Test Exam Dynamics II (Summer 2021)

Lecturer: Prof. Dr. G. Lohmann

Time: July 5, 2021, 15:00-17:00h

Instructions before you start: *The perfect score for the exam is* 100 *points, although the sum of the problems is* 150*. Therefore, you can choose among the problems to solve.* 50 *points are necessary for the course. Keep in mind that each problem has a different number of points.*
You are allowed to use a calculator & pen. Collaboration or use of alternative sources of information is not allowed. Good luck!

1. **Questions about fluid mechanics (*6 points*, for each Q *2 points*).**

   a) Name three different dimensionless parameters which can characterize the flow.

   b) ν denotes the kinematic viscosity, L a length-scale L determined by the geometry of the flow, and U a characteristic velocity.
   Is the dimensionless Reynolds number given as $Re = U/(L\nu)$?

   In which context does the Reynolds number play a role?

   c) Describe in words the Rayleigh-Bénard instability (Fig. 1). The basic state possesses a steady-state solution in which there is no motion, and the temperature varies linearly with depth:

$$u = w = 0 \tag{1}$$

$$T_{eq} = T_0 + \left(1 - \frac{z}{H}\right)\Delta T \tag{2}$$

   When this solution becomes unstable, ... (please continue)

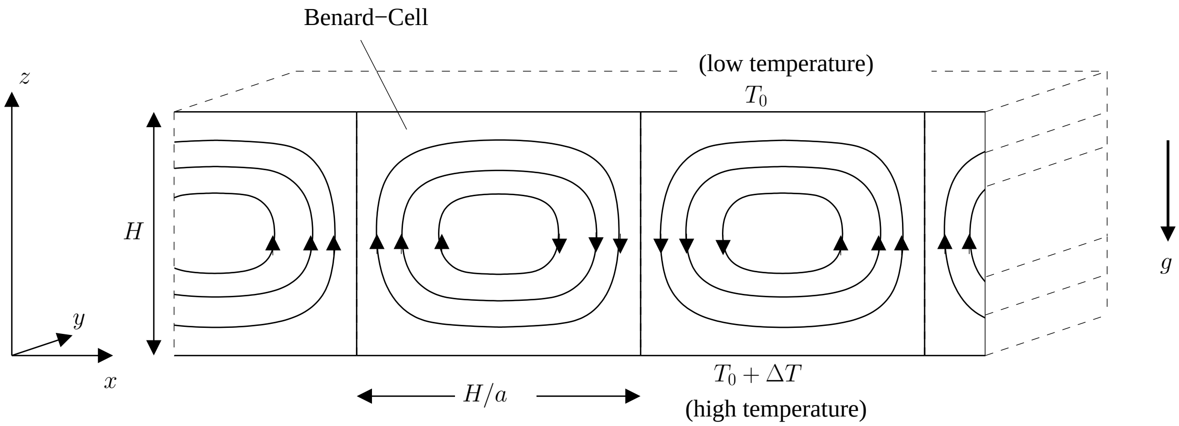

Figure 1: Geometry of the Rayleigh-Bénard system.

2. **Concept of dynamic similarity** *(8 points)*

For the case of an incompressible flow, assuming the temperature effects are negligible and external forces are neglected, the Navier-Stokes equations consist of conservation of mass

$$\nabla \cdot \mathbf{u} = 0 \tag{3}$$

and conservation of momentum

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu \nabla^2 \mathbf{u} \tag{4}$$

where $\mathbf{u}$ is the velocity vector and p is the pressure, ν denotes the kinematic viscosity.

a) Show: The equations (3,4) can be made dimensionless by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. For example: $u = U \cdot u_d$.

Note: the units of $[\rho_0] = kg/m^3$, $[p] = kg/(ms^2)$, and $[p]/[\rho_0] = m^2/s^2$. Therefore the pressure gradient term in (4) has the scaling U^2/L.

b) Show: The scalings vanish completely in front of the terms except for the $\nabla^2 \mathbf{u_d}$-term! The dimensionless parameter is the Reynolds number and the only parameter left!

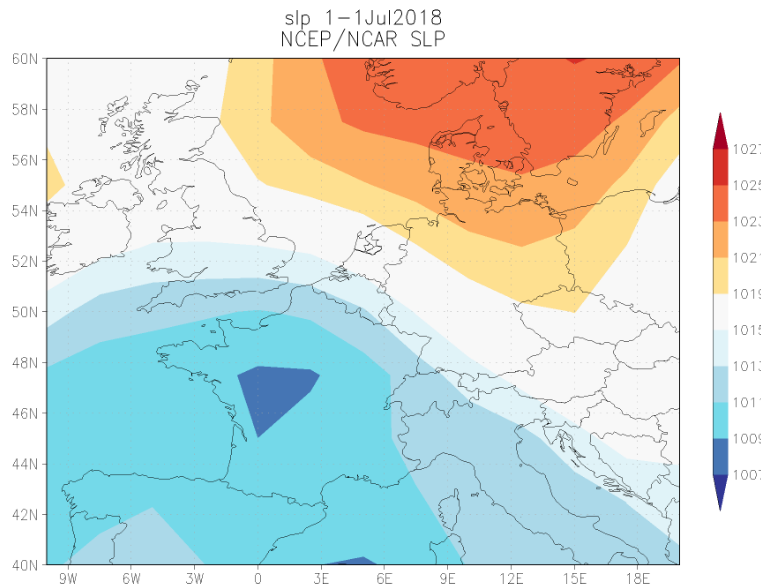

Figure 2: Sea level pressure (hPa) field for July 1, 2018. Source: NCEP/NCAR reanalysis.

3. **Questions about atmosphere-ocean dynamics** (*16 points*, **for each** *2 points*).

a) Please clarify: On the Northern Hemisphere, particles tend to go to the right or left relative to the direction of motion due to the Coriolis force?

b) How is the Coriolis parameter f defined ?

c) What is the hydrostatic approximation in the momentum equations?

d) Please write down the barotropic potential vorticity equation for large-scale motion! Which quantity is conserved along the streamlines?

e) What are the two dominant terms in the horizontal momentum balance for the large-scale dynamics at mid-latitudes for the atmosphere and ocean flow ?

f) Draw the direction of large-scale motions in the atmosphere in Fig. 2 using the geostrophic balance.

g) Explain the Taylor-Proudman Theorem! (remember $f = f_0$, barotropic circulation)

h) Draw a schematic figure of the Atlantic Ocean meridional overturning! Include the directions (N,S), (E,W), depth in your sketch.

4. **Elimination of the pressure term** *(10 points)*

   Assume a 2D flow without non-linear terms and friction, where the equations reduce to:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{5}$$

$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} \quad . \tag{6}$$

   a) Eliminate the pressure in (5,6) .

   b) Show: Defining the stream function ψ through

$$u = -\frac{\partial \psi}{\partial y} \quad ; \quad v = \frac{\partial \psi}{\partial x} \tag{7}$$

   (mass continuity being unconditionally satisfied), the dynamics degrade into one equation:

$$\partial_t \left(\nabla^2 \psi \right) = 0 \tag{8}$$

   c) We now consider the rotating framework and add the Coriolis terms $-\rho f v$ and $\rho f u$ to the left hand sides of (5,6). Show that (8) changed into

$$\partial_t \left(\nabla^2 \psi \right) + \beta v = 0 \tag{9}$$

   d) Consider the non-linear case, is the following correct ?

$$D_t \left(\nabla^2 \psi \right) + \beta v = 0 \tag{10}$$

	Quantity	Atmosphere	Ocean
horizontal velocity	U	$10\ \ ms^{-1}$	$10^{-1}\, ms^{-1}$
horizontal length	L	$10^6\, m$	$10^6\, m$
vertical length	H	$10^4\, m$	$10^3\, m$
horizonal Pressure changes	δP (horizontal)	$10^3\, Pa$	$10^4\, Pa$
time scale	T	$10^5\, s$	$10^7\, s$
Coriolis parameter at 45°N	$f_0 = 2\Omega \sin \varphi_0$	$10^{-4}\, s^{-1}$	$10^{-4}\, s^{-1}$
density	ρ	$1\ \ kgm^{-3}$	$10^3\, kgm^{-3}$
viscosity (turbulent)	ν	$10^{-5}\, kgm^{-3}$	$10^{-6}\, kgm^{-3}$

Table 1: Table shows the typical scales in the atmosphere and ocean system.

5. **Scaling of the dynamical equations in the atmosphere and ocean** *(4 points)*

We work in the rotating frame of reference of the Earth. The equation can be scaled by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. We can estimate the relative contributions in units of m/s^2 in the horizontal momentum equations:

$$\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{U/T \sim 10^{-8}} + \underbrace{\mathbf{v} \cdot \nabla \mathbf{v}}_{U^2/L \sim 10^{-8}} = \underbrace{-\frac{1}{\rho} \nabla p}_{\delta \mathrm{P}/(\rho \mathrm{L}) \sim 10^{-5}} + \underbrace{2\Omega \times \mathbf{v}}_{f_0 \mathrm{U} \sim 10^{-5}} + \underbrace{fric}_{\nu U/H^2 \sim 10^{-13}} \tag{11}$$

where fric denotes the contributions of friction due to eddy stress divergence (usually $\sim \nu \nabla^2 \mathbf{v}$). Typical values are given in Table 1. The values have been taken for the ocean.

a) Please repeat the estimate for the atmosphere using Table 1.

b) The Rossby number Ro is the ratio of inertial (the left hand side) to Coriolis (second term on the right hand side) in (11): terms

$$Ro = \frac{(U^2/L)}{(fU)} = \frac{U}{fL} \quad . \tag{12}$$

Ro is small when the flow is in a so-called geostrophic balance. Please calculate Ro for the atmosphere and ocean using Table 1.

6. **Wind-driven ocean circulation** *(9 points)*

The **Sverdrup transport** V for the depth-integrated flow is calculated by

$$\rho_0 \beta V = \frac{\partial}{\partial x}\tau_y - \frac{\partial}{\partial y}\tau_x \tag{13}$$

where τ_x and τ_y are the components of the wind stress.

The **Ekman transports** V_E, U_E describe the dynamics in the upper mixed layer:

$$f V_E = -\tau_x/\rho_0 \qquad , \qquad f U_E = \tau_y/\rho_0 \tag{14}$$

where $U_E = \int_{-E}^{0} u dz$ and $V_E = \int_{-E}^{0} v dz$ are the depth-integrated velocities in the thin friction-dominated Ekman layer at the sea surface.

Ekman vertical velocity w_E: Using the continuity equation, the divergence of the Ekman transports leads to a vertical velocity w_E at the bottom of the Ekman layer:

$$w_E = -\int_{-E}^{0} \frac{\partial w}{\partial z} dz = \frac{\partial}{\partial x} U_E + \frac{\partial}{\partial y} V_E = \frac{\partial}{\partial x}\left(\frac{\tau_y}{\rho_0 \, f}\right) - \frac{\partial}{\partial y}\left(\frac{\tau_x}{\rho_0 \, f}\right) \quad . \tag{15}$$

a) Assume that the windstress is only zonal with

$$\tau_x = -\tau_0 \cos(\pi y/B) \qquad \text{for an ocean basin with } \; 0 < x < L, \; 0 < y < B. \tag{16}$$

Calculate the Sverdrup transport, Ekman transports, and Ekman pumping velocity for this special case.

b) Make a schematic diagram of the windstress, Sverdrup transport, Ekman transports, and Ekman pumping velocity.

c) Using a), at what latitudes y are $|V|$ and $|V_E|$ maximum? Calculate their magnitudes. Take constant $f = 10^{-4}\,\mathrm{s}^{-1}$ and $\beta = 1.8 \cdot 10^{-11}\,\mathrm{m}^{-1}\mathrm{s}^{-1}$ and $B = 5000\,\mathrm{km}$, $\tau_0/\rho_0 = 10^{-4}\,\mathrm{m}^2\mathrm{s}^{-2}$.

d) Using the values in b), calculate the maximum of w_E for constant f.

7. **Rossby, gravity, and Kelvin waves** *(9 points)*

Start with the shallow water equations

$$\begin{aligned} \frac{\partial u}{\partial t} - fv &= -g\frac{\partial \eta}{\partial x} &(17)\\ \frac{\partial v}{\partial t} + fu &= -g\frac{\partial \eta}{\partial y} &(18)\\ \frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) &= 0 &(19) \end{aligned}$$

with H=const. as mean depth and η as surface anomaly.

a) With the elimination of the fast gravity waves in equation (19)

$$\frac{\partial \eta}{\partial t} = 0$$

derive the dispersion relation for divergence-free Rossby waves! Ansatz: Introduce a stream-function for u,v: $\Psi \sim \exp(ikx + ily - i\omega t)$

b) With the assumption of $f = f_0 = 0$ derive the dispersion relation for gravity waves! The restoring force is related to gravity.

Ansatz: Start with the equation (19) and derive the solution.

c) Kelvin waves:

Derive the dispersion relation for Kelvin waves?

Why are Kelvin waves trapped along the equator and the coasts? Make a sketch!

8. **Rossby wave formula (long waves in the westerlies)** *(7 points)*

a) Assume a mean flow with constant zonal velocity $u = U = const > 0$ and a varying north-south component $v = v(x,t)$ which gives the total motion a wave-like form. Furthermore, h =const.

Write down the vorticity equation for this specific flow! Remember that the vorticity equation is

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0 \tag{20}$$

b) Use a) and the ansatz

$$v(x,t) = A\cos[(kx - \omega t)] \tag{21}$$

to determine the disperion relation $\omega(k)$, the group velocity $\frac{\partial \omega}{\partial k}$, and the phase velocity $c = \omega/k$.

c) Derive the wavelength $L = 2\pi/k$ of the stationary wave given by $c = 0$.

9. **Potential vorticity in the atmosphere:** *(6 points)*

a) An air column at $53°$N with $\zeta = 0$ initially streches from the surface to a fixed tropopause at 8 km height. If the air column moves until it is over a mountain barrier of 3 km height at $30°$N, what is its absolute vorticity and relative vorticity as it passes the mountain top?

b) An air column at $53°$N with $\zeta = 0$ initially streches from the surface to a fixed tropopause at 8 km height. If the air column moves until it is over a mountain barrier of 2 km height at $64°$N, what is its absolute vorticity and relative vorticity as it passes the mountain top?

c) Draw a scetch for a) and b).

Assume: $\sin 53° = 0.8; \sin 30° = 0.5; \sin 64° = 0.9$

The angular velocity of the Earth $\Omega = 2\pi/(1 \text{ day})$.

Use the vorticity equation $D_t\left(\frac{\zeta+f}{h}\right) = 0$

10. **Questions about advection** *(9 points)*

a) A ship is steaming northward at a rate of 10 km/h. The surface pressure increases toward the northwest at a rate of 5 Pa/km. What is the pressure tendency recorded at a nearby island station if the pressure aboard the ship decreases at a rate of 100Pa/3h?

b) The temperature at a point 50 km north of a station is 3°C cooler than at the station. If the wind is blowing from the northeast at 20m/s and the air is being heated by radiation at a rate of 1°C/h, what is the local temperature change at the station?

c) The following data were received from 50 km to the east, north, west and south of a station, respectively: 90 degrees, 10m/s; 120 degrees, 4m/s; 90 degrees, 8m/s; 60 degrees, 4m/s. Given are the angle and absolute value of the wind speed. Calculate the approximate horizontal divergence at the station.

11. **Estimates of overturning** *(6 points)*

It is observed that water sinks in to the deep ocean in polar regions of the Atlantic basin at a rate of 15 $\boldsymbol{Sv}$. (Atlantic basin: $\mathbf{80,000,000}\,\boldsymbol{km^2 area} \times \mathbf{4}\,\boldsymbol{km}$ depth.) a) How long would it take to 'fill up' the Atlantic basin?

b) Supposing that the local sinking is balanced by large-scale upwelling, estimate the strength of this upwelling. Hint: Upwelling $= \boldsymbol{area} \times \boldsymbol{w}.$ Express your answer in $\boldsymbol{m\ y^{-1}}$.

c) Compare this number with that of the Ekman pumping ! The order of magnitude of the Ekman vertical velocity $\boldsymbol{w_E}$ can be estimated as from a typical wind stress variation of $\mathbf{0.2}\boldsymbol{Nm^{-2}}$ per $\mathbf{2000}$ km in y-direction:

$$\boldsymbol{w_E \simeq -\frac{\Delta \tau_x}{\rho\, f_0 \Delta y}} \tag{22}$$

12. **Angular momentum and Hadley Cell** *(10 points)*

Consider a zonally symmetric circulation (i.e., one with no longitudinal variations) in the atmosphere. In the inviscid upper troposphere one expects such a flow to conserve absolute angular momentum, i.e.,

$$\frac{DA}{Dt} = 0, \tag{23}$$

where A is the absolute angular momentum per unit mass (parallel to the Earth's rotation axis). Denoting Ω is the Earth rotation rate, u the eastward wind component, $r = R\cos\varphi$ the distance from the rotation axis, R the Earth's radius, and φ latitude, it follows

$$A = r\,(\Omega r + u) = \Omega R^2 \cos^2\varphi + uR\cos\varphi \quad . \tag{24}$$

a) Show, for inviscid zonally symmetric flow, that the relation $\frac{DA}{Dt} = 0$ is consistent with the zonal component of the equation of motion

$$\frac{Du}{Dt} + \frac{uv}{R}\tan\varphi - fv = 0 \tag{25}$$

where $y = R\varphi$. We assume that $-\frac{1}{\rho}\frac{\partial p}{\partial x} = 0$

b) In the upper troposphere, the flow leaves the rising branch of the Hadley Cell at the equator with angular momentum density $A_0 = \Omega R^2$, assuming that the flow rises from the ground there with no relative motion. Show that the zonal flow can then be described as $u = \Omega R^2 \sin^2\varphi / \cos\varphi$.

c) Show that the zonal flow will be greatest at the edge of the cell, where φ is greatest, thus producing the subtropical jet. If the Hadley circulation is symmetric about the equator, and its edge is at $30°$ latitude, determine the strength of the subtropical jet. Use: $\Omega R^2 = \frac{2\pi}{86400\,s} \cdot (6.371 \cdot 10^6 m)^2 = 3 \cdot 10^9 m^2 s^{-1}$.

d) Consider the tropical Hadley circulation in northern winter. The circulation rises at $10°S$, moves northward across the equator and sinks at $20°N$. Assuming that the air leaves the boundary layer at $10°S$ with zonal velocity u = 0, calculate the zonal wind in the upper troposphere and provide the numbers for the equator, $10°N$, and $20°N$.

13. **Questions about tools** (*12 points*, **for each** *2 points*).

a) Please write down the Euler forward numerical scheme for $\frac{d}{dt}x = f(x)$!
Consider also the special case $f(x) = rx - x^3$.

b) What is the necessary condition for stability in a linear system $\frac{d}{dt}x = Ax$ with real vector x and $n \times n$ matrix A?

c) What is the Fourier transform of a function $x(t)$?
What is the Fourier transform of the $\delta(t)$-function?

d) What is the definition of auto-correlation and auto-covariance?
How is the Fourier transformation of the auto-covariance called?

e) The Laplace transform is given by

$$\mathcal{L}\{x(t)\} = L(s) = \int_0^\infty e^{-st} x(t) dt \tag{26}$$

Show that integration by parts leads to

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sL(s) - x(0) \tag{27}$$

f) Show that

$$\mathcal{L}\{\exp(-at)\} = \frac{1}{s+a} \tag{28}$$

14. **Bifurcations** *(10 points)*

Consider the dynamical system

$$\frac{dx}{dt} = b + x^2 \tag{29}$$

a) Analyze the stability/instability of the equilibria through linearization. The control parameter is b.

b) Explain the graphical method to obtain stability or instability (Fig. 3) !
like ... filled circles with positive slope are unstable ...

c) Calculate the potential and show the stability with a diagram for $b = -1$!

d) Draw the bifurcations as in Fig. 3 for $\frac{dx}{dt} = bx(1-x)$. Analyze the stability/instability of the equilibria through linearization.

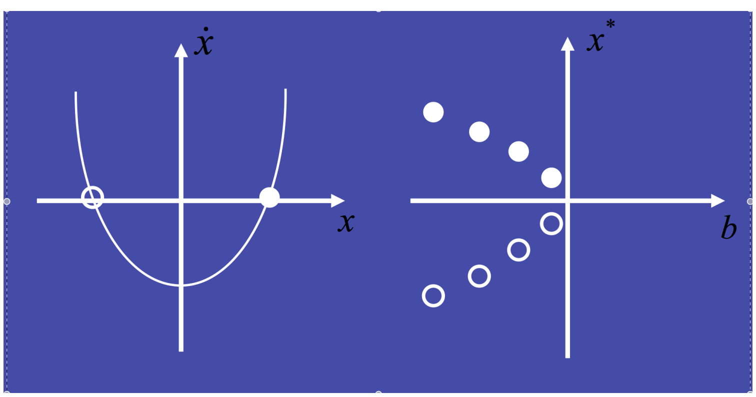

Figure 3: Saddle-node bifurcation diagram using the graphical method.

15. **Spectrum of the stochastic climate model** *(8 points)*

Imagine that the temperature of the ocean mixed layer of depth h is governed by

$$\frac{dT}{dt} = -\lambda T + Q_{net}\,, \tag{30}$$

where λ is the typical damping rate of a temperature anomaly and Q_{net} is given by the stochastic heat flux divided by the heat capacity. The air-sea fluxes are represented by a white-noise process $Q_{net} = \hat{Q}_\omega e^{i\omega t}$ where $\hat{Q}_\omega$ is the amplitude of the random forcing at frequency ω. $\hat{Q}^*$ is the complex conjugate of $\hat{Q}$.

a) What is a white-noise process? Remember that

$$\int_R \exp(i\omega t)\delta(t-0)dt = 1 \tag{31}$$

and use the Fourier transformation.

b) Solve (30) for the temperature response $T = \hat{T}_\omega e^{i\omega t}$ and hence show that:

$$\hat{T}_\omega = \frac{\hat{Q}_\omega}{(\lambda + i\omega)} \tag{32}$$

c) Show that it has a spectral density $\hat{T}_\omega\,\hat{T}_\omega^*$ is given by:

$$\hat{T}\,\hat{T}^* = \frac{\hat{Q}\,\hat{Q}^*}{(\lambda^2 + \omega^2)} \tag{33}$$

and the spectrum

$$S(\omega) = <\hat{T}\,\hat{T}^*> = \frac{1}{(\lambda^2 + \omega^2)}. \tag{34}$$

The brackets $< \cdots >$ denote the ensemble mean. $<\hat{Q}\,\hat{Q}^*> = 1$

d) Make a sketch of the spectrum using a log-log plot and show that fluctuations with a frequency greater than λ are damped.

16. **Short programming questions.** *(5 points)*

    Write down the output for the following R-commands:

    a) `a<-c(0,-5,4,20); mean(a)`

    b) `max(a)-min(a)`

    c) `a*2+c(3,1,-1,0)`

    d) `my.fun<-function(n){return(n*n+n-1)}`
       `my.fun(6)-my.fun(1)`

CDO-related part of the exam:

Assume that you have been asked to do some research on climate model output that has been provided to you in a file named ’output.nc’. This file contains various climate-related quantities, including variables ’siced’, ’snacl’, ’u’, ’precip’, and ’tsurf’. We assume that all data present in the file has global distribution (0-360 E, -90-90 N), the time resolution is one time step per month, and each time step represents a monthly mean. For each of your answers, give the full CDO command that you would employ when performing this task at your computer. Apply the correct CDO syntax. Since there is more than one variable present in the file, for some of the problems it may be necessary to select a specific variable in your CDO command to derive the correct result. Single quotes given in the problems highlight the exact variable name or file name and should be ommitted in your commands. Avoid creating intermediate files by using the CDO piping syntax (a hyphen before the operator) wherever possible. Allowed supporting material to be used by you: the CDO cheat sheat provided with this exam.

17. **cdo question 1** *(5 points)*

    Derive information on the climate model output in file 'output.nc' using the climate data operators (CDO). Explain how you would identify the following listed items of information:
    a) the unit of variable 'siced'
    b) information on the dates for which climate data is present in the file.

18. **cdo question 2** *(5 points)*

    Compute spatial and temporal averages of climate model output in file 'output.nc' using the climate data operators (CDO).
    Explain how you would compute the following characteristics of climate data:
    a) the global average of variable 'precip'
    b) the time average of variable 'tsurf' averaged over the longitudes 0-10 E and 0-20 N

19. **cdo question 3** *(5 points)*

    Devise CDO commands that compute characteristics of climate model output in file 'output.nc' in order to answer some scientific questions. For each of the problems, note down how you would further analyse the output of the CDO command devised by you in order to answer the question. The problems to solve are:
    a) Which is the coldest month in the spatial average over the Southern Hemisphere (longitudes 0-360 E, latitudes -90-0 N) based on the data in variable 'tsurf'?
    b) What is the time average temperature at the location of Bremen (geographical coordinates 53.0793 N, 8.801 E) based on the data in variable 'tsurf'? Note: In case you need to employ remapping to interpolate the global data set to the geolocation of Bremen, apply nearest neighbour remapping (CDO operator remapnn).

```
#derive information on the time axis of a file
cdo showdate input.nc

#derive information on the spatial organisation of data (i.e. on the grid) of a file
cdo griddes input.nc

#derive information on the vertical organisation of data (i.e. the levels) of a file
cdo showlevel input.nc

#derive information on the climate data (variables) in a file
cdo pardes input.nc

#extract a variable named "varname" from file input.nc
cdo selvar,varname input.nc output.nc

#extract the first month of all years in file input.nc
cdo selmon,1 input.nc output.nc

#calculate a time average over a time series input.nc
cdo timmean input.nc output.nc

#generate a seasonal mean from input.nc
cdo seasmean input.nc output.nc

#generate an annual mean from input.nc
cdo yearmean input.nc output.nc

#calculate an average annual cycle from file input.nc
cdo ymonmean input.nc output.nc

#select a region from input.nc, spreading from longitude "a" to "b", and from latitude "c" to "d"
cdo sellonlatbox,a,b,c,d input.nc output.nc

#calculate a spatial average of field input.nc
cdo fldmean input.nc output.nc

#write the output of a cdo operator "a" to the screen (omits creation of an output file)
cdo output -a input.nc

#calculate the difference between two NetCDF files input1.nc and input2.nc
cdo sub input1.nc input2.nc output.nc

#multiply two fields input1.nc and input2.nc
cdo mul input1.nc input2.nc output.nc

#add a skalar constant value "a" to field input.nc
cdo addc,a input.nc output.nc

#select only regions of input2.nc, for which mask input1.nc is true (i.e. 1), represents an if-then programming
cdo ifthen input1.nc input2.nc output.nc

#use input2.nc, where mask input1.nc is true - otherwise use input3.nc, represents an if-then-else programming
cdo ifthenelse input1.nc input2.nc input3.nc output.nc

#reduce a data range (a,b) in input.nc to the constant value "c"
cdo setrtoc,a,b,c input.nc output.nc

#replace a data range (a,b) in input.nc by the missing value ("NaN")
cdo setrtomiss,a,b input.nc output.nc

#calculate trend of time series input.nc; offset "a" and slope "b" of the regression line; stored in a.nc, b.nc
cdo trend input.nc a.nc b.nc

#calculate the horizontal area covered by each grid cell of input.nc, store the result in file output.nc
cdo gridarea input.nc output.nc

#interpolate a spatial data input.nc using nearest neighbor to a specific geolocation of longitude X and latitude Y,
    store the result in file output.nc (X and Y being geolocations in degrees)
cdo remapnn,lon=X/lat=Y input.nc output.nc
```