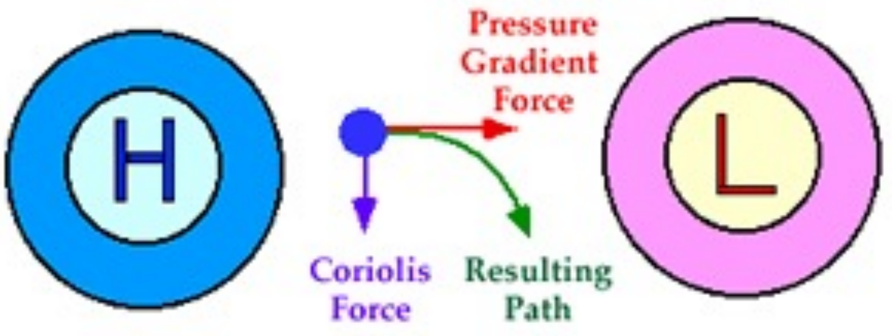
Lecture 7
Waves

Dynamics II, 7.6.2021, 14:00

Gerrit Lohmann

Surface winds

An air parcel initially at rest will move from high pressure to low pressure (**pressure gradient force**)



Geostrophic wind blows parallel to the isobars because the **Coriolis force** and **pressure gradient force** are in balance.

Rossby waves and the westerly wind belt

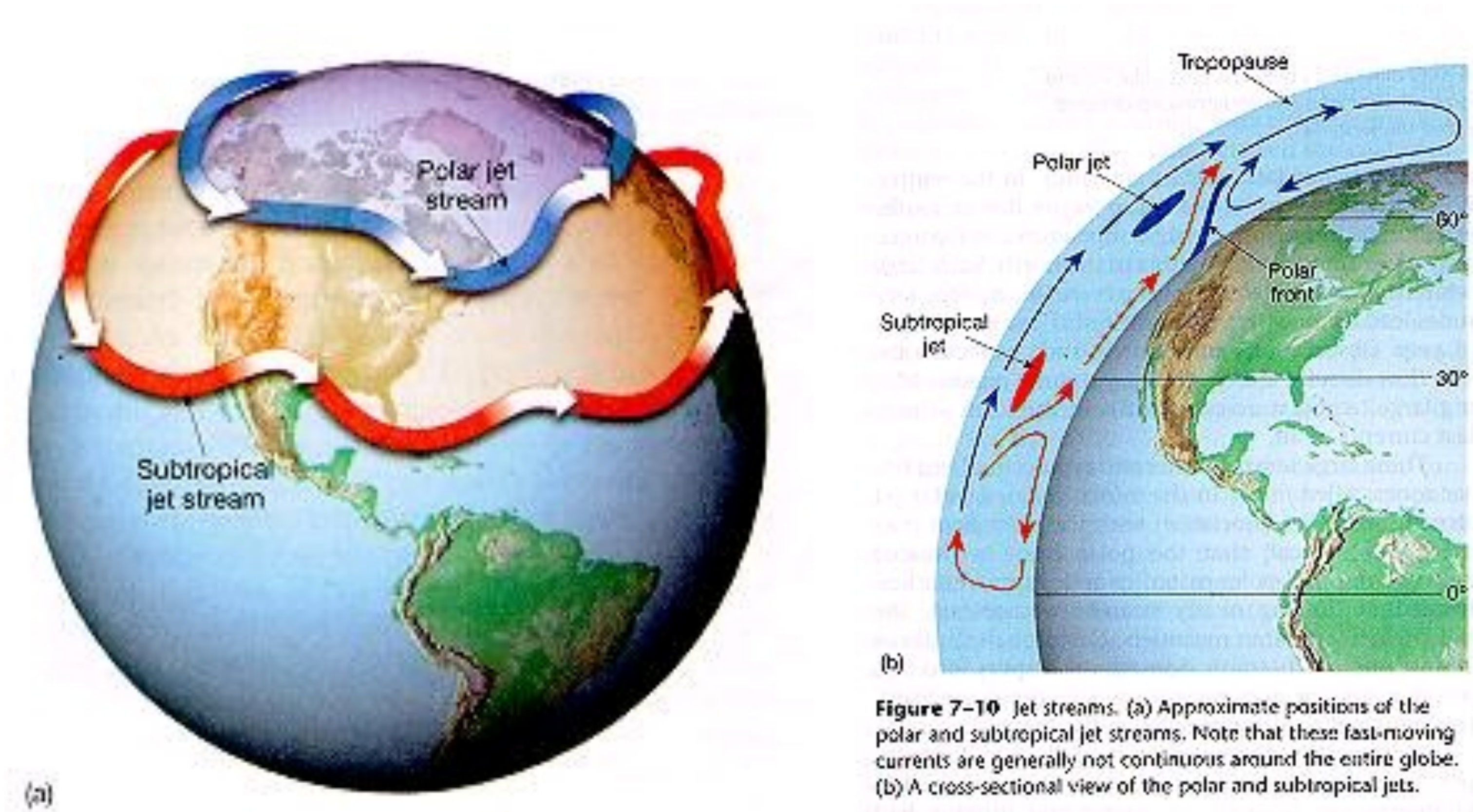
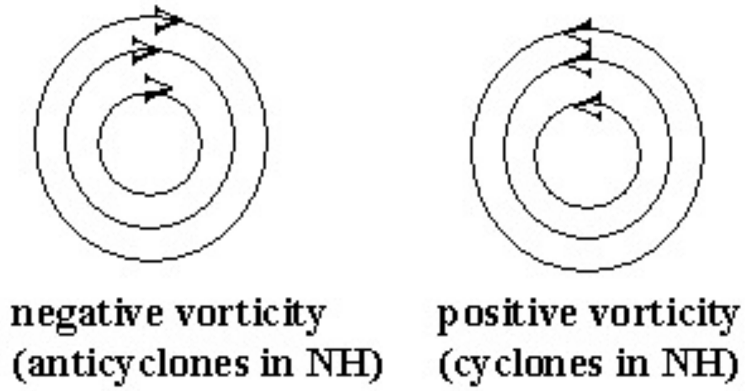


Figure 7–10 Jet streams. (a) Approximate positions of the polar and subtropical jet streams. Note that these fast-moving currents are generally not continuous around the entire globe. (b) A cross-sectional view of the polar and subtropical jets.

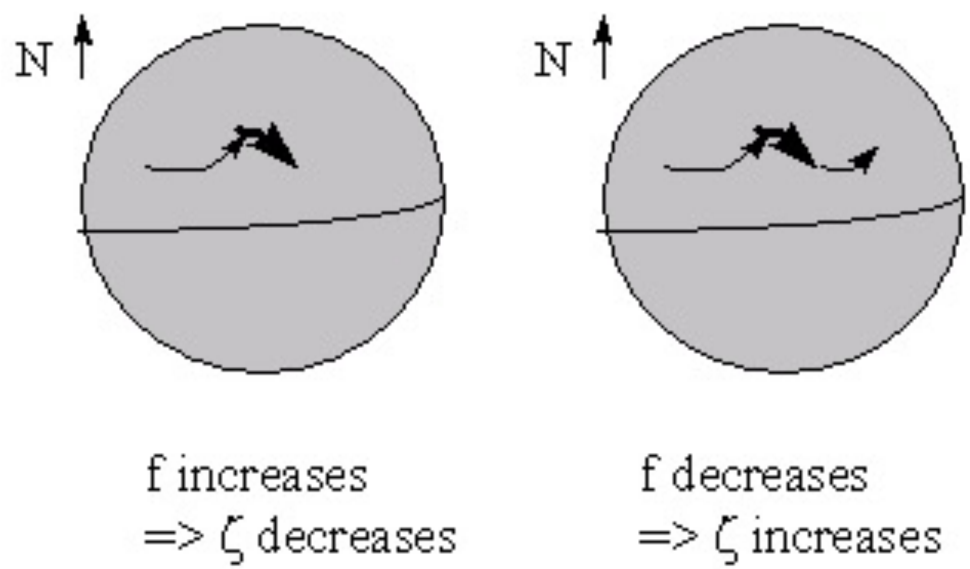
The jet stream is closely linked to the position of **Rossby waves**.

Rossby waves

Vorticity - the tendency to spin about an axis



On the spinnng Earth there is vorticity from the Earth's spin (**planetary vorticity**) and local vorticity due to cyclonic/anticyclonic behaviour (**relative vorticity**)
The absolute vorticity is conserved: *zeta* + *f* = *constant*



Oscillations: Rossby waves

Topographic Rossby waves: standing wave fixed to a permanent forcing location

Rossby waves atmosphere

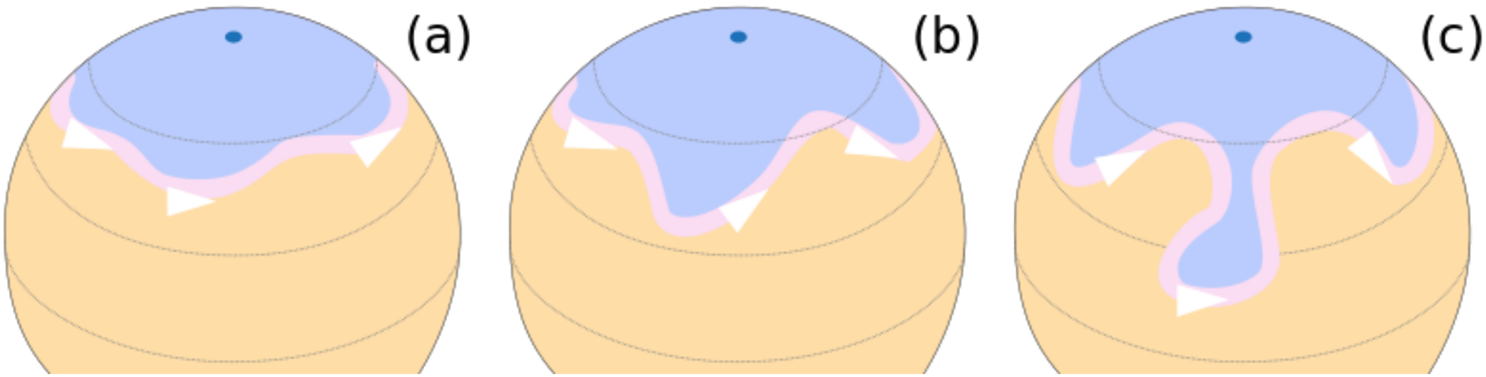


Figure 7.1: Meanders (Rossby Waves) of the Northern Hemisphere's polar jet stream developing (a), (b); then finally detaching a "drop" of cold air (c). Orange: warmer masses of air; pink: jet stream.

https://www.youtube.com/watch?v=Lg91eowtfbw&ab_channel=MetOffice-LearnAboutWeather

Waves: 2D equations

pressure in the vertically homogenous ocean is $p = g\rho(H + \eta)$

$$\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv &= -g\frac{\partial \eta}{\partial x} \\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -g\frac{\partial \eta}{\partial y}
\end{aligned}$$

Waves: 2D equations

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$$
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\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu &= -g\frac{\partial \eta}{\partial y}
\end{aligned}
$$

$$
\frac{\partial}{\partial t}\Big(\rho\,(H+\eta)\Big) + \frac{\partial}{\partial x}\Big(u\rho\,(H+\eta)\Big) + \frac{\partial}{\partial y}\Big(v\rho\,(H+\eta)\Big) = 0
$$

and since the density is constant it reads

$$
\frac{\partial}{\partial t}\eta + u\frac{\partial}{\partial x}\eta + v\frac{\partial}{\partial y}\eta + \frac{\partial}{\partial x}(Hu) + \frac{\partial}{\partial y}(Hv) = 0
$$

Waves: 2D equations

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$$\frac{\partial}{\partial t}\Big(\rho\,(H+\eta)\Big) + \frac{\partial}{\partial x}\Big(u\rho\,(H+\eta)\Big) + \frac{\partial}{\partial y}\Big(v\rho\,(H+\eta)\Big) = 0$$

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$$\frac{\partial}{\partial t}\eta + u\frac{\partial}{\partial x}\eta + v\frac{\partial}{\partial y}\eta + \frac{\partial}{\partial x}(Hu) + \frac{\partial}{\partial y}(Hv) = 0$$

Shallow Water Model

Linear equations

$$(\eta << H)$$

$$
\begin{aligned}
\partial_t u &= f\ v - g\,\partial_x \eta \\
\partial_t v &= -f\ u - g\,\partial_y \eta \\
\partial_t \eta &= -\partial_x (H u) - \partial_y (H v)
\end{aligned}
$$

Plain waves

f and β are taken as fixed parameters in the equations. Then, the wave equations can be reduced to plain waves with eigenfunctions

$\sim \exp(ikx + ily - i\omega t)$

In fluid dynamics, **dispersion** of water waves generally refers to frequency dispersion, which means that waves of different wavelengths travel at different phase speeds.

The function $\omega(k)$, which gives ω as a function of k, is known as the dispersion relation.

7.3.1 Inertial Waves

$\partial_x \eta, \partial_y \eta = 0$ and $f = f_0 =$ const.
(no pressure gradients and constant f)

$$\begin{aligned} \frac{\partial u}{\partial t} - f_0 v &= 0 \\ \frac{\partial v}{\partial t} + f_0 u &= 0 \end{aligned}$$

$$\Longrightarrow \quad \frac{\partial^2 u}{\partial t^2} = -f_0^2 u$$

Inertial waves

$$u(t) = u(0)\sin(f_0 t)$$
$$v(t) = u(0)\cos(f_0 t)$$

Figure 7.4: Schematic representation of inertial circles of air masses in the absence of other forces, calculated for a wind speed of approximately 50 to 70 m/s. Note that the rotation is exactly opposite of that normally experienced with air masses in weather systems around depressions.

Seen in drifting buoys

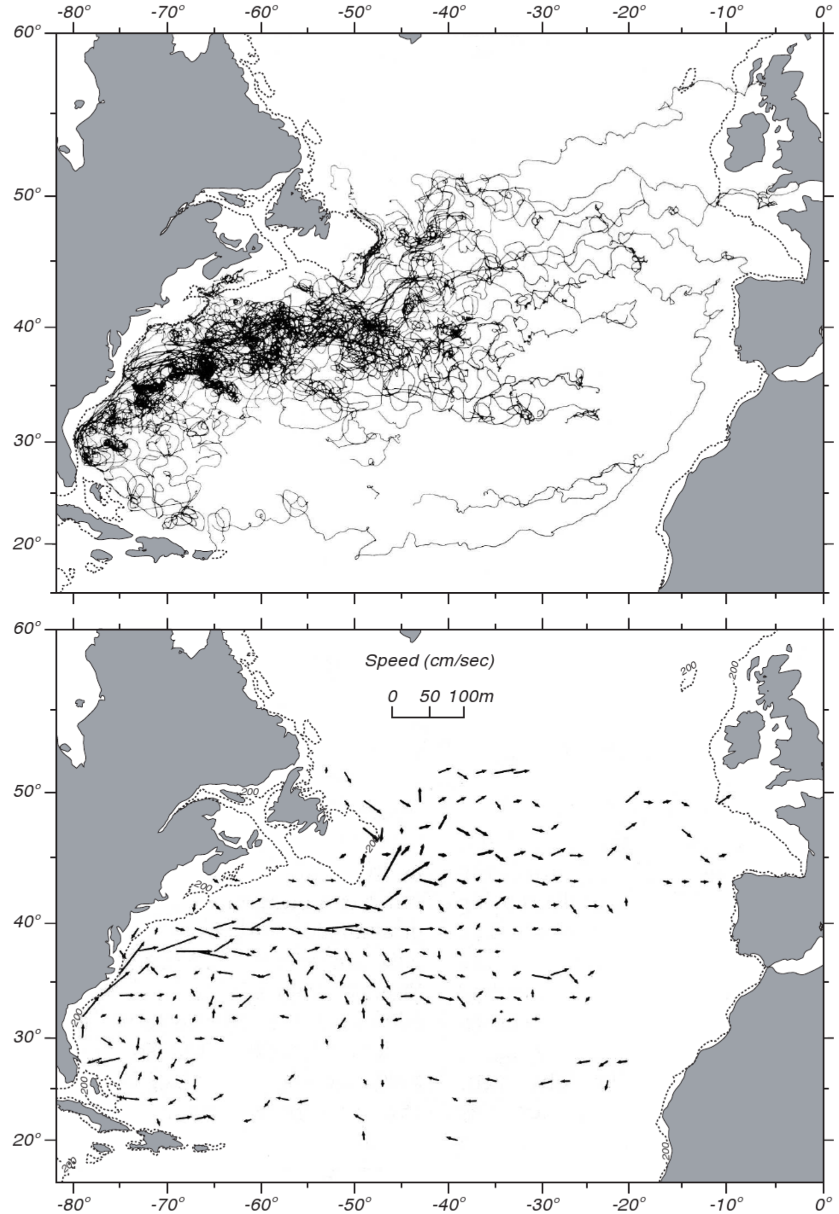

Figure 7.5: **Top:** Tracks of 110 drifting buoys deployed in the western north Atlantic. **Bottom:** Mean velocity of currents in $2^\circ \times 2^\circ$ boxes calculated from tracks above. Boxes with fewer than 40 observations were omitted. Length of arrow is proportional to speed. Maximum values are near $0.6ms$ in the Gulf Stream near 37°N 71°W. After Richardson (1981).

7.3.2 Gravity Waves

Shallow-water gravity waves are defined through their dynamics without the effect of the Earth's rotation, i.e. $f = 0$:

$$\frac{\partial^2 \eta}{\partial t^2} = gH \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \eta \tag{7.28}$$

$$\begin{aligned}
\partial_t u &= \cancel{f\, v} - g\, \partial_x \eta \\
\partial_t v &= -\cancel{f\, u} - g\, \partial_y \eta \\
\partial_t \eta &= -\partial_x (Hu) - \partial_y (Hv)
\end{aligned}$$

7.3.2 Gravity Waves

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$$\frac{\partial^2 \eta}{\partial t^2} = gH \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \eta \qquad (7.28)$$

With the ansatz

$$\eta = \exp(ikx + ily - i\omega t) \qquad (7.29)$$

ω is given by

$$\omega(k, l) = \pm\sqrt{gH\left(k^2 + l^2\right)}, \qquad (7.30)$$

where k and l are the zonal and meridional wavenumbers. Since there is no preferred direction in the (x,y) coordinate, we simply drop the y-dependence and introduce the phase speed

$$c = \omega / k = \pm\sqrt{gH} \quad . \qquad (7.31)$$

Numerical solution

Exercise 48 – Numerical solution of shallow-water gravity waves

- open shallow1D.R
- Identify the lines of the code in which the momentum equation and in which the continuum equation are solved.
- Run the program. Which type of waves do you see?
- Change the constants of water depth H, gravity g, describe your observations!
- Can you roughly estimate the phase speed of the waves?

Script of the lecture

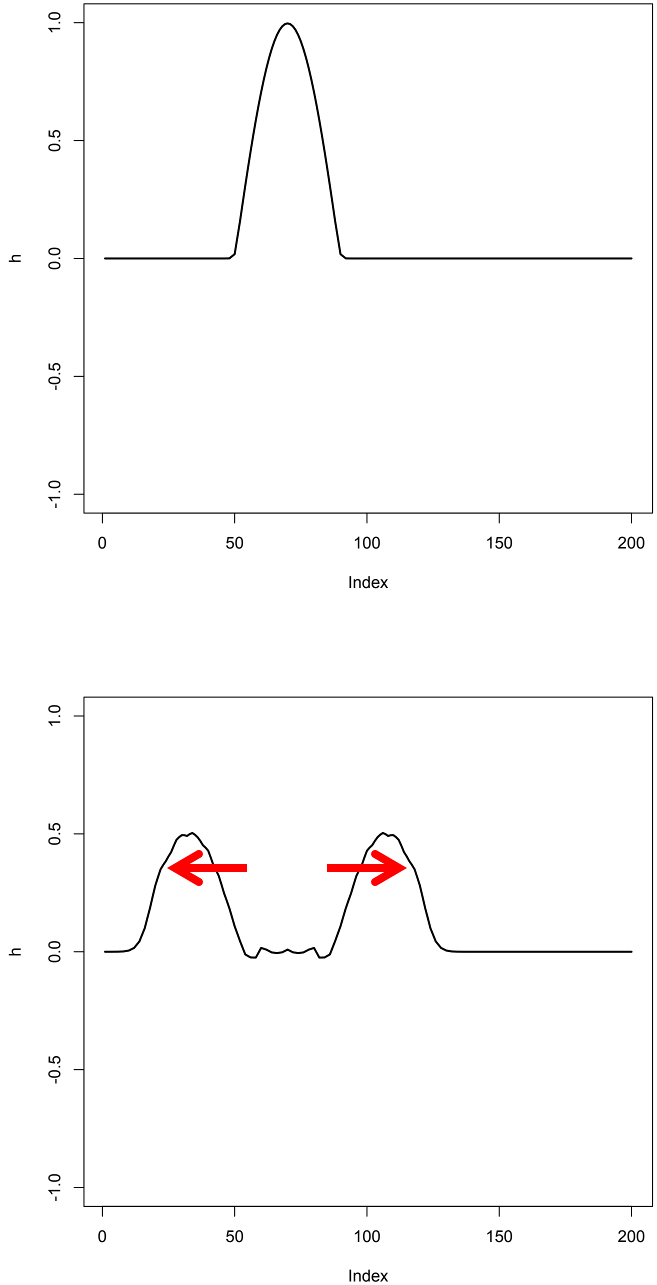


Figure 7.2: Numerical solution of 1D shallow water equation in exercise 48. Upper panel: initial condition. Lower panel: time snap shot.

open shallow2D_rossby.R

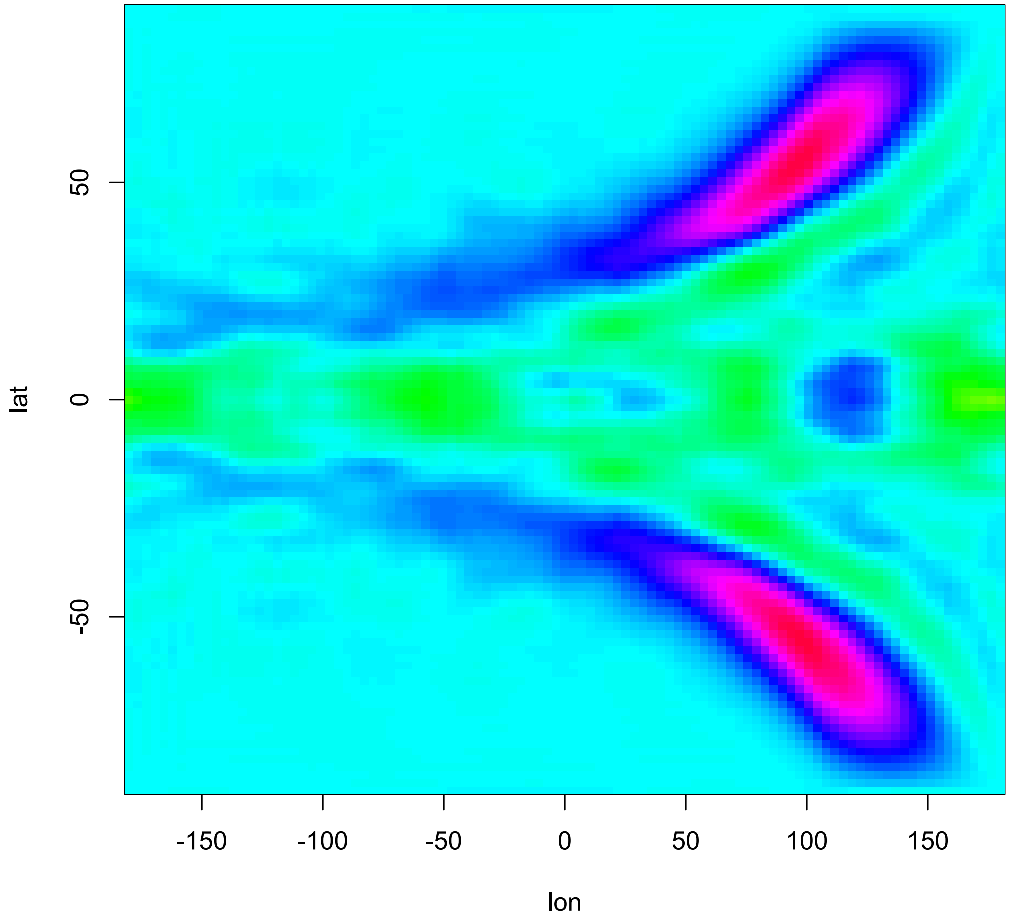


Figure 7.3: Global Rossby and Kelvin wave signatures in the exercise 49.

Shallow Water Model

Linear equations

$$(\eta << H)$$

$$\begin{aligned}
\partial_t u &= f\ v - g\,\partial_x \eta \\
\partial_t v &= -f\ u - g\,\partial_y \eta \\
\partial_t \eta &= -\partial_x (Hu) - \partial_y (Hv)
\end{aligned}$$

7.3.3 Extratropical Rossby Waves

From the equations (7.7,7.8,7.9), we drop the term $\partial_t \eta$ and introduce the stream function ψ through

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \tag{7.36}$$

such that (7.9) is fulfilled. Taking $\frac{\partial}{\partial y}$ of (7.7) and subtract $\frac{\partial}{\partial x}$ of (7.8) elimintates the η term as in section 1.3:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = -\beta \frac{\partial \psi}{\partial x} \tag{7.37}$$

Cont. Rossby Waves

With the ansatz

$$\psi = \exp(ikx + ily - i\omega t) \tag{7.38}$$

and assumption that β is just a parameter, ω is given by

$$\omega(k,l) = -\frac{\beta k}{k^2 + l^2}, \tag{7.39}$$

where k and l are the zonal and meridional wavenumbers. Again, β is used as a parameter (also called Rossby parameter) and is not expressed in terms of y:

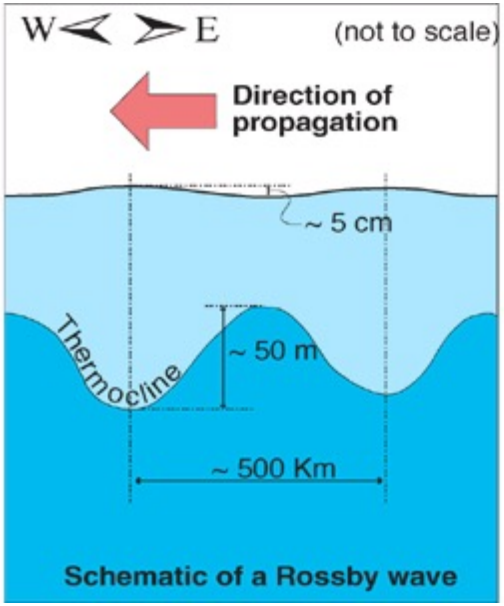
$$\beta = \frac{df}{dy} = \frac{1}{R}\frac{d}{d\varphi}(2\Omega\sin\varphi) = \frac{2\Omega\cos\varphi}{R} \tag{7.40}$$

where φ is the latitude, Ω is the angular speed of the Earth's rotation, and R is the mean radius of the Earth. The wave speed $c = \omega/k = -\beta\left(k^2 + l^2\right)^{-1}$. The feature that the phase speed is faster at low latitudes can be also seen in Fig. 7.3 using the full dynamics.

https://paleodyn.uni-bremen.de/study/Dyn2/dynamics_7.html

Rossby waves: Ocean

Schematic view "first-mode baroclinic" Rossby wave



Existence in the oceans (Carl-Gustav Rossby, 1930s) has been only indirectly confirmed before the advent of satellite oceanography.

Why is it so difficult to observe them?

It is the big difference in the horizontal and vertical scale of these waves which makes them so difficult to observe.

speed varies with latitude and increases equatorward, order of just a few cm/s

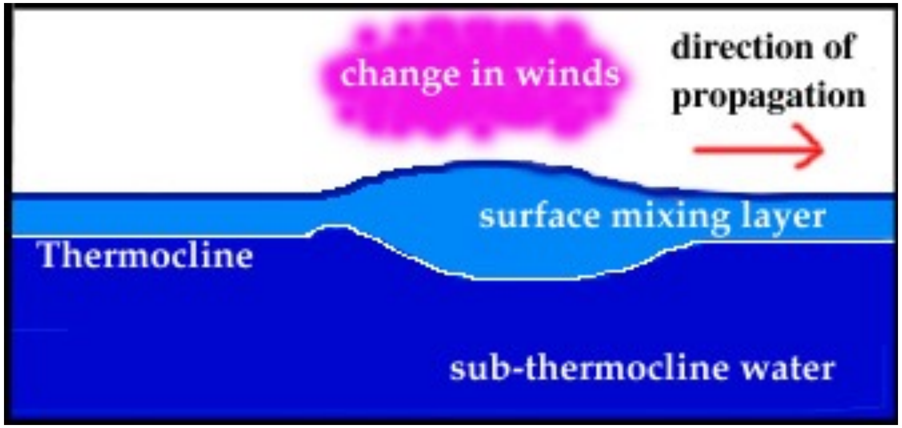
baroclinic

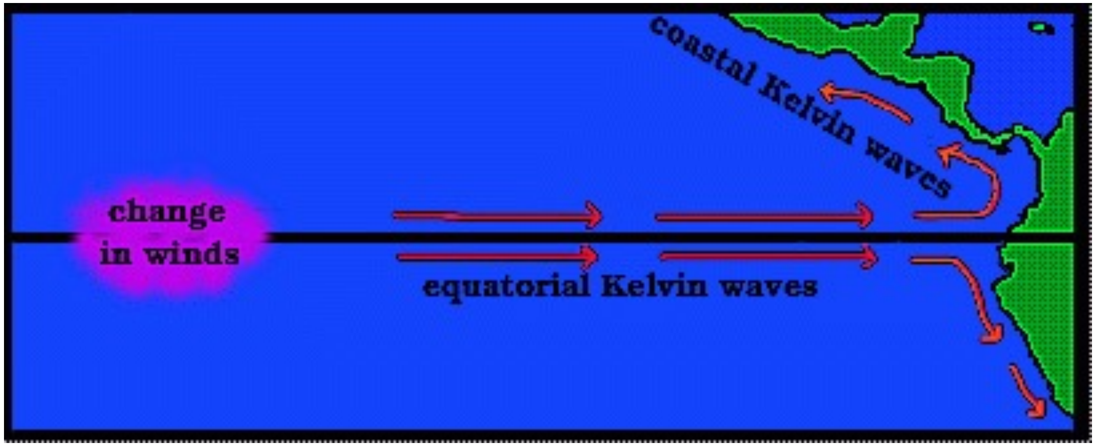
$$
\begin{aligned}
\partial_t u &= f\, v - g\, \partial_x \eta \\
\partial_t v &= -f\, u - g\, \partial_y \eta \\
\partial_t \eta &= -\partial_x (H u) - \partial_y (H v)
\end{aligned}
$$

Exercise 51 – Baroclinic shallow-water gravity waves

In case we have a layed ocean, we consider the so-called baroclinic dynamics with the modified gravity $g' = \frac{\rho_1 - \rho_2}{\rho_1}$ using the densities $\rho_{1,2}$. Task: Derive the baroclinic dynamics using the shallow water equations for 2 different layers and subtract the equations from each other!

Kelvin wave





These waves, especially the surface waves are very fast moving, typically with speeds of ~2.8 m/s, or about 250 kilometers in a day. A Kelvin wave would take about 2 months to cross the Pacific from New Guinea to South America.

Shallow Water Model

Towards Kelvin waves

Solid
wall

$$
\begin{aligned}
\cancel{\partial_t u} &= f\,v - g\,\partial_x \eta \\
\partial_t v &= -f\,\cancel{u} - g\,\partial_y \eta \\
\partial_t \eta &= -\cancel{\partial_x (H u)} - \partial_y (H v)
\end{aligned}
$$

7.4 Kelvin waves

7.4.1 Coastal Kelvin waves

A Kelvin wave is a wave in the ocean or atmosphere that balances the Coriolis force against a topographic boundary such as a coastline. If one assumes that the Coriolis coefficient f is constant along the right boundary conditions, $\boldsymbol{u = 0}$, and the zonal wind speed is set equal to zero, then the equations become the following:

$$\frac{\partial \eta}{\partial t} = -H\frac{\partial v}{\partial y} \tag{7.45}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y} \tag{7.46}$$

and therefore

$$\frac{\partial^2 \eta}{\partial t^2} = gH \quad \frac{\partial^2}{\partial y^2}\eta \tag{7.47}$$

The solution to these equations yields the following phase speed: $c^2 = gH$ and $\omega = \pm cl$, which is the same speed as for shallow-water gravity waves without the effect of Earth's rotation. We see that η and v have also an x-dependence

$$\eta(x,y,t) = \tilde{\eta}(x)\exp(ily - i\omega t) \quad (7.48)$$

$$v(x,y,t) = \tilde{v}(x)\exp(ily - i\omega t) \quad . \quad (7.49)$$

Using (7.46), we obtain

$$-i\omega\,\tilde{v}(x) = -gil\,\tilde{\eta}(x) \quad \text{and therefore } \tilde{v}(x) = \frac{g}{\omega}l\,\tilde{\eta}(x) = \pm\frac{g}{c}\,\tilde{\eta}(x) \quad (7.50)$$

$$\text{From the u-momentum equation} \quad \frac{\partial\eta}{\partial x} = \frac{f}{g}v \quad (7.51)$$

$$\text{we obtain therefore} \quad \frac{\partial\tilde{\eta}}{\partial x} = \pm\frac{f}{c}\tilde{\eta} \quad (7.52)$$

where only the minus sign provides a useful solution (not blowing up). The solution has an exponential decay of $\tilde{\eta}(x) = \exp(-x/L_r)$ on the scale of the Rossby radius $L_r = c/f$. The wave has a trapped character along the boundary. It is important to note that for an observer traveling with

7.4.2 Equatorial Kelvin waves

Analogous we have Equatorial Kelvin waves: assume $v = 0$, then the equations become the following:

$$\frac{\partial \eta}{\partial t} = -H \frac{\partial u}{\partial y} \tag{7.53}$$

$$\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial y} \tag{7.54}$$

and therefore again

$$\frac{\partial^2 \eta}{\partial t^2} = gH \quad \frac{\partial^2}{\partial y^2} \eta \tag{7.55}$$

The solution to these equations yields the phase speed: $c^2 = gH$ and $\omega = ck,$ which is the same speed as for shallow-water gravity waves without the effect of Earth's rotation. We see that η and u have also an x-dependence

7.5 Equatorial waves: Theory of Matsuno

We consider the equations (7.7,7.8,7.9) on the equatorial $\beta-$plane. In the equatorial region, the fluid dynamical system is described as

$$\partial_t u = \beta y \, v - g \, \partial_x \eta \tag{7.61}$$

$$\partial_t v = -\beta y \, u - g \, \partial_y \eta \tag{7.62}$$

$$\partial_t \eta = -\partial_x (H u) - \partial_y (H v) \quad . \tag{7.63}$$

We non-dimensionalize the system through the parameters listed in Table 7.1. In the non-dimensional form (and dropping the stars in Table 7.1), the system reads then

$$\partial_t u = y \, v - \partial_x \eta \tag{7.64}$$

$$\partial_t v = -y \, u - \partial_y \eta \tag{7.65}$$

$$\partial_t \eta = -\partial_x u - \partial_y v \quad . \tag{7.66}$$

-> script

7.5 Equatorial waves: Theorv of Matsuno

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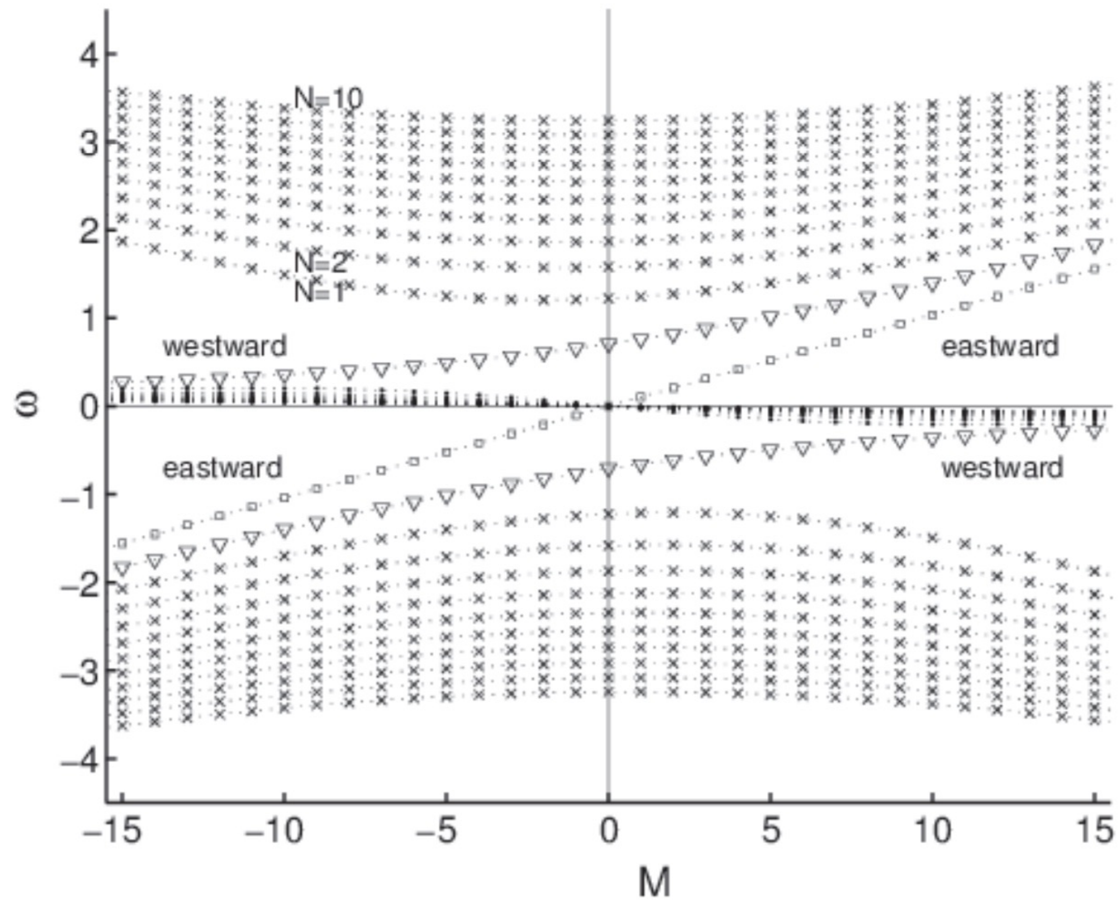

Figure 7.7: Dispersion relation for equatorial waves. Curves show dependence of frequency on zonal wave number M for mode numbers $N \leq 10$. Kelvin waves propagate eastward, Rossby waves (●) westward, while gravity waves (×) exist for both directions. Yanai waves (▽) behave Rossby-like for $M < 0$ and gravity-like for $M \geq 0$.

Zoo of waves

- Gravity wave
- Inertial wave
- Rossby wave
- Kelvin wave

<u>Not here:</u>
Breaking waves
Tsunamis
Non-linear waves

http://www.altimetry.info/thematic-use-cases/examples-of-altimetry-data-use/monitoring-el-nino-rossby-and-kelvin-waves/