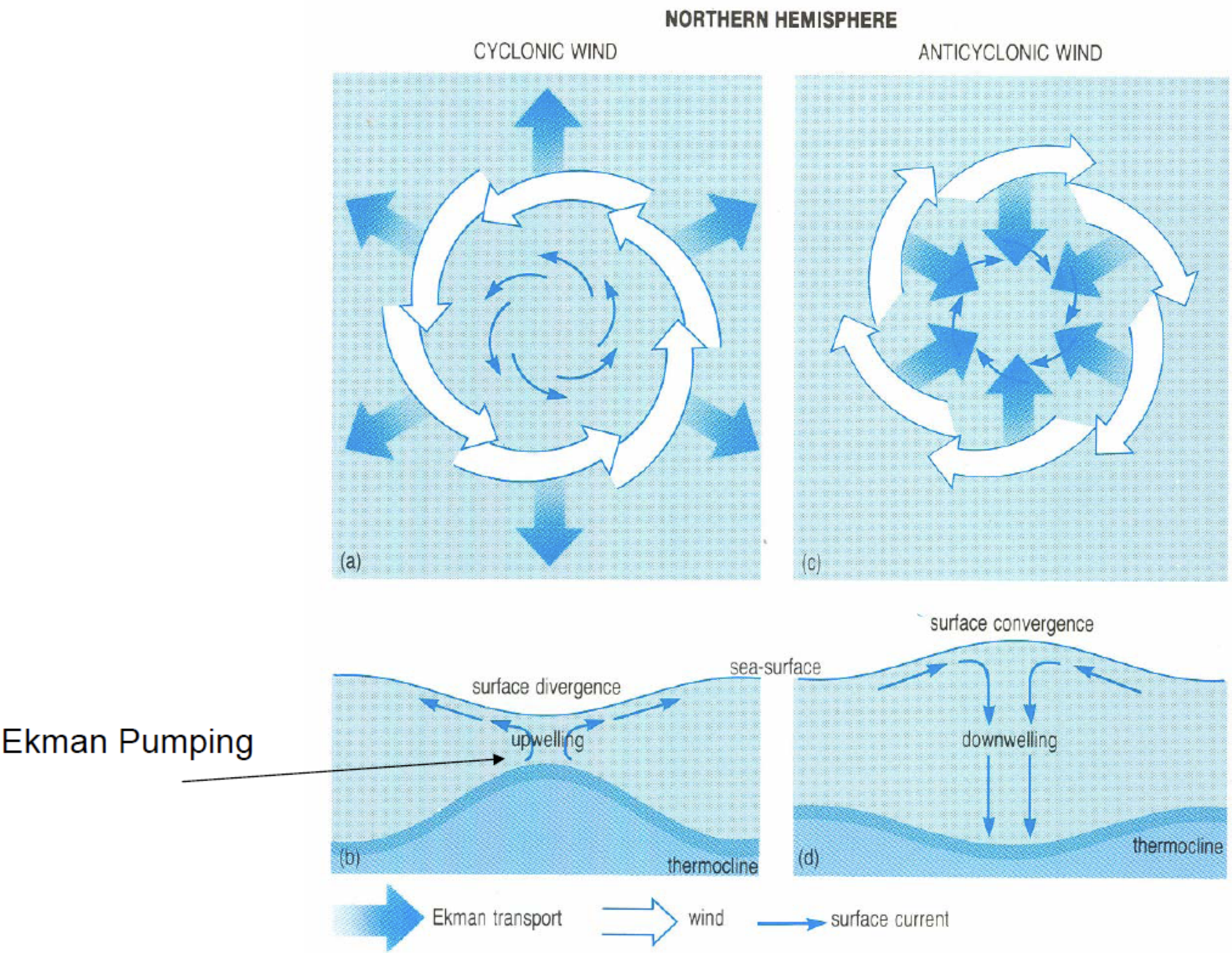
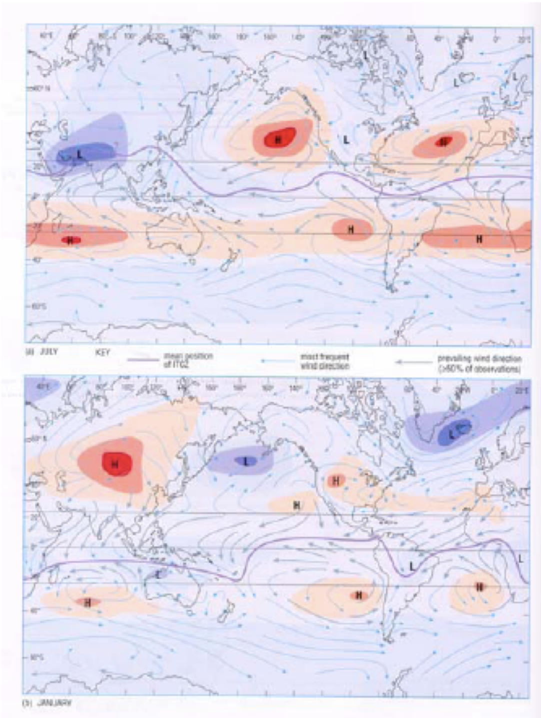
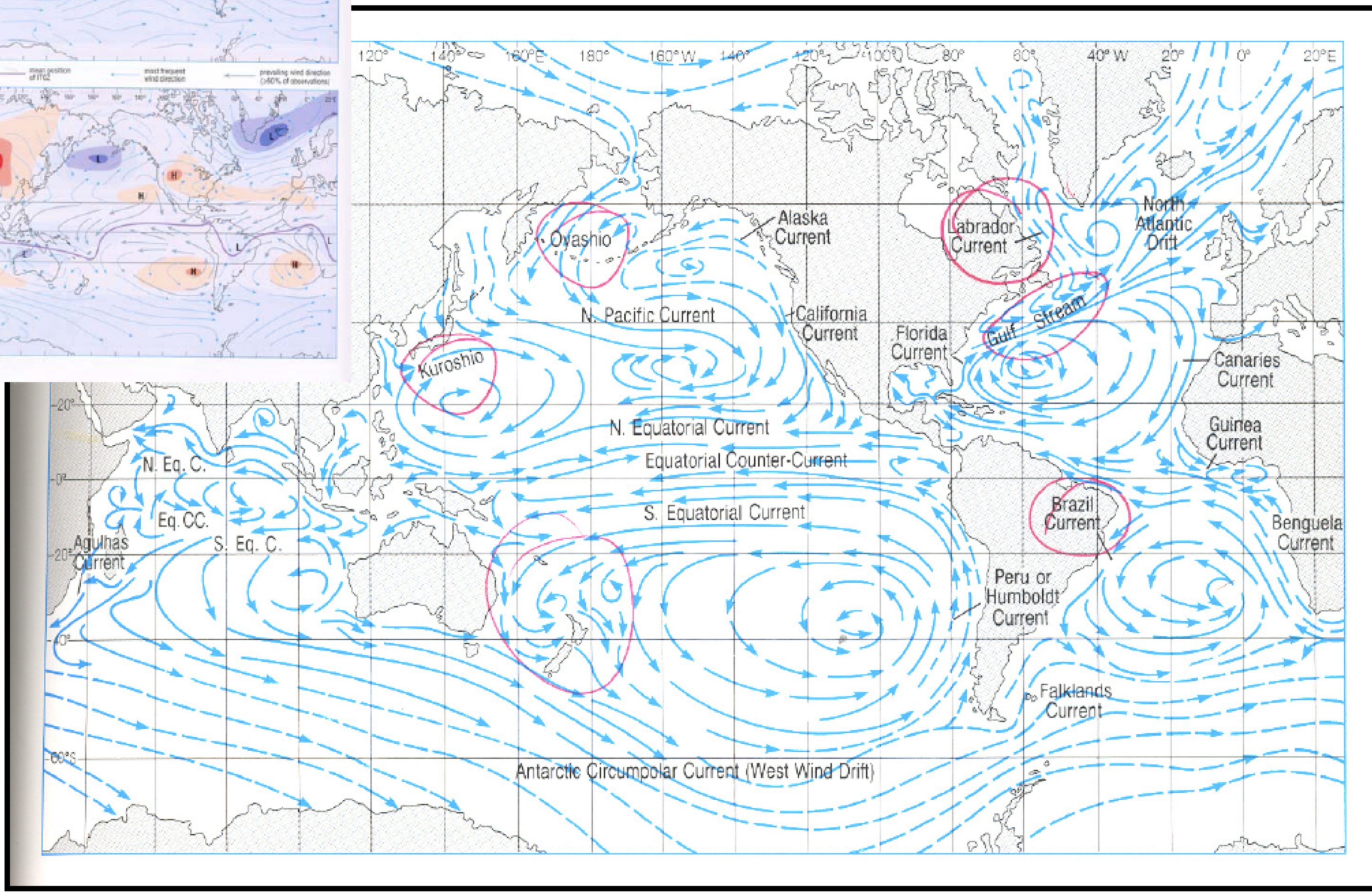


Figure 10.21: (left) The zonal-average of the zonal wind stress over the Pacific Ocean. (middle) The Sverdrup transport stream function (in $Sv = 10^6\ m^3\ s^{-1}$) obtained by evaluation of Eq. 10-20 using climatological wind stresses, Fig. 10.2. Note that no account has been made of islands; we have just integrated right through them. The transport of the western boundary currents (marked by the $N \leftrightarrow S$ arrows) can be read off from Ψ_{west_bdy}. (right) The zonal-average zonal current over the Pacific obtained from surface drifter data shown in Fig. 9.14. Key features corresponding to Fig. 9.13 are indicated.

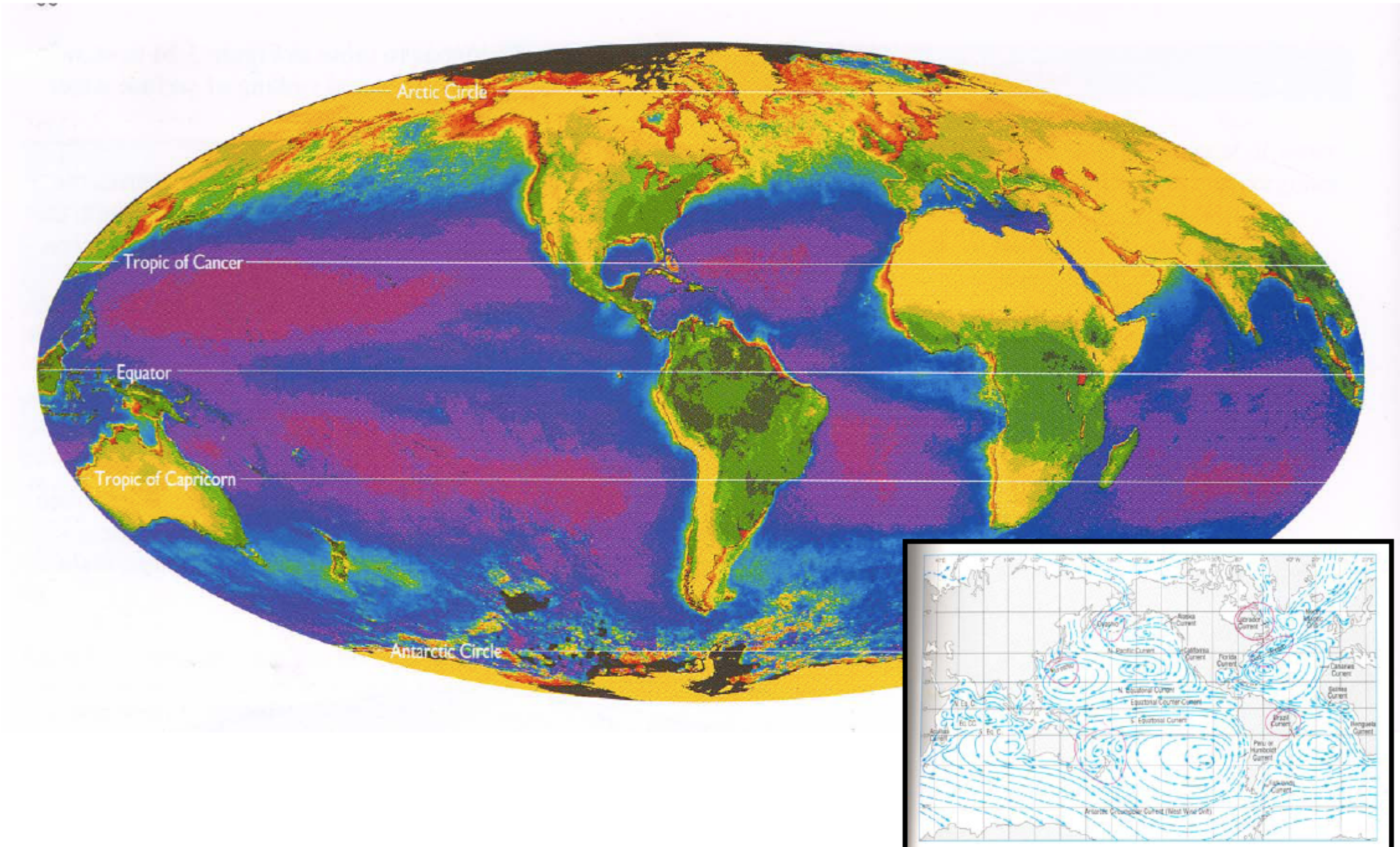

NORTHERN HEMISPHERE
CYCLONIC WIND
ANTICYCLONIC WIND
(a)
(c)
surface convergence
sea-surface
surface divergence
upwelling
downwelling
Ekman Pumping
(b)
thermocline
(d)
thermocline
Ekman transport
wind
surface current

Subtropical/Subpolar gyres

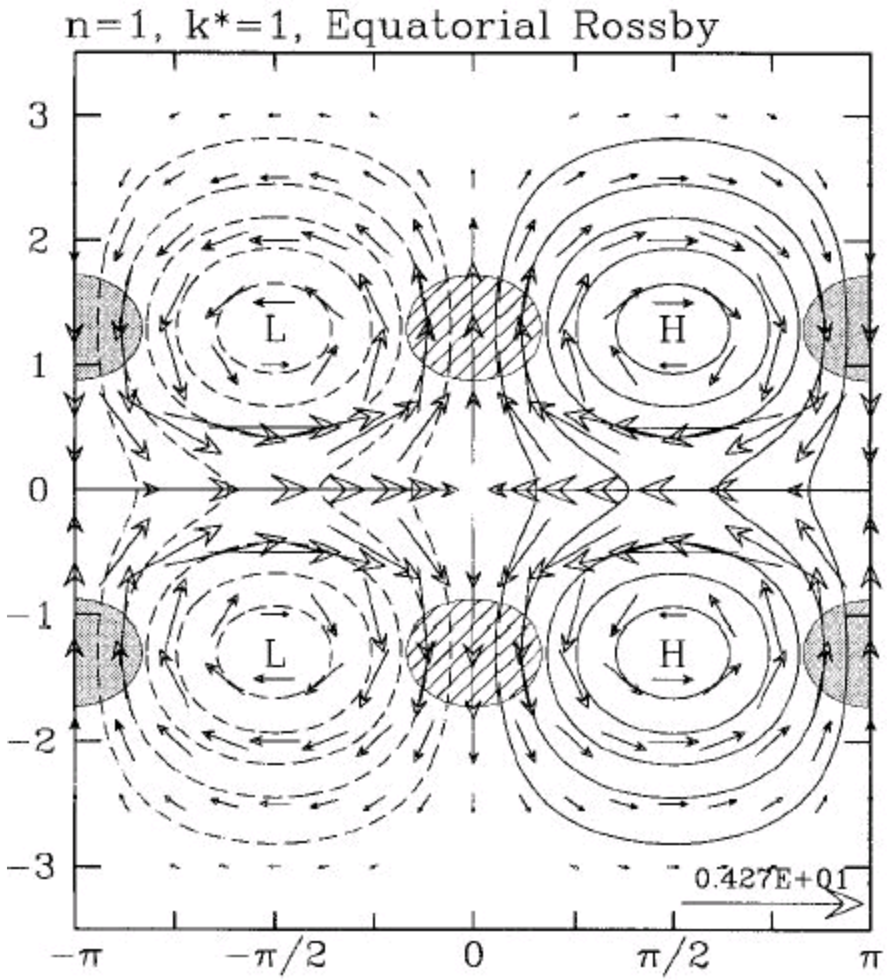




Distribution of Primary Production: red represents regions of high productivity and purple indicating areas of low productivity

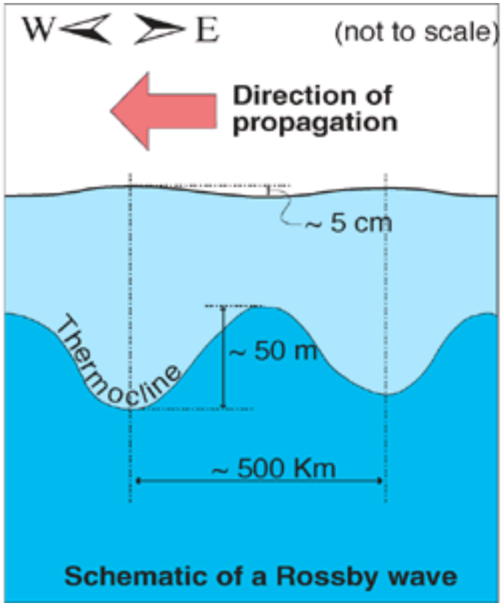


N=1, k=1 Rossby wave



Rossby waves: Ocean

Schematic view "first-mode baroclinic" Rossby wave



Existence in the oceans (Carl-Gustav Rossby, 1930s) has been only indirectly confirmed before the advent of satellite oceanography.

Why is it so difficult to observe them?

It is the big difference in the horizontal and vertical scale of these waves which makes them so difficult to observe.

speed varies with latitude and increases equatorward, order of just a few cm/s

8.3.3 Extratropical Rossby Waves

From the equations (8.7,8.8,8.9), we drop the term $\partial_t \eta$ and introduce the stream function ψ through

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \tag{8.36}$$

such that (8.9) is fulfilled. Taking $\frac{\partial}{\partial y}$ of (8.7) and subtract $\frac{\partial}{\partial x}$ of (8.8) elimintates the η term as in section 4.4:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = -\beta \frac{\partial \psi}{\partial x} \tag{8.37}$$

With the ansatz

$$\psi = \exp(ikx + ily - i\omega t) \tag{8.38}$$

and assumption that β is just a parameter, ω is given by

$$\omega(k,l) = -\frac{\beta k}{k^2 + l^2}, \tag{8.39}$$

Exercise 71 – Rossby waves

Consider the vorticity equation

$$\frac{D}{Dt}[(\zeta + f)/h] = 0 \tag{8.41}$$

with h =const., u and v are the velocity components.

1. Assume a mean flow with constant zonal velocity U

$$u = U = const > 0 \tag{8.42}$$

and a varying north-south component

$$v = v(x, t) \tag{8.43}$$

Shallow Water Model

Case b=0

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} - f v &= -g\,\frac{\partial h}{\partial x}\\
\frac{\partial v}{\partial t} + u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} + f u &= -g\,\frac{\partial h}{\partial y}\\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0.
\end{aligned}$$

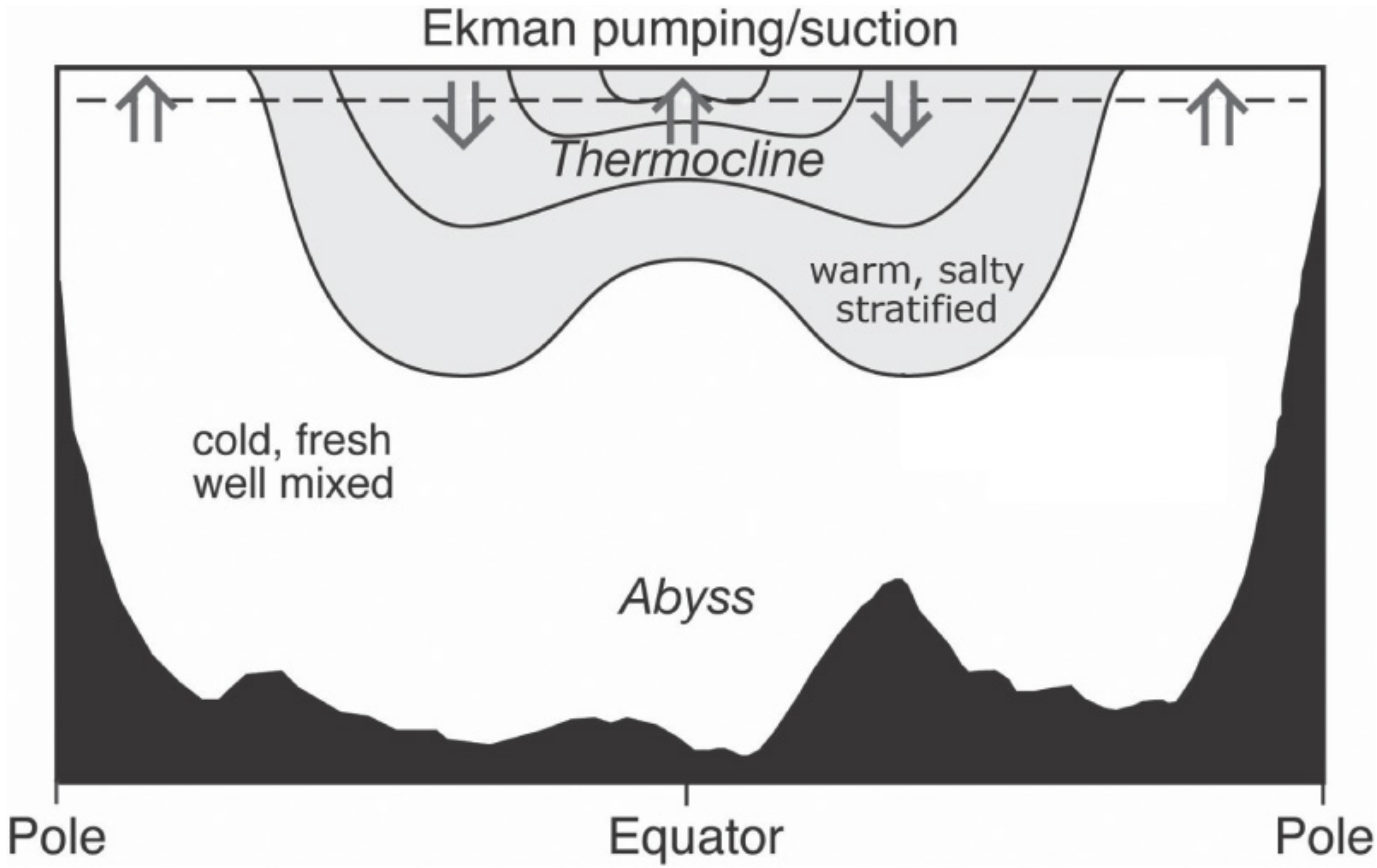
This is a formulation that we will encounter in layered models !

Shallow Water Model

Case b=0

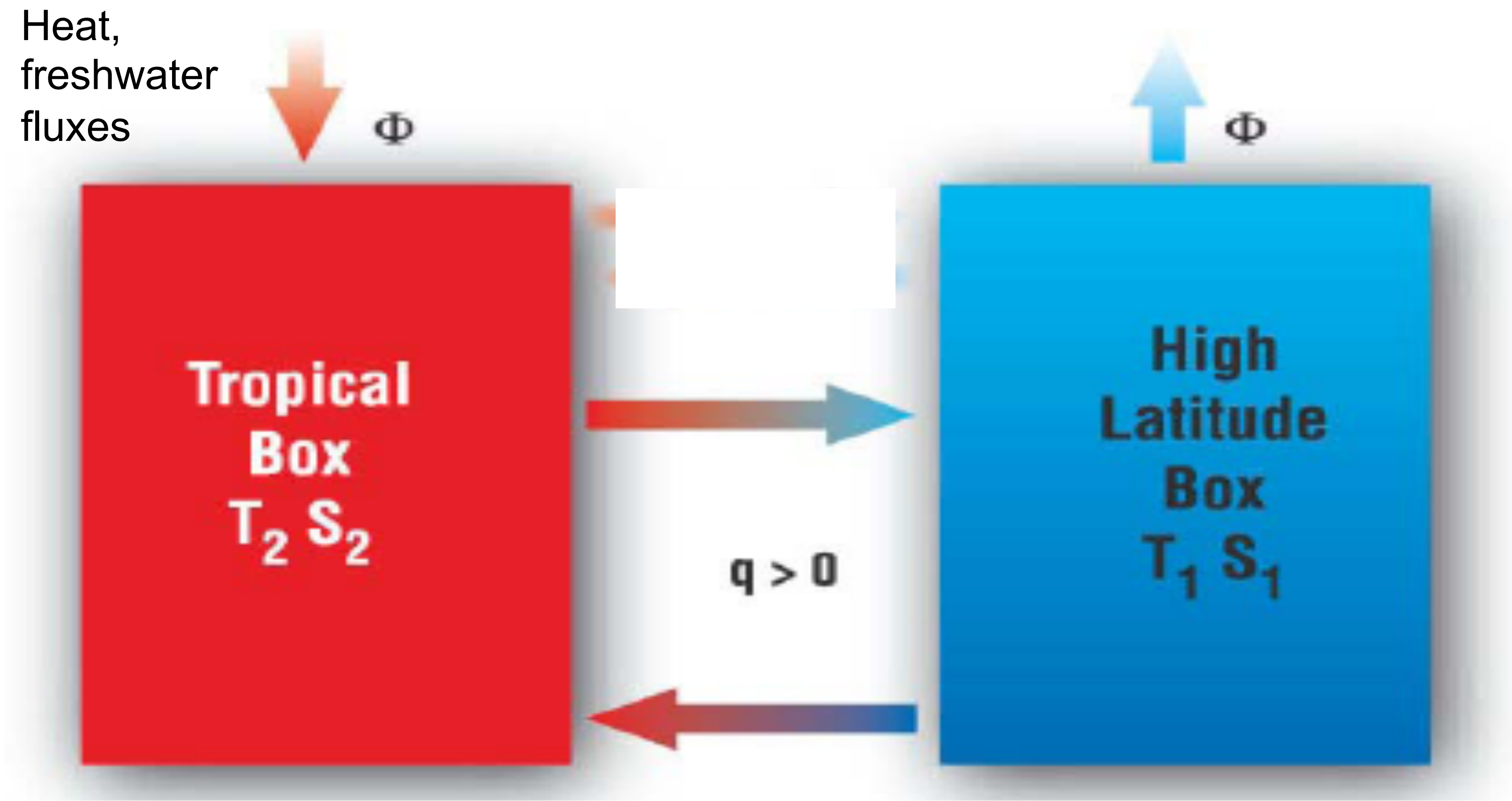
$$
\begin{aligned}
\frac{\partial u}{\partial t} \quad - f v &= -g \frac{\partial h}{\partial x} \\
\frac{\partial v}{\partial t} \quad + f u &= -g \frac{\partial h}{\partial y} \\
\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) + \frac{\partial}{\partial y}(hv) &= 0.
\end{aligned}
$$

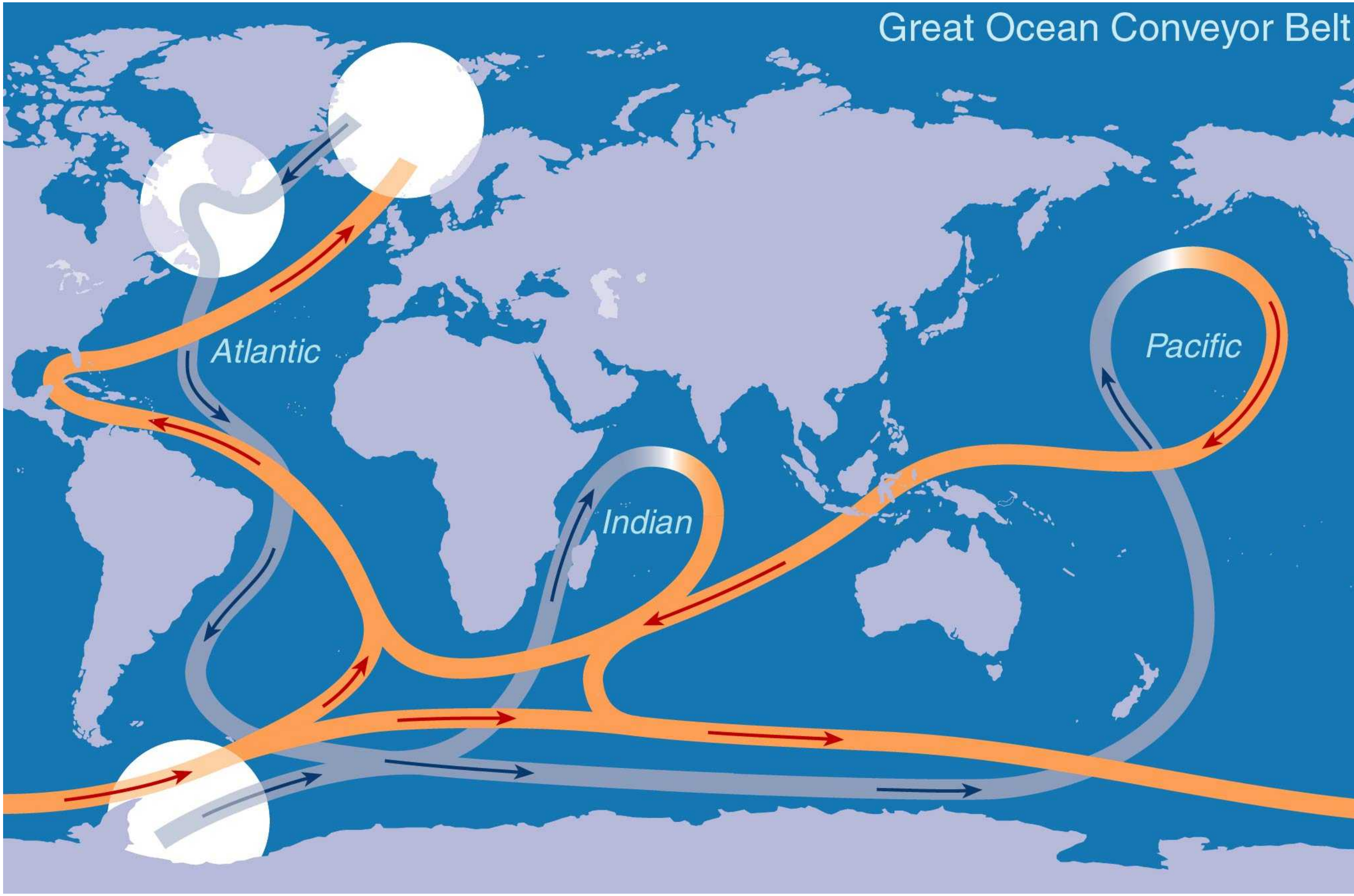
Ocean layers



- Steady winds blowing on the sea surface produce a thin, horizontal boundary layer, the *Ekman layer.*
- *thin: a few-hundred meters thick, compared with the depth of the water in the deep ocean.*
- *A similar boundary layer exists at the bottom of the ocean and at the bottom of the atmosphere just above the sea surface, the planetary boundary layer or frictional layer .*

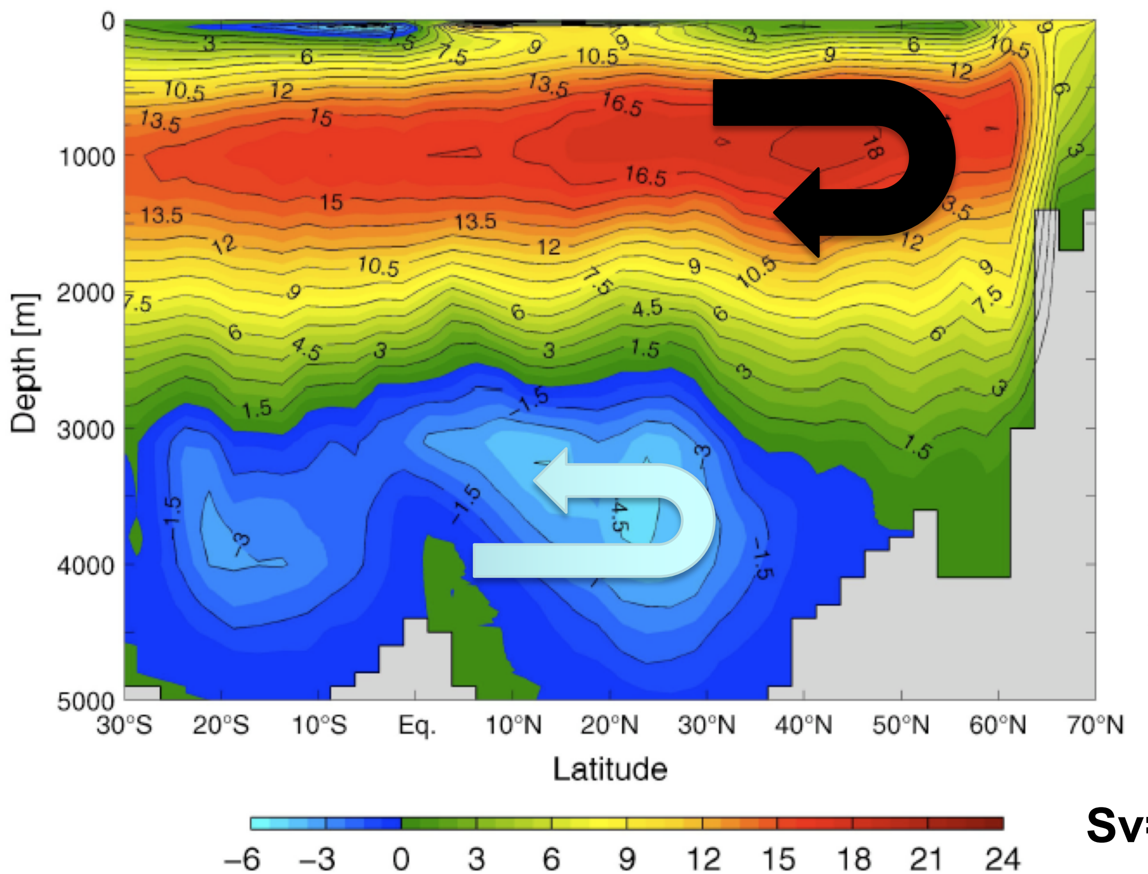
Stommel (1961) Box Model




Great Ocean Conveyor Belt
Atlantic
Pacific
Indian

Meridional overturning circulation

Atlantic Ocean deep sea circulation



NADW: 18 Sv

AABW: 4 Sv

Sv=10^6 m^3/s

Symmetric solution

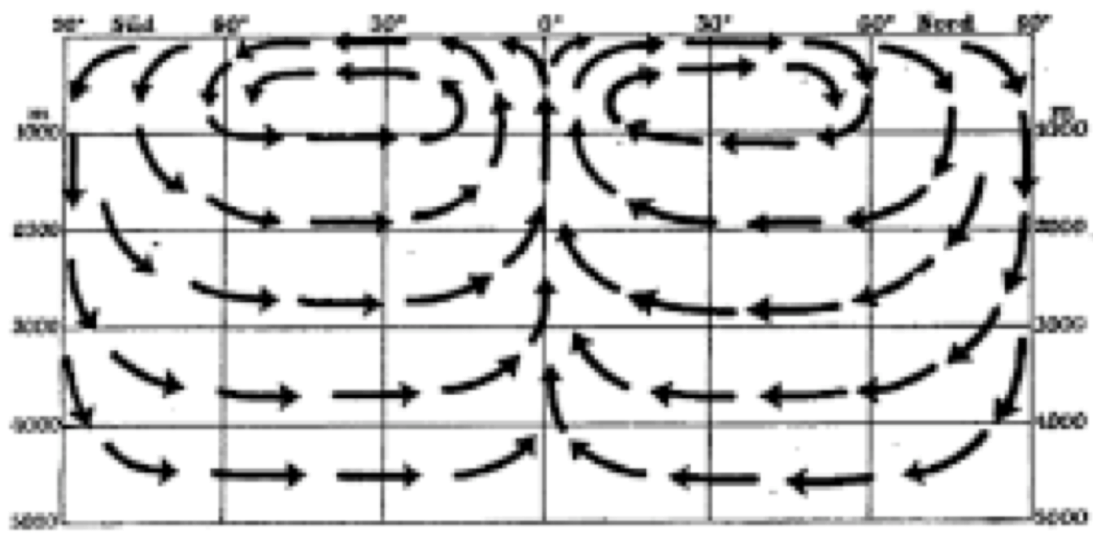


Figure 2.15: Atlantic circulation model according to (von Lenz, 1847a, b), figure after (Merz and Wüst, 1922)

Simple Model of MOC

It is instructive to derive a simple concept of the meridional overturning based on vorticity dynamics in the (y,z)-plane. The dynamical model in two dimensions read

$$\frac{d}{dt}v = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu - \kappa v \tag{2.92}$$

$$\frac{d}{dt}w = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g}{\rho_0}(\rho - \rho_0) - \kappa w \tag{2.93}$$

with κ as parameter for Rayleigh friction. Using the continuity equation

$$0 = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{2.94}$$

one can introduce a streamfunction $\Phi(y,z)$ with $v = \partial_z \Phi$ and $w = -\partial_y \Phi$. The associated vorticity equation in the (y,z)-plane is therefore

$$\frac{d}{dt}\nabla^2\Phi = -f\frac{\partial u}{\partial z} + \frac{g}{\rho_0}\frac{\partial \rho}{\partial y} - \kappa\nabla^2\Phi \tag{2.95}$$

We can choose the ansatz[†] satisfying that the normal velocity at the boundary vanishes, $\Phi = 0$:

$$\Phi = \Phi_{max}\ \sin(\pi y/L) \times \sin(\pi z/D) \tag{2.97}$$

MOC continued

The parameters L and D dentote the meridional and depth extend. With the assumption that the Coriolis term is absorbed into the viscous terms, we derive:

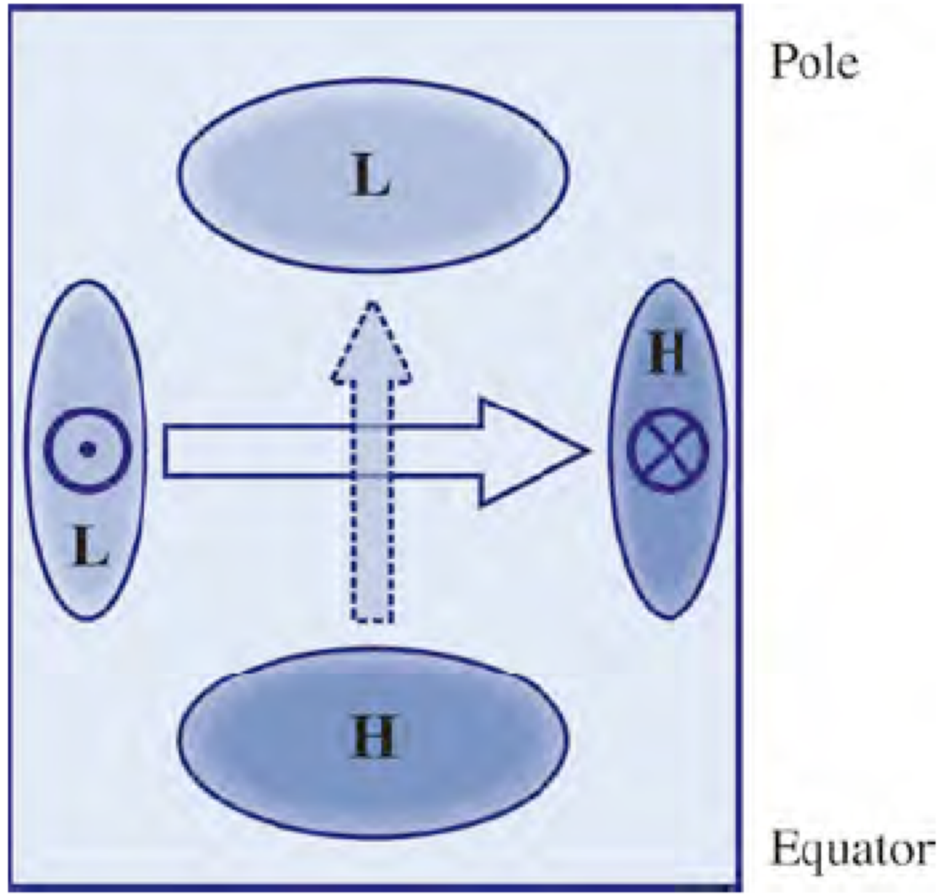
$$\frac{d}{dt}\Phi_{max} \quad = \quad \frac{a}{\rho_0}(\rho_{north} - \rho_{south}) \; - \; \kappa\Phi_{max} \tag{2.98}$$

with $a = gLD^2/4(L^2 + D^2)$.[‡]

This shows that the overturning circulation depends on the density differences on the right and left boxes. In the literature, (2.98) is simplified to a diagnostic relation

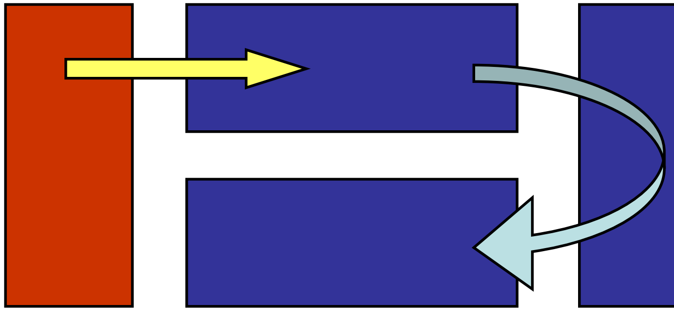
$$\Phi_{max} = \frac{a}{\rho_0\,\kappa}\,(\rho_{north} - \rho_{south}) \tag{2.99}$$

because the adjustement of Φ_{max} is quasi-instantaneous due to Kelvin waves (section 2.3).



Schematic of the surface flow driven by a north-south density gradient in an ocean basin. The primary north-south gradient – as a result of the surface forcing – is in balance with an eastward geostrophic current which generates a secondary high and low pressure system. This, in turn, drives a northward geostrophic current, the upper branch of the

Conceptual Model of MOC



South- Equator North- Atlantic

T and S

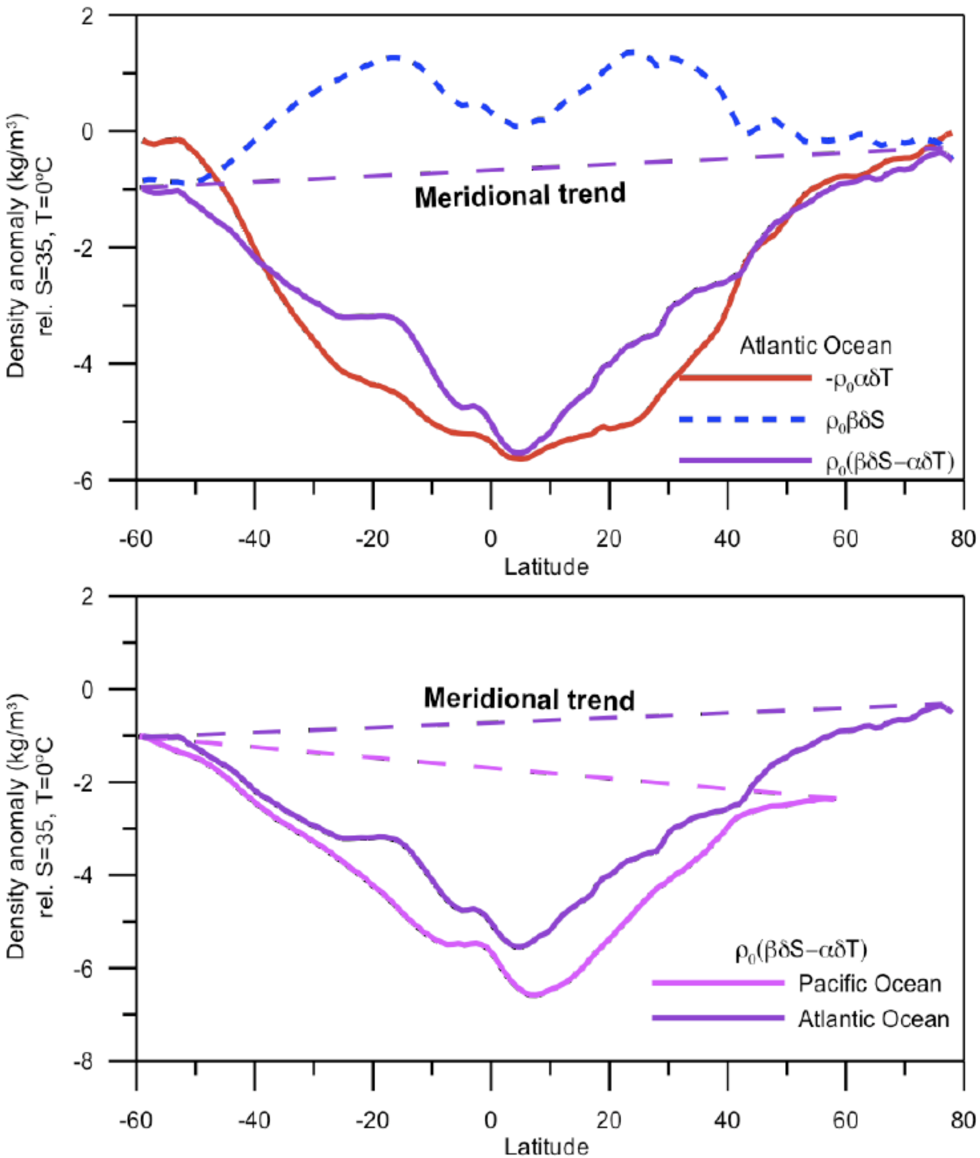
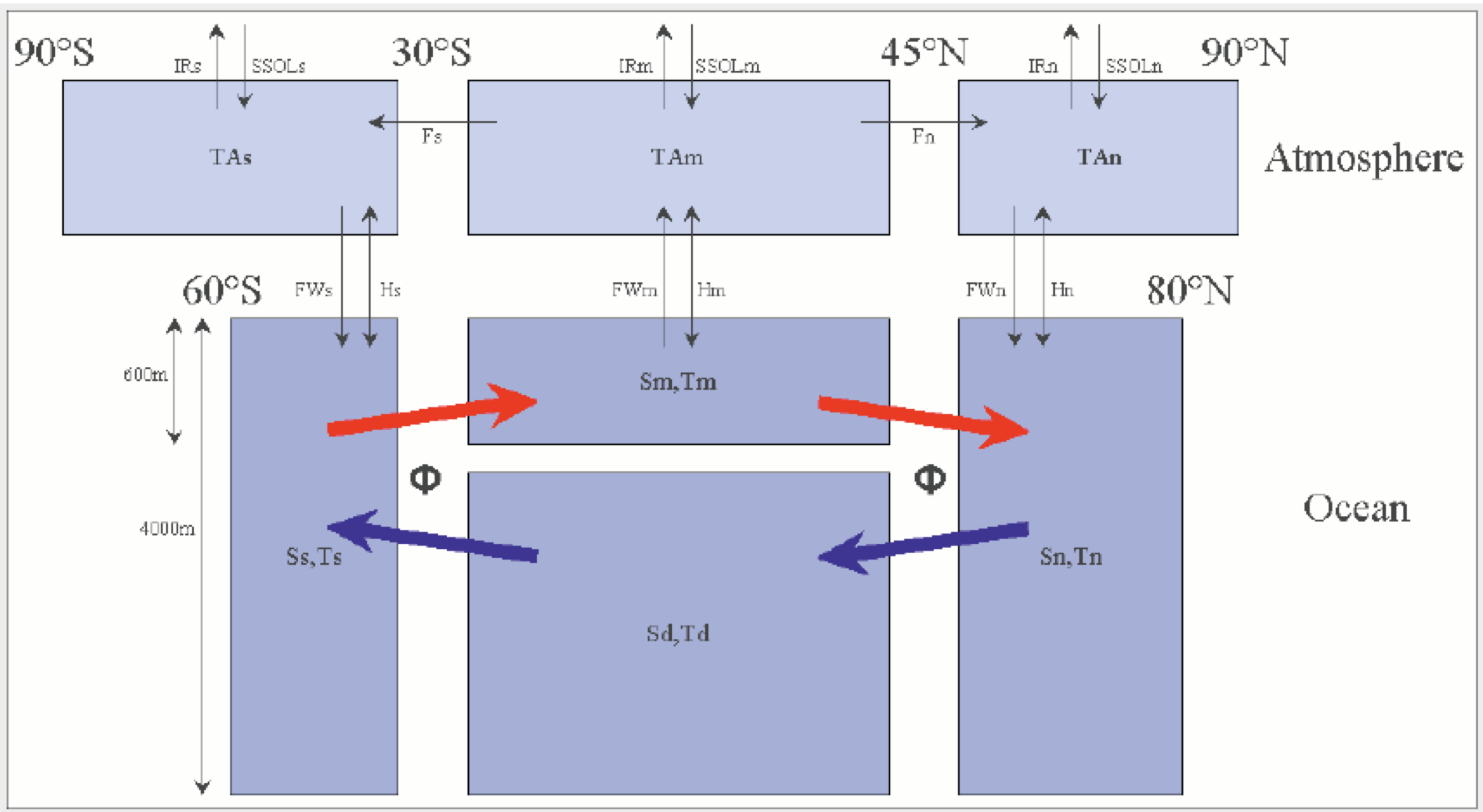


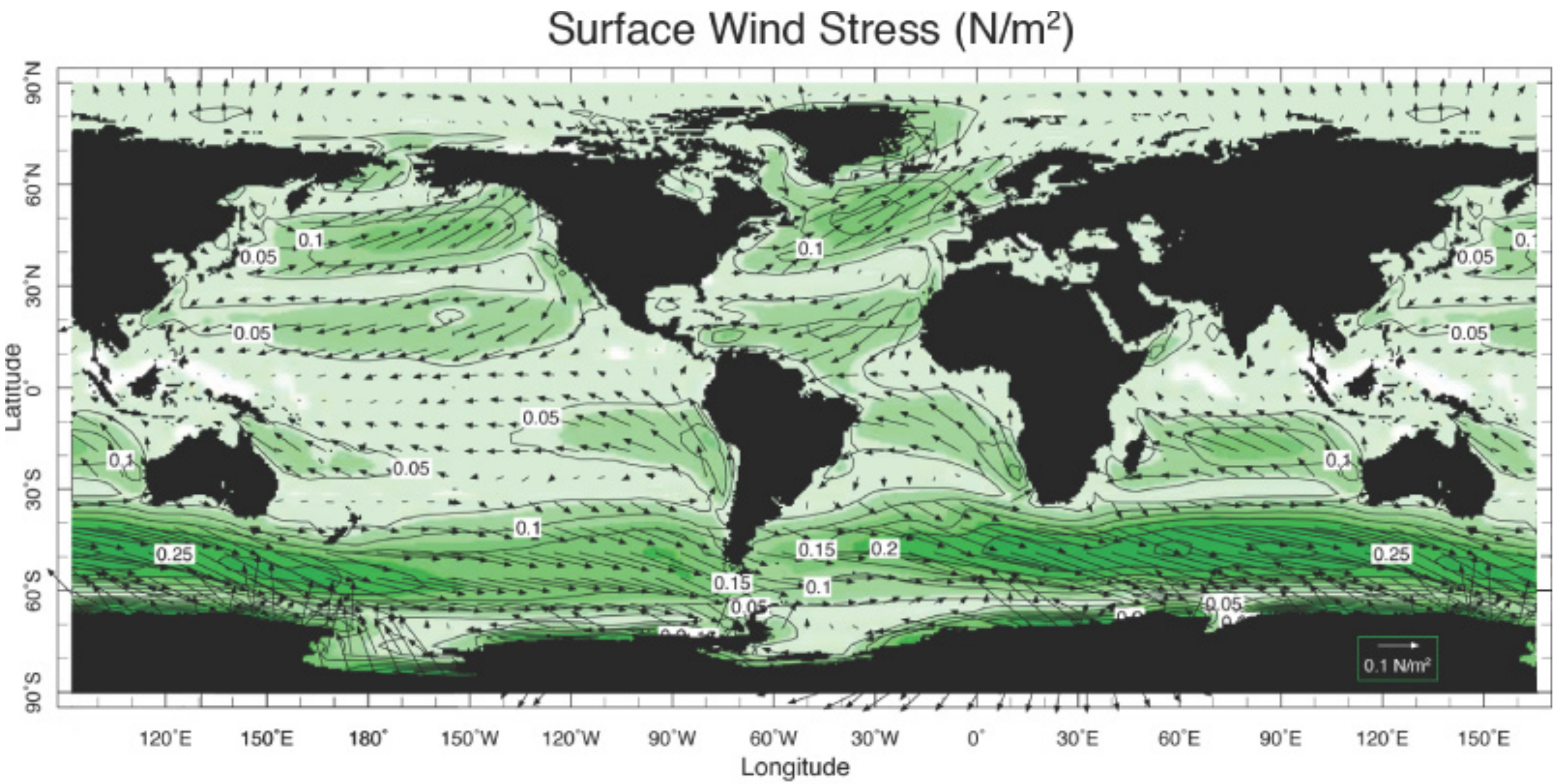
Figure 2.16: a) The Atlantic surface density is mainly related to temperature differences. b) But the pole-to-pole differences are caused by salinity differences.

Box Models

- Stommel ‘s model almost completey ignored (25 years)
- Rooth, 1982: Two hemisphere counterpart,
- Unaware of Stommel (1961) model
- Rooth suggested to F. Bryan: test with a GCM

Application: Climate-Box-Model



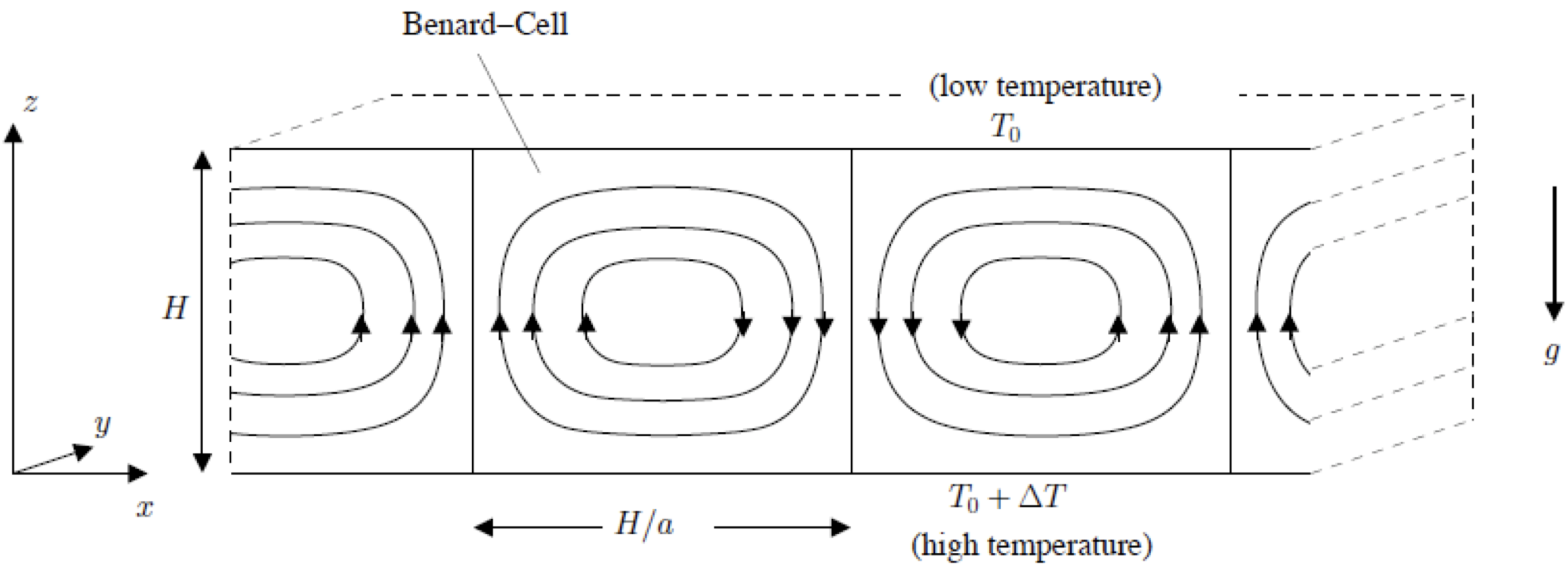


Surface Wind Stress (N/m²)
Latitude
90°N
60°N
30°N
0°
30°S
60°S
90°S
120°E
150°E
180°
150°W
120°W
90°W
60°W
30°W
0°
30°E
60°E
90°E
120°E
150°E
Longitude
0.1
0.05
0.25
0.15
0.2
0.1 N/m²

others

- Rayleigh-Bénard convection and the Lorenz system
  Bifurcations
- Deep water circulation

Applicaton: Rayleigh-Bénard convection



$$T(x, y, z = H) = T_0$$

$$T(x, y, z = 0) = T_0 + \Delta T$$

Zero Solution

Such a system possesses a steady-state solution in which there is no motion, and the temperature varies linearly with depth:

$$\begin{aligned} \boldsymbol{v} &= \mathbf{0} \\ T &= T_0 + \left(1 - \frac{z}{H}\right) \Delta T \end{aligned} \tag{2.3}$$

When this solution becomes unstable, convection should develop.

Then, the dynamics can be formulated for Ψ and Θ, which is the departure of temperature from that occurring in the state of no convection (2.3):

$$\partial_t \nabla^2 \Psi = -\frac{\partial(\Psi, \nabla^2 \Psi)}{\partial(x, z)} + \nu \nabla^4 \Psi + g\alpha \frac{\partial \Theta}{\partial x} \tag{2.10}$$

$$\partial_t \Theta = -\frac{\partial(\Psi, \Theta)}{\partial(x, z)} + \frac{\Delta T}{H} \frac{\partial \Psi}{\partial x} + \kappa \nabla^2 \Theta \quad . \tag{2.11}$$

The notation

$$\frac{\partial(\Psi, \nabla^2 \Psi)}{\partial(x, z)} \quad ,$$

known as the determinant of the Jacobian matrix (or simply "the Jacobian"), stands for

$$\frac{\partial \Psi}{\partial x} \frac{\nabla^2 \Psi}{\partial z} - \frac{\partial \Psi}{\partial z} \frac{\nabla^2 \Psi}{\partial x} \quad .$$

The constants g, α, ν, and κ denote, respectively, the acceleration of gravity, the coefficient of thermal expansion, the kinematic viscosity, and the thermal conductivity of the fluid. The problem is most tractable (analytically) when both the upper- and the lower-boundaries are taken to be free, in which case Ψ and $\nabla^2 \Psi$ vanish at both boundaries.

Laplace transform

The Fourier transform is intimately related with the Laplace transform $F(s)$, which is also used for the solution of differential equations and the analysis of filters (`https://en.wikipedia.org/wiki/Laplace_transform`). We introduce the complex variable $s = -i\omega$.

$$\mathcal{L}\{x(t)\} = F(s) = \int_0^\infty e^{-st} x(t) dt \tag{1.31}$$

It follows (integration by parts for 1.32)

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sF(s) - x(0) \tag{1.32}$$

1.3 Covariance and spectrum

A stationary process exhibits an autocovariance function of the form

$$Cov(\tau) = \langle (x(t+\tau) - \langle x \rangle)(x(t) - \langle x \rangle) \rangle \tag{1.58}$$

where $\langle \ldots \rangle$ denotes the statistical ensemble mean.[6] Normalized to the variance (i.e. the autocovariance function at $\tau = 0$) one gets the autocorrelation function $C(\tau)$:

$$C(\tau) = Cov(\tau)/Cov(0) \quad . \tag{1.59}$$

Many stochastic processes in nature exhibit short-range correlations, which decay exponentially:

$$C(\tau) \sim \exp(-\tau/\tau_0), \text{ for } \tau \to \infty \tag{1.60}$$

The Fourier transformation of the random variable x is

$$\hat{x}(\omega) = \int_{\mathsf{R}} x(t) e^{i\omega t}\, dt \quad = \lim_{T \to \infty} \int_{-T/2}^{T/2} x(t) e^{i\omega t}\, dt \tag{1.62}$$

and is also a ramdom variable, but its power spectral density $S(\omega)$ is not:

$$S(\omega) \;\; := \;\; \left\langle \hat{x}\hat{x}^{+} \right\rangle = \left\langle |\hat{x}(\omega)|^2 \right\rangle \quad . \tag{1.63}$$

$$S(\omega) = \widehat{Cov(\tau)} \quad ,$$