Climate Dynamics:

Concepts, Scaling and Multiple Equilibria

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Contents

General framework: Climate dynamics

Over the last century, humans have altered the composition of the Earth's atmosphere and surface to the extent that these factors measurably affect current climate conditions. Paleoclimate reconstructions, in particular from ice cores have also shown that climate can change over relatively short periods such as a few years to decades. The objective of the book is to examine fundamental concepts used to understand climate dynamics. Here, we will approach climate dynamics from a fluid dynamics and complex systems point of view.

The script has several parts, an application follows after every theoretical section. The content (part I-III) is designed for 12 lessons for a master course at the University of Bremen (Dynamics II).

Part I deals with the general structure of fluid dynamical models. Like the ocean, the atmosphere is considered as a Newtonian Fluid. The concepts of scaling and vorticity are introduced. Ice dynamics is not explicitely considered here although it is an important part of the Earth system. One application deals with the Rayleigh-Bénard convection.

In the script I provide a framework to analyze the stability of dynamical systems (Part II). These systems provide the prototype of nonlinear dynamics, bifurcations, multiple equilibria. A bifurcation occurs when a parameter change causes the stability of an equilibrium. In his classic studies of chaotic systems, Lorenz has proposed a deterministic theory of climate change with his concept of the 'almost-intransitivity' of the highly non-linear climate systems. In the Lorenz equations exist the possibility of multiple stable solutions and internal variability, even in the absence of any variations in external forcing [Lorenz, 1976]. More complex models, e.g. Bryan [1986]; Dijkstra et al. [2004] also demonstrated this possibility.

In Part III, I introduce the basic concepts of large-scale meteorology and oceanography. The Coriolis effect is one of the dominating forces for the large-scale dynamics of the oceans and the atmosphere. In meteorology and ocean science, it is convenient to use a rotating frame of reference where the Earth is stationary. The resulting flow can be derived from scaling arguments. Several approximations can be done since the scales of the components in the dynamical equations differ in the orders of magnitude. One fundamental aspect of ocean dynamics are waves. A short theory is given and numerical examples are provided. Furthermore, the deep ocean circulation is studied in a conceptual box model. Here, we introduce an interhemispheric box model of the deep ocean circulation to study the feedbacks in the climate system.

Part IV is part of the course Climate II with advanced participants. Several tasks are included for those who have some additional time (part V and several chapters with a star). The numerical examples in the chapters may be helpful for the students who are already familiar with programming (they can improve the code and follow the main ideas of the code etc.), for those who are not familiar they can use it more as a black box and as a starting point for more research. Several task do not require that the complete code is understood, but one can change initial conditions or parameters in the problems. I mainly use R iwhich s an open source programme avaible through R.

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Part I

First part: Fluid Dynamics

Chapter 1

Basics of Fluid Dynamics

Our starting point is a mathematical model for the system of interest. In physics a model typically describes the state variables, plus fundamental laws and equations of state. These variables evolve in space and time. For the ocean circulation, we proceed as follows:

- State variables: Velocity (in each of three directions), pressure, temperature, salinity, density
- Fundamental laws: Conservation of momentum, conservation of mass, conservation of temperature and salinity
- Equations of state: Relationship of density to temperature, salinity and pressure, and perhaps also a model for the formation of sea-ice

The state variables for the ocean model are expressed as a continuum in space and time, and the fundamental laws as partial differential equations[1]. Even at this stage, though, simplifications may be made. For example, it is common to treat seawater as incompressible. Furthermore, equations of state are often specified by empirical relationships or laboratory experiments.

In the following, the general structure of ocean circulation, atmospheric energy balance as well as ice sheet models are described. The dynamics of flow are based on the Navier-Stokes equations. The derivation of the Navier-Stokes equations begins with an application of Newton's second law: conservation of momentum (often alongside mass and energy conservation) being written for an

[1]If the atmosphere is becoming too thin in the upper levels, a more molecular, statistical description is appropiate (section 5)

arbitrary control volume. In an inertial frame of reference, the general form of the equations of fluid motion is:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \nabla \cdot \mathbb{T} + \mathbf{F}, \tag{1.1}$$

where $\mathbf{u}$ is the flow velocity (a vector), ρ is the fluid density, p is the pressure, $\mathbb{T}$ is the 3×3 (deviatoric) stress tensor, and $\mathbf{F}$ represents body forces (per unit volume) acting on the fluid and ∇ is the nabla operator. This is a statement of the conservation of momentum in a fluid and it is an application of Newton's second law to a continuum; in fact this equation is applicable to any non-relativistic continuum and is known as the Cauchy momentum equation (e.g., Landau and Lifshitz [1959]).

This equation is often written using the substantive derivative, making it more apparent that this is a statement of Newton's second law:

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \nabla \cdot \mathbb{T} + \mathbf{F}. \tag{1.2}$$

The left side of the equation describes acceleration, and may be composed of time dependent or advective effects (also the effects of non-inertial coordinates if present). The right side of the equation is in effect a summation of body forces (such as gravity) and divergence of stress (pressure and stress). A very significant feature of the Navier-Stokes equations is the presence of advective acceleration: the effect of time independent acceleration of a fluid with respect to space, represented by the nonlinear quantity $\mathbf{u} \cdot \nabla \mathbf{u}$. A general framework can be generally formulated as a transport phenomenon, see section 3.4.

1.1 Material laws

The effect of stress in the fluid is represented by the ∇p and $\nabla \cdot \mathbb{T}$ terms, these are gradients of surface forces, analogous to stresses in a solid. ∇p is called the pressure gradient and arises from the isotropic part of the stress tensor. This part is given by normal stresses that turn up in almost all situations, dynamic or not. The anisotropic part of the stress tensor gives rise to $\nabla \cdot \mathbb{T}$, which conventionally describes viscous forces. For incompressible flow, this is only a shear effect. Thus, $\mathbb{T}$ is the deviatoric stress tensor, and the stress tensor is equal to:

$$\boldsymbol{\sigma} = -p\,\mathbb{I} + \mathbb{T} \tag{1.3}$$

where $\mathbb{I}$ is the 3×3 identity matrix. Interestingly, only the gradient of pressure matters, not the pressure itself. The effect of the pressure gradient is that fluid flows from high pressure to low pressure.

The stress terms p and $\mathbb{T}$ are yet unknown, so the general form of the equations of motion is not usable to solve problems. Besides the equations of motion -Newton's second law- a force model is needed relating the stresses to the fluid motion. For this reason, assumptions on the specific behavior of a fluid are made (based on observations) and applied in order to specify the stresses in terms of the other flow variables, such as velocity and density.

The Cauchy stress tensor can be also written in matrix form:

$$\mathbb{T} = \begin{pmatrix} \mathbf{T}^{(\mathrm{e}_1)} \\ \mathbf{T}^{(\mathrm{e}_2)} \\ \mathbf{T}^{(\mathrm{e}_3)} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \equiv \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \equiv \begin{pmatrix} \sigma_{x} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_{y} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_{z} \end{pmatrix} \tag{1.4}$$

where $\boldsymbol{\sigma}$ are the normal stresses and $\boldsymbol{\tau}$ are the shear stresses. From the Newton's third law (actio est reactio) the stress vectors $\mathbf{T}^{(\mathrm{e}_i)} = \frac{d\mathbf{F}}{dA}$ with $\mathbf{e}_i$ as normalvector acting on opposite sides of the same surface are equal in amount and opposite in direction ($-\mathbf{T}^{(\mathrm{e}_i)} = \mathbf{T}^{(-\mathrm{e}_i)}$). According to conservation of angular momentum, summation of moments is zero. Thus the stress tensor is symmetrical: $\mathbb{T} = \mathbb{T}^{T}$. In Fig. 1.1 the stress vectors $\mathbf{T}^{(\mathrm{e}_i)}$ can be decomposed in one normal stress and two shear stress components.

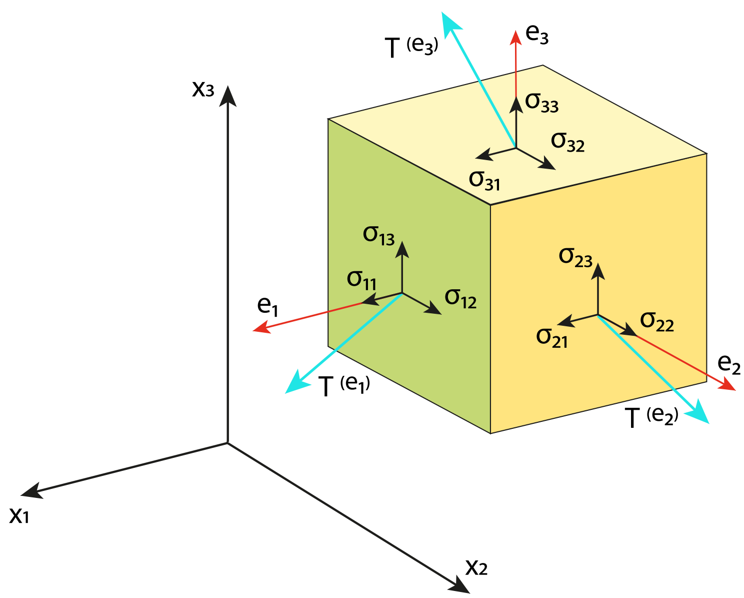

Figure 1.1: Components of stress in three dimensions.

1.2 Navier-Stokes equations

The so-called Navier-Stokes equations result from the following assumptions on the deviatoric stress tensor $\mathbb{T}$:

- the deviatoric stress vanishes for a fluid at rest, and by Galilean invariance also does not depend directly on the flow velocity itself, but only on spatial derivatives of the flow velocity
- in the Navier-Stokes equations, the deviatoric stress is expressed as the product of the tensor gradient $\nabla \mathbf{v}$ of the flow velocity with a viscosity tensor $\mathbb{A}$, i.e. $\mathbb{T} = \mathbb{A}\,(\nabla \mathbf{v})$
- the fluid is assumed to be isotropic, as valid for gases and simple liquids, and consequently $\mathbb{A}$ is an isotropic tensor; furthermore, since the deviatoric stress tensor is symmetric, it turns out that it can be expressed in terms of two scalar dynamic viscosities μ and μ'' : $\mathbb{T} = 2\mu\mathbb{E} + \mu''(\nabla \cdot \mathbf{v})\mathbb{I}$, where $\mathbb{E} = \frac{1}{2}\,(\nabla \mathbf{v}) + \frac{1}{2}\,(\nabla \mathbf{v})^{\mathrm{T}}$ is the rate-of-strain tensor and $\nabla \cdot \mathbf{v}$ is the rate of expansion of the flow
- the deviatoric stress tensor has zero trace, so for a three-dimensional flow $2\mu + 3\mu'' = 0$

As a result, in the Navier-Stokes equations the deviatoric stress tensor has the following form:

$$\mathbb{T} = 2\mu\left(\mathbb{E} - \tfrac{1}{3}(\nabla \cdot \mathbf{u})\mathbb{I}\right), \tag{1.5}$$

with the quantity between brackets the non-isotropic part of the rate-of-strain tensor $\mathbb{E}$. The dynamic viscosity μ does not need to be constant - in general it depends on conditions like temperature and pressure, and in turbulence modelling the concept of eddy viscosity is used to approximate the average deviatoric stress.

The Navier-Stokes equations are strictly a statement of the conservation of momentum. In order to fully describe fluid flow, more information is needed (how much depends on the assumptions made), this may include boundary data (no-slip, capillary surface, etc), the conservation of mass, the conservation of energy, and/or an equation of state. Regardless of the flow assumptions, a statement of the conservation of mass is generally necessary. This is achieved through the mass continuity equation, given in its most general form as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \tag{1.6}$$

or, using the substantive derivative:

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot \mathbf{u}) = 0. \tag{1.7}$$

A simplification of the resulting flow equations is obtained when considering an incompressible flow of a Newtonian fluid. The assumption of incompressibility rules out the possibility of sound or shock waves to occur; so this simplification is invalid if these phenomena are important. The incompressible flow assumption typically holds well even when dealing with a "compressible" fluid -such as air at room temperature- at low Mach numbers (even when flowing up to about Mach 0.3).[2] Taking this into account and assuming constant viscosity, the Navier-Stokes equations will read, in vector form:

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{F}. \tag{1.8}$$

The vector field $\mathbf{F}$ represents "other" (body force) forces. Typically this is only gravity, but may include other fields (such as electromagnetic). In a non-inertial coordinate system, other "forces" such as that associated with rotating coordinates may be inserted[3]. Often, these forces may be represented as the gradient of some scalar quantity. Gravity in the z direction, for example, is the gradient of $-\rho g z$. Since pressure shows up only as a gradient, this implies that solving a problem

[2]The density and pressure fields can be expressed as a perturbation from a hydrostatically balanced state around a reference density $\rho_r(z)$ (e.g. a horizontal mean of density in the area of interest) and associated pressure $p_r(z)$ which are linked through $dp_r/dz = -g\rho_r$ and $p_r(z = 0) = 0$. Sound waves are filtered by realizing that the time rate of change of density due to diabatic effects and compressibility is much smaller than that due to change of volume.

[3]We will see later that the Coriolis force will be one of the main contributions in the rotating Earth system (section 6.1)

without any such body force can be mended to include the body force by modifying pressure. The shear stress term $\nabla\mathbb{T}$ becomes the useful quantity $\mu\nabla^2\mathbf{u}$ when the fluid is assumed incompressible and Newtonian, where μ is the dynamic viscosity.

It's well worth observing the meaning of each term (compare to the Cauchy momentum equation):

$$\overbrace{\rho\Big(\underbrace{\frac{\partial\mathbf{u}}{\partial t}}_{\substack{\text{Unsteady}\\\text{acceleration}}}+\underbrace{\mathbf{u}\cdot\nabla\mathbf{u}}_{\substack{\text{Advective}\\\text{acceleration}}}\Big)}^{\text{Inertia (per volume)}}=\overbrace{\underbrace{-\nabla p}_{\substack{\text{Pressure}\\\text{gradient}}}+\underbrace{\mu\nabla^2\mathbf{u}}_{\text{Viscosity}}}^{\text{Divergence of stress}}+\underbrace{\mathbf{F}}_{\substack{\text{Other}\\\text{body}\\\text{forces}}}. \tag{1.9}$$

Note that only the advection terms are nonlinear for incompressible Newtonian flow. This acceleration is an acceleration caused by a (possibly steady) change in velocity over position, for example the speeding up of fluid entering a converging nozzle. Though individual fluid particles are being accelerated and thus are under unsteady motion, the flow field (a velocity distribution) will not necessarily be time dependent.

Another important observation is that the viscosity is represented by the vector Laplacian of the velocity field. This implies that Newtonian viscosity is diffusion of momentum, this works in much the same way as the diffusion of heat seen in the heat equation (which also involves the Laplacian).

If temperature effects are also neglected, the only "other" equation (apart from initial/boundary conditions) needed is the mass continuity equation. Under the incompressible assumption, density is a constant and it follows that the equation will simplify to:

$$\nabla\cdot\mathbf{u}=\mathbf{0} \qquad . \tag{1.10}$$

This is more specifically a statement of the conservation of volume (see divergence). These equations are commonly used in 3 coordinates systems: Cartesian, cylindrical, and spherical. While the Cartesian equations seem to follow directly from the vector equation above, the vector form of the Navier-Stokes equation involves some tensor calculus which means that writing it in other coordinate systems is not as simple as doing so for scalar equations (such as the heat equation).

1.3 Some exercises for Chapter 1

Exercise 1 – Questions about advection

1. A ship is steaming northward at a rate of 10 km/h. The surface pressure increases toward the northwest at a rate of 5 Pa/km. What is the pressure tendency recorded at a nearby island station if the pressure aboard the ship decreases at a rate of 100Pa/3h?
2. The temperature at a point 50 km north of a station is 3°C cooler than at the station. If the wind is blowing from the northeast at 20m/s and the air is being heated by radiation at a rate of 1°C/h, what is the local temperature change at the station?
3. The following data were received from 50 km to the east, north, west and south of a station, respectively: 90 degree, 10m/s; 120 degree,4m/s; 90degree,8m/s; 60 degree, 4m/s. Given are the angle and absolute value of the wind speed. Calculate the approximate horizontal divergence at the station.
4. Let the $x = (x_1, x_2, x_3)$ coordinates be inertial. What are the necessary and sufficient conditions that the coordinates $y_i = A_{ij}x_j + v_j(x,t)t$ be inertial for constant matrix $A = (A_{ij})$?
5. How can the movement of fluid particel be descibed in accordance with Newton's first law? Which forces can create accelerations or decelerations? Please use the definition of specific forces, that is, the force per unit mass: $f = F/m$.
6. The potential temperature in the atmosphere is defined as

$$\Theta = T(p_0/p)^{R/c_p} \tag{1.11}$$

With $p_0 = \text{const.}$ Calculate the vertical temperature gradient

$$\gamma = -\frac{dT}{dz} \tag{1.12}$$

What is the result when assuming the hydrostatic equillibrium

$$\frac{dp}{dz} = -g\rho$$

with $g = 9.81m/s^2$? What is the condition for which the the potential temperature is constant in the vertical?

Solution of 2. Temperature Advection

The total change of temperature is given by

$$\begin{aligned} \frac{\mathrm{d}T}{\mathrm{d}t} &= \frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \dot{q} \\ \Leftrightarrow \quad \frac{\partial T}{\partial t} &= -\mathbf{u} \cdot \nabla T + \dot{q} \end{aligned}$$

Here we use the velocity

$$u = -20\frac{m}{s} \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \nabla T = \frac{3°C}{50km}\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \qquad \dot{q} = 1\frac{°C}{h}$$

	Horizontal Length L	Velocity V	Time T
Microturbulence	1-10 cm	1-10 cm/s	seconds
Thunderstorms	1-10 km	10 m/s	hours
Weather patterns	100-1000 km	1-10 m/s	days to weeks
Climatic variations	global	1-10 m/s	decades and beyond

Table 1.1: Table shows the typical scales in the environmental, atmosphere, ocean and climate system. Using these orders of magnitude, one can derive estimates of the timescales.

Then we calculate

$$\begin{aligned}\frac{\partial T}{\partial t} &= -\mathbf{u} \cdot \nabla T + \dot{q} \\ &= 20\frac{m}{s}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix} \cdot \begin{pmatrix}0\\-1\\0\end{pmatrix}\frac{3°C}{50km} + 1\frac{°C}{h} \\ &\approx -2.1\frac{°C}{h}\end{aligned}$$

Exercise 2 – Typical scales

Table 2 lists typical velocity, length and time scales of some fluid processes and systems. Not surprisingly, larger systems evolve on longer time scales. Depending on the size of the system under consideration, the spatial scale can be regional, continental or even global. Using the length and velocity scales (L and V), determine a typical time scale (T=L/V)! (Rough estimates are given in the last column in Table 2.)

Exercise 3 – Weather chart

From the weather chart (Figure 1.2), identify the horizontal extent of a major atmospheric sea level pressure and the associated wind speed. Determine a typical time scale T !

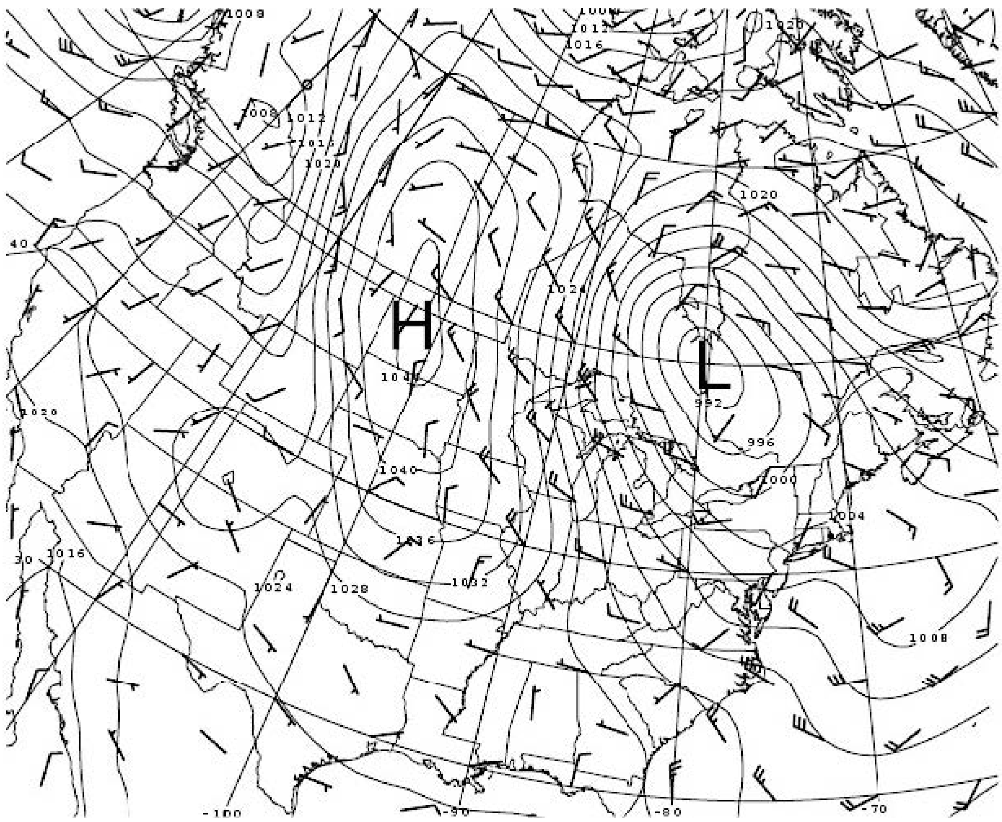

Figure 1.2: Surface pressure field and surface wind on 10th February 2008 at 12GMT. The contour interval is 4mbar. High and low pressure systems are marked as H and L. The dark segments represent wind arrows, whose arrowhead is not drawn in meteorological plots, by convention. The reader should imagine arrowhead at the end of segment that has no quivers. The quivers are drawn at only one side, at the tail end. The wind blows in the direction of the quiver base to the arrowhead. One full quiver represents a wind of 5m/s.

1.4 Integral and differential formulation*

On a volume work two types of forces:

1. The force $\vec{F}$ on each volume element. For gravity holds: $\vec{F} = \varrho\vec{g}$.
2. Surface forces working only on the margins: $\vec{t}$. For these holds: $\vec{t} = \vec{n}\,\sigma$, where σ is the *stress tensor*.

σ can be split in a part $p\,\mathsf{I}$ representing the normal tensions and a part $\mathbb{T}$ representing the shear stresses: $\sigma = \mathbb{T} + p\,\mathsf{I}$, where I is the unit tensor or identity matrix. When viscous aspects can be ignored holds:

$$div\,\sigma = -\nabla p \quad . \tag{1.13}$$

When the flow velocity is $\vec{v}$ at position $\vec{r}$ holds on position $\vec{r} + d\vec{r}$:

$$\vec{v}(\vec{r} + d\vec{r}\,) = \underbrace{\vec{v}(\vec{r}\,)}_{\text{translation}} + \underbrace{d\vec{r} \cdot (\nabla\vec{v}\,)}_{\text{rotation, deformation, dilatation}}$$

The quantity $\mathsf{L} := \nabla\vec{v}$ can be split in a symmetric part D and an antisymmetric part W. $\mathsf{L} = \mathsf{D} + \mathsf{W}$ with

$$D_{ij} := \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \;,\; W_{ij} := \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$$

When the rotation or *vorticity* $\vec{\omega} = \mathrm{rot}\vec{v}$ is introduced holds: $W_{ij} = \frac{1}{2}\varepsilon_{ijk}\omega_k$. $\vec{\omega}$ represents the local rotation velocity: $\vec{dr} \cdot \mathsf{W} = \frac{1}{2}\omega \times \vec{dr}$.

For a *Newtonian liquid* holds: $\mathbb{T} = 2\eta\mathsf{D}$. Here, η is the dynamical viscosity. This is related to the shear stress τ by:

$$\tau_{ij} = \eta\frac{\partial v_i}{\partial x_j}$$

For compressible media can be stated: $\mathbb{T} = (\eta'\mathrm{div}\vec{v}\,)\mathsf{I} + 2\eta\mathsf{D}$. From equating the thermodynamical and mechanical pressure it follows: $3\eta' + 2\eta = 0$. If the viscosity is constant holds:

$\mathrm{div}(2\mathsf{D}) = \nabla^2\vec{v} + \mathrm{grad}\ \mathrm{div}\vec{v}$.

The conservation laws for mass, momentum and energy for continuous media can be written in both integral and differential form. They are:

Integral notation:

1. Conservation of mass: $\dfrac{\partial}{\partial t}\iiint \varrho d^3V + \oiint \varrho(\vec{v}\cdot\vec{n}\,)d^2A = 0$
2. Conservation of momentum: $\dfrac{\partial}{\partial t}\iiint \varrho\vec{v} d^3V + \oiint \varrho\vec{v}(\vec{v}\cdot\vec{n}\,)d^2A = \iiint f_0 d^3V + \oiint \vec{n}\cdot T d^2A$
3. Conservation of energy: $\dfrac{\partial}{\partial t}\iiint (\tfrac{1}{2}v^2 + e)\varrho d^3V + \oiint (\tfrac{1}{2}v^2 + e)\varrho(\vec{v}\cdot\vec{n}\,)d^2A =$

$$-\oiint (\vec{q}\cdot\vec{n}\,)d^2A + \iiint (\vec{v}\cdot\vec{f}_0)d^3V + \oiint (\vec{v}\cdot\vec{n}\ \mathsf{T})d^2A$$

Differential notation:

1. Conservation of mass: $\dfrac{\partial\varrho}{\partial t} + \mathrm{div}\cdot(\varrho\vec{v}\,) = 0$
2. Conservation of momentum: $\varrho\dfrac{\partial\vec{v}}{\partial t} + (\varrho\vec{v}\cdot\nabla)\vec{v} = \vec{f}_0 + \mathrm{div}\mathsf{T} = \vec{f}_0 - \mathrm{grad}p + \mathrm{div}\mathsf{T}'$
3. Conservation of energy: $\varrho T\dfrac{ds}{dt} = \varrho\dfrac{de}{dt} - \dfrac{p}{\varrho}\dfrac{d\varrho}{dt} = -\mathrm{div}\vec{q} + \mathbb{T}:\mathsf{D}$

Here, e is the internal energy per unit of mass E/m and s is the entropy per unit of mass S/m. $\vec{q} = -\kappa\vec{\nabla}T$ is the heat flow. Further holds:

$$p = -\frac{\partial E}{\partial V} = -\frac{\partial e}{\partial 1/\varrho}\ ,\quad T = \frac{\partial E}{\partial S} = \frac{\partial e}{\partial s}$$

so

$$C_V = \left(\frac{\partial e}{\partial T}\right)_V \quad \text{and} \quad C_p = \left(\frac{\partial h}{\partial T}\right)_p$$

with $h = H/m$ the enthalpy per unit of mass.

From this one can derive the *Navier-Stokes* equations for an incompressible, viscous and heat-conducting medium:

$$
\begin{aligned}
\text{div}\vec{v} &= 0 \\
\varrho\frac{\partial\vec{v}}{\partial t} + \varrho(\vec{v}\cdot\nabla)\vec{v} &= \varrho\vec{g} - \text{grad}p + \eta\nabla^2\vec{v} \\
\varrho C\frac{\partial T}{\partial t} + \varrho C(\vec{v}\cdot\nabla)T &= \kappa\nabla^2 T + 2\eta\mathsf{D}:\mathsf{D}
\end{aligned}
$$

with C the thermal heat capacity. The force $\vec{F}$ on an object within a flow, when viscous effects are limited to the boundary layer, can be obtained using the momentum law. If a surface A surrounds the object outside the boundary layer holds:

$$\vec{F} = -\oint\!\!\!\oint [p\vec{n} + \varrho\vec{v}(\vec{v}\cdot\vec{n}\,)]d^2A$$

1.5 Elimination of the pressure term

Taking the curl of the Navier-Stokes equation results in the elimination of pressure. This is especially easy to see if 2D Cartesian flow is assumed ($w = 0$ and no dependence of anything on z), where the equations reduce to:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{1.14}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \quad . \tag{1.15}$$

Differentiating the first with respect to y, the second with respect to x and subtracting the resulting equations will eliminate pressure and any potential force. Defining the stream function ψ through

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \tag{1.16}$$

results in mass continuity being unconditionally satisfied (given the stream function is continuous), and then incompressible Newtonian 2D momentum and mass conservation degrade into one equation:

$$\frac{\partial}{\partial t}\left(\nabla^2\psi\right) + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\nabla^2\psi\right) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\left(\nabla^2\psi\right) = \nu\nabla^4\psi \tag{1.17}$$

or using the total derivative

$$D_t\left(\nabla^2\psi\right) = \nu\nabla^4\psi \tag{1.18}$$

where ∇^4 is the (2D) biharmonic operator and ν is the kinematic viscosity $\nu = \frac{\mu}{\rho}$. This single equation together with appropriate boundary conditions describes 2D fluid flow, taking only kinematic viscosity as a parameter. Note that the equation for creeping flow results when the left side is assumed zero. In axisymmetric flow another stream function formulation, called the Stokes stream function, can be used to describe the velocity components of an incompressible flow with one scalar function. The concept of taking the curl of the flow will become very important in ocean dynamics (section 6.5). The term $\zeta = \nabla^2\psi$ is called relative vorticity, its dynamics can be described as

$$D_t\zeta = \nu\nabla^2\zeta \quad . \tag{1.19}$$

1.6 Non-dimensional parameters: The Reynolds number

For the case of an incompressible flow in the Navier-Stokes equations, assuming the temperature effects are negligible and external forces are neglected, they consist of conservation of mass

$$\nabla \cdot \mathbf{u} = 0 \tag{1.20}$$

and conservation of momentum (1.8).

$$\partial_t\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu\nabla^2\mathbf{u} \tag{1.21}$$

where $\mathbf{u}$ is the velocity vector and p is the pressure, ν denotes the kinematic viscosity. The equations can be made dimensionless by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. For inter-comparison of analytical solutions, numerical results, and of experimental measurements, it is useful to report the results in a dimensionless system. This is justified by the important concept of dynamic similarity (Buckingham [1914]). The main goal for using this system is to replace physical or numerical parameters with some dimensionless numbers, which completely determine the dynamical behavior of the system[4]. The procedure for converting

[4] It is this fact that allows engineers to make solid predictions of how a large-scale system would perform based on a miniature model. The dimensionless quantities can often be kept constant when the size of the system is changed by using a fluid with a different viscosity during the tests. The miniature and the "real" flows are then equivalent. The Buckingham π theorem is a key theorem in dimensional analysis. It is a formalization of Rayleigh's method of

to this system first implies, first of all, the selection of some representative values for the physical quantities involved in the original equations (in the physical system). For our current problem, we need to provide representative values for velocity (U), time (T), distances (L). From these, we can derive scaling parameters for the time-derivatives and spatial-gradients also. Using these values, the values in the dimensionless-system (written with subscript d) can be defined:

$$u = U \cdot u_d \tag{1.22}$$

$$t = T \cdot t_d \tag{1.23}$$

$$x = L \cdot x_d \tag{1.24}$$

with $U = L/T$. From these scalings, we can also derive

$$\partial_t = \frac{\partial}{\partial t} = \frac{1}{T} \cdot \frac{\partial}{\partial t_d} \tag{1.25}$$

$$\partial_x = \frac{\partial}{\partial x} = \frac{1}{L} \cdot \frac{\partial}{\partial x_d} \tag{1.26}$$

Note furthermore the units of $[\rho_0] = kg/m^3$, $[p] = kg/(ms^2)$, and $[p]/[\rho_0] = m^2/s^2$. Therefore the pressure gradient term in (1.8) has the scaling U^2/L. Furthermore, devide the equation (1.8) by U^2/L and the scalings vanish completely in front of the terms except for the $\nabla_d^2 \mathrm{u_d}$-term! This procedure yields therefore for (1.20,1.21):

$$\nabla_d \cdot \mathrm{u_d} = 0 \tag{1.27}$$

dimensional analysis. Loosely, the theorem states that if there is a physically meaningful equation involving a certain number n of physical variables, then the original equation can be rewritten in terms of a set of $p = n - k$ dimensionless parameters constructed from the original variables where k is the number of physical dimensions involved. For the system (1.20,1.21), $n = 4$ for velocity, density, pressure, ν; $k = 3$ for mass, length and time; $p = 4 - 3 = 1$ one dimensionless parameter, the Reynolds number.

and conservation of momentum

$$\frac{\partial}{\partial t_d}\mathbf{u_d} + (\mathbf{u_d} \cdot \nabla_d)\mathbf{u_d} = -\nabla_d p_d + \frac{1}{Re}\nabla_d^2 \mathbf{u_d} \tag{1.28}$$

The dimensionless parameter $Re = UL/\nu$ is the Reynolds number and the only parameter left! For large Reynolds numbers, the flow is turbulent. In most practical flows Re is rather large $(10^4 - 10^8)$, large enough for the flow to be turbulent. A large Reynolds number allows the flow to develop steep gradients locally. The typical length-scale corresponding to these steep gradients can become so small that viscousity is not negligible. So the dissipation takes place at small scales. In this way different lengthscales are present in a turbulent flow, which range from L to the Kolmogorov length scale. This length scale is the typical length of the smallest eddy present in a turbulent flow. In the climate system, this dissipation by turbulence is modeled via eddy terms.

In the literature, the term "equations have been made dimensionless", means that this procedure is applied and the subscripts d are dropped.

Remark: For inter-comparison of analytical solutions, numerical results, and of experimental measurements, it is useful to report the results in a dimensionless system. The main goal for using this system is to replace physical or numerical parameters with some dimensionless numbers, which completely determine the dynamical behavior of the system.

Exercise 4 – Repeat: Concept of dynamic similarity

1. Show: The equations (1.20,1.21) can be made dimensionless by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U.
2. What is the charateristic number? Discuss that it is $\frac{\text{Convective Inertial Force}}{\text{Shear Force}}$. When the number is large, it shows that the flow is dominated by convective inertial effects. When the number is small, it shows that the flow is dominated by shear effects.
3. Please start from the potential vorticity dynamics (1.19) instead of (1.20,1.21). Derive the non-dimensionalized potential vorticity dynamics.

Remark: Later we will include the Coriolis effect (exercise 40).

1.7 Characterising flows by dimensionless numbers

The advantage of dimensionless numbers is that they make model experiments possible: one has to make the dimensionless numbers which are important for the specific experiment equal for both model and the real situation. One can also deduce functional equalities without solving the differential equations. Some dimensionless numbers are given by:

Strouhal:	$\mathrm{Sr} = \dfrac{\omega L}{v}$	Froude:	$\mathrm{Fr} = \dfrac{v^2}{gL}$	Mach:	$\mathrm{Ma} = \dfrac{v}{c}$
Fourier:	$\mathrm{Fo} = \dfrac{a}{\omega L^2}$	Péclet:	$\mathrm{Pe} = \dfrac{vL}{a}$	Reynolds:	$\mathrm{Re} = \dfrac{vL}{\nu}$
Prandtl:	$\mathrm{Pr} = \dfrac{\nu}{a}$	Nusselt:	$\mathrm{Nu} = \dfrac{L\alpha}{\kappa}$	Eckert:	$\mathrm{Ec} = \dfrac{v^2}{c\Delta T}$

Here, $\nu = \eta/\varrho$ is the *kinematic viscosity*, c is the speed of sound and L is a characteristic length of the system. α follows from the equation for heat transport $\kappa\partial_y T = \alpha\Delta T$ and $a = \kappa/\varrho c$ is the thermal diffusion coefficient.

These numbers can be interpreted as follows:

- Re: (stationary inertial forces)/(viscous forces)
- Sr: (non-stationary inertial forces)/(stationary inertial forces)
- Fr: (stationary inertial forces)/(gravity)
- Fo: (heat conductance)/(non-stationary change in enthalpy)
- Pe: (convective heat transport)/(heat conductance)
- Ec: (viscous dissipation)/(convective heat transport)
- Ma: (velocity)/(speed of sound): objects moving faster than approximately Ma = 0,8 produce shockwaves which propagate with an angle θ with the velocity of the object. For this angle holds Ma= $1/\arctan(\theta)$.

- Pr and Nu are related to specific materials.

Now, the dimensionless Navier-Stokes equation becomes, with $x_d = x/L$, $\vec{v}_d = \vec{v}/V$, $\nabla_d = L\nabla_d$, $\nabla_d^2 = L^2\nabla^2$ and $t_d = t\omega$:

$$\mathrm{Sr}\frac{\partial \vec{v}_d}{\partial t_d} + (\vec{v}_d \cdot \nabla_d)\vec{v}_d = -\nabla_d p_d + \frac{\vec{g}}{\mathrm{Fr}} + \frac{\nabla_d^2 \vec{v}_d}{\mathrm{Re}} \quad (1.29)$$

1.8 Dynamic similarity: Application in engineering*

Engineering models are used to study complex fluid dynamics problems where calculations and computer simulations are not reliable. Models are usually smaller than the final design, but not always. Scale models allow testing of a design prior to building, and in many cases are a critical step in the development process. Construction of a scale model, however, must be accompanied by an analysis to determine what conditions it is tested under. While the geometry may be simply scaled, other parameters, such as pressure, temperature or the velocity and type of fluid may need to be altered. Similitude is achieved when testing conditions are created such that the test results are applicable to the real design. The following criteria are required:

1. Geometric similarity: The model is the same shape as the application, usually scaled.
2. Kinematic similarity: Fluid flow of both the model and real application must undergo similar time rates of change motions. (fluid streamlines are similar)
3. Dynamic similarity: Ratios of all forces acting on corresponding fluid particles and boundary surfaces in the two systems are constant.

Dimensional analysis is used to express the system with as few independent variables and as many dimensionless parameters as possible. The values of the dimensionless parameters are held to be the same for both the scale model and application. The design of marine vessels remains more of an art than a science in large part because dynamic similitude is especially difficult to attain for a vessel that is partially submerged: a ship is affected by wind forces in the air above it, by hydrodynamic forces within the water under it, and especially by wave motions at the interface

Variable	Application	Scaled model	Units
L (diameter of submarine)	1	1/40	(m)
V (speed)	5	calculate	(m/s)
ρ (density)	1028	988	$(kg/m3)$
μ (dynamic viscosity)	$1.88 \cdot 10^{-3}$	$1.0 \cdot 10^{-3}$	$Pa \cdot s(Ns/m^2)$
F (force)	calculate	to be measured	N (kgm/s^2)

Table 1.2: Table shows the typical scales for the submarine model.

between the water and the air. The scaling requirements for each of these phenomena differ, so models cannot replicate what happens to a full sized vessel nearly so well as can be done for an aircraft or submarin–each of which operates entirely within one medium.

As an example, consider a submarine modeled at 1/40th scale. The application operates in sea water at $0.5°C$, moving at $5m/s$. The model will be tested in fresh water at $20°C$. Find the power required for the submarine to operate at the stated speed. A free body diagram is constructed and the relevant relationships of force and velocity are formulated. The variables which describe the system are listed in Table 1.2. This example has five independent variables and three fundamental units. The fundamental units are: metre, kilogram, second. Invoking the Buckingham π theorem shows that the system can be described with two dimensionless numbers and one independent variable. Dimensional analysis is used to re-arrange the units to form the Reynolds number (Re) and so-called pressure coefficient (pc). The pressure coefficient is a parameter for studying the flow of incompressible fluids such as water, and also the low-speed flow of compressible fluids such as air. The relationship between the dimensionless coefficient and the dimensional numbers is

$$pc = \frac{p - p_\infty}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{p - p_\infty}{p_0 - p_\infty} \tag{1.30}$$

where:

p is the static pressure at the point at which pressure coefficient is being evaluated

p_∞ is the static pressure in the freestream (i.e. remote from any disturbance)

p_0 is the stagnation pressure in the freestream (i.e. remote from any disturbance)

ρ_∞ is the freestream fluid density

V_∞ is the freestream velocity of the fluid, or the velocity of the body through the fluid.

Scaling laws:

$$Re = \left(\frac{\rho V L}{\mu}\right) \qquad \longrightarrow V_{\text{model}} = V_{\text{application}} \times \left(\frac{\rho_a}{\rho_m}\right) \times \left(\frac{L_a}{L_m}\right) \times \left(\frac{\mu_m}{\mu_a}\right) \tag{1.31}$$

$$pc = \left(\frac{2\Delta p}{\rho V^2}\right), F = \Delta p L^2 \quad \longrightarrow F_{\text{application}} = F_{\text{model}} \times \left(\frac{\rho_a}{\rho_m}\right) \times \left(\frac{V_a}{V_m}\right)^2 \times \left(\frac{L_a}{L_m}\right)^2. \tag{1.32}$$

The pressure (p) is not one of the five variables, but the force (F) is. The pressure difference has thus been replaced with (F/L^2) in the pressure coefficient. This gives a required test velocity of:

$$V_{\text{model}} = V_{\text{application}} \times 21.9.$$

A model test is then conducted at that velocity and the force that is measured in the model (F_{model}) is then scaled to find the force that can be expected for the real application $(F_{application})$:

$$F_{\text{application}} = F_{\text{model}} \times 3.44$$

The power P in Watt required by the submarine is then:

$$P[\text{W}] = F_{\text{application}} \times V_{\text{application}} = F_{\text{model}}[\text{N}] \times 17.2 \text{ m/s}$$

Note that even though the model is scaled smaller, the water velocity needs to be increased for testing. This remarkable result shows how similitude in nature is often counterintuitive.

Similitude has been well documented for a large number of engineering problems and is the basis of many textbook formulas and dimensionless quantities. These formulas and quantities are easy to use without having to repeat the laborious task of dimensional analysis and formula derivation. Similitude can be used to predict the performance of a new design based on data from an existing, similar design. In this case, the model is the existing design. Another use of similitude and models is in validation of computer simulations with the ultimate goal of eliminating the need for physical models altogether. Another application of similitude is to replace the operating fluid with a different test fluid. Wind tunnels, for example, have trouble with air liquefying in certain conditions so helium is sometimes used. Other applications may operate in dangerous or expensive fluids so the testing is carried out in a more convenient substitute.

Part II

Second part: Dynamical systems

Part III

Third part: Dynamics of the climate system

Part IV

Fourth part: Climate change

Bibliography

Abramowitz, M. and Stegun, I. A. (1965). Handbook of mathematical functions with formulas, graph, and mathematical tables. *Applied Mathematics Series*, 55:1046.

Arnold, L. (1995). *Random dynamical systems*. Springer.

Arnold, L. (2001). *Hasselmann's program revisited: The analysis of stochasticity in deterministic climate models*, volume 49. Birkhäuser, Boston.

Baker, G. L. and Blackburn, J. A. (2005). *The pendulum: a case study in physics*, volume 8. Oxford University Press Oxford.

Barber, D., Dyke, A., Hillaire-Marcel, C., Jennings, J., Andrews, J., Kerwin, M., Bilodeau, G., McNeely, R., Southon, J., Morehead, M., and Gagnonk, J.-M. (1999). Forcing of the cold event of 8,200 years ago by catastrophic drainage of laurentide lakes. *Nature*, 400(6742):344–348.

Bhatnagar, P., Gross, E. P., and Krook, M. K. (1954). A model for collision process in gases. i. small amplitude processes in charged and neutral one-component system. *Phys. Rev*, 94:511.

Boltzmann, L. (1896). *Vorlesungen über Gastheorie : 2 Volumes (in German)*. Leipzig 1895/98 UB: O 5262-6.

Boltzmann, L. (1995). *Lectures on Gas Theory*. Dover Publ. New York. ISBN 978-0486684550.

Broecker, S. and Peng, T.-H. (1982). *Tracers in the Sea*. Columbia University.

Broecker, W. S. (1987). The biggest chill. *Natural History*, 97(2):74–82.

Broecker, W. S. et al. (1991). The great ocean conveyor. *Oceanography*, 4(2):79–89.

Brüning, R. and Lohmann, G. (1999). Charles s. peirce on creative metaphor: a case study on the conveyor belt metaphor in oceanography. *Foundations of science*, 4(4):389–403.

Bryan, F. (1986). High latitude salinity effects and inter-hemispheric thermohaline circulations. *Nature*, 323(3):301–304.

Buckingham, E. (1914). On physically similar systems; illustrations of the use of dimensional equations. *Physical Review*, 4(4):345–376.

Budyko, M. I. (1969). The effect of solar radiation variations on the climate of earth. *Tellus*, 21:611–619.

Busch, W. (1865). *Max und Moritz (in German); Max and Maurice, a Juvenile History in Seven Tricks* . Braun und Schneider, München.

Cercignani, C. (1987). *The Boltzmann equation and its applications*. Springer New York. ISBN 978-0387966373.

Cercignani, C. (1990). *Mathematical methods in kinetic theory*. Plenum, 2 edition. ISBN 978-0306434600.

Chelton, D. B. and Schlax, M. G. (1996). Global Observations of Oceanic Rossby Waves. *Science*, 272:234–238.

Chen, D., Gerdes, R., and Lohmann, G. (1995). A 1-d atmospheric energy balance model developed for ocean modelling. *Theoretical and Applied Climatology*, 51:25–38.

Chorin, A. J. and Hald, O. H. (2006). Stochastic tools in mathematics and science. surveys and tutorials in the applied mathematical sciences, vol. 1.

Chorin, A. J., Kast, A. P., and Kupferman, R. (1999). Unresolved computation and optimal predictions. *Communications on pure and applied mathematics*, 52(10):1231–1254.

Chorin, A. J., Kupferman, R., and Levy, D. (2000). Optimal prediction for hamiltonian partial differential equations. *Journal of Computational Physics*, 162(1):267–297.

Courant, R., Friedrichs, K., and Lewy, H. (1928). Über die partiellen Differenzengleichungen der mathematischen Physik. *Mathematische Annalen*, 100:32–74.

Courant, R., Friedrichs, K., and Lewy, H. (1967). On the partial difference equations of mathematical physics. *IBM J. Res. Dev.*, 11(2):215–234.

Dansgaard, W., Johnsen, S., Clausen, H., Dahl-Jensen, D., Gundestrup, N., Hammer, C., C.S. Hvidberg, J. S., Sveinbjoernsdottir, A., Jouzel, J., and Bond, G. (1993). Evidence for general instability of past climate from a 250-kyr ice-core record. *Nature*, 364:218–220.

d'Humieres, D., Bouzidi, M., and Lallemand, P. (2001). Thirteen-velocity three-dimensional lattice boltzmann model. *PRE*, 63(6, Part 2).

Dijkstra, H., Raa, L. T., and Weijer, W. (2004). A systematic approach to determine thresholds of the ocean's thermohaline circulation. *Tellus A*, 56 (4):362.

Doedel, E. J., Champneys, A. R., Fairgrieve, T. F., Kuznetsov, Y. A., Sandstede, B., and Wang, X. (1997). Continuation and bifurcation software for ordinary differential equations (with homcont). *Available by anonymous ftp from ftp cs concordia ca, directory pub/doedel/auto*.

Egger, J. (2001). Master equations for climatic parameter sets. *Climate Dynamics*, 18(1-2):169–177.

Einstein, A. (1905). Investigations on the theory of the brownian movement. *Ann. der Physik*, 17:549–560.

Einstein, A. (1926). Die Ursache der Mäanderbildung der Flußläufe und des sogenannten Baerschen Gesetzes. *Naturwissenschaften*, 14:223–224.

Evans, D. J. and Morriss, G. (2008). *Statistical mechanics of nonequilibrium liquids*. Cambridge University Press.

Fairbanks, R. G. (1989). A 17, 000-year glacio-eustatic sea level record: influence of glacial melting rates on the younger dryas event and deep-ocean circulation. *Nature*, 342(6250):637–642.

Feigenbaum, M. J. (1980). The transition to aperiodic behaviour in turbulent systems. *Commun. Math. Phys.*, 77.

Flammer, C. (1957). *Spheroidal wave functions*. Stanford University Press.

Frisch, U. (1996). *Turbulence: the legacy of A.N. Kolmogorov*. Cambridge University Press. ISBN 0-521-45103-5.

Gerkema, T., Zimmerman, J., Maas, L., and Van Haren, H. (2008). Geophysical and astrophysical fluid dynamics beyond the traditional approximation. *Reviews of Geophysics*, 46(2).

Gill, A. E. (1982). *Atmosphere-ocean dynamics*, volume 30. Academic Press. International Geophysics Series.

Givon, D., Kupferman, R., and Stuart, A. (2004). Extracting macroscopic dynamics: model problems and algorithms. *Nonlinearity*, 17(6):R55.

Gottwald, G. (2010). On recent trends in climate dynamics. *AMS Gazette*, 37(5).

Grassberger, P. and Procaccia, I. (1983). Measuring the strangeness of strange attractors. *Physica D: Nonlinear Phenomena*, 9(2):189–208.

Haken, H. (1996). Slaving principle revisited. *Physica D: Nonlinear Phenomena*, 97(1):95–103.

Haney, R. L. (1971). Surface thermal boundary conditions for ocean circulation models. *Journal of Physical Oceanography*, 1:241–248.

Hasselmann, K. (1976). Stochastic climate models. Part I. Theory. *Tellus*, 6:473–485.

He, X. and Luo, L. S. (1997). Theory of the lattice Boltzmann method: From the Boltzmann equation to the lattice Boltzmann equation. *Phys. Rev. E*, 56(6):6811–6817.

Holton, J. R. (2004). *An Introduction to Dynamic Meteorology*. Elsevier Academic Press, Burlington, MA.

Kambe, T. (2007). *Elementary Fluid Mechanics*. World Scientific Publishing.

Kuznetsov, Y. A. (1998). *Elements of applied bifurcation theory*, volume 112. Springer, New York.

Landau, L. D. and Lifshitz, E. M. (1959). *Fluid Mechanics*, volume 6 of *Course of Theoretical Physics*. Pergamon Press, Oxford.

Langevin, P. (1908). On the theory of brownian motion. *Comptes Rendues*, 146:530–533.

Leith, C. (1975). Climate response and fluctuation dissipation. *Journal of the Atmospheric Sciences*, 32(10):2022–2026.

Lohmann, G. (2003). Atmospheric and oceanic freshwater transport during weak atlantic overturning circulation. *Tellus A*, 55(5):438–449.

Longuet-Higgins, M. S. (1968). The eigenfunctions of laplace's tidal equations over a sphere. *Philosophical Transactions for the Royal Society of London. Series A, Mathematical and Physical Sciences*, pages 511–607.

Lorenz, E. (1982). Atmospheric predictability experiments with a large numerical model. *Tellus A*, 34:505–513.

Lorenz, E. N. (1960). Maximum simplification of the dynamic equations. *Tellus*, 12(3):243–254.

Lorenz, E. N. (1963). Deterministic nonperiodic flow. *Journal of the atmospheric sciences*, 20(2):130–141.

Lorenz, E. N. (1976). Nondeterministic theories of climatic change. *Quaternary Research*, 6(4):495–506.

Lorenz, E. N. (1984). Irregularity: a fundamental property of the atmosphere*. *Tellus A*, 36(2):98–110.

Lucarini, V., Blender, R., Herbert, C., Pascale, S., Ragone, F., and Wouters, J. (2014). Mathematical and physical ideas for climate science. *Rev. Gephys.*

Maas, L. R. (1994). A simple model for the three-dimensional, thermally and wind-driven ocean circulation. *Tellus A*, 46(5):671–680.

Manabe, S. and Stouffer, R. (1993). Century-scale effects of increased atmospheric CO_2 on the ocean atmosphere system. *Nature*, 364:215–218.

Mandelbrot, B. B. (1967). How long is the coast of britain: Statistical self-similarity and fractal dimension. *Science*, 155:636–638.

Mandelbrot, B. B. (1983). *The fractal geometry of nature*. Macmillan.

Matsuno, T. (1966). Quasi-geostrophic motions in the equatorial area. *J. Meteor. Soc. Japan*, 44(1):25–43.

Mori, H. (1965). Transport, collective motion, and brownian motion. *Progress of Theoretical Physics*, 33(3):423–455.

Mori, H., Fujisaka, H., and Shigematsu, H. (1974). A new expansion of the master equation. *Progress of Theoretical Physics*, 51(1):109–122.

Müller and Maier-Reimer (2000). Trapped rossby waves. *Phys. Rev. E*, 61:1468 – 1485.

Müller, D., Kelly, B., and O'brien, J. (1994). Spheroidal eigenfunctions of the tidal equation. *Physical review letters*, 73(11):1557.

Müller, D. and O'Brien, J. (1995). Shallow water waves on the rotating sphere. *Physical Review E*, 51(5):4418.

Olbers, D. (2001). A gallery of simple models from climate physics. *In: Stochastic Climate Models, Progress in Probability (Eds.: P. Imkeller and J. von Storch)*, 49:3–63.

Peitgen, H.-O. and Richter, P. (1986). *The Beauty of Fractals*. Heidelberg: Springer-Verlag.

Proudman, J. (1916). On the motion of solids in a liquid possessing vorticity. *Proc. R. Soc. Lond. A*, 92:408–424.

Rahmstorf, S. (1996). On the freshwater forcing and transport of the Atlantic thermohaline circulation. *Climate Dynamics*, 12:799–811.

Rayleigh, L. (1916). On convection currents in a horizontal layber of fluid, when the higher temperature is on the under side. *Phil. Mag.*, 6:529–546.

Rooth, C. (1982). Hydrology and ocean circulation. *Progress in Oceanography*, 11:131–149.

Rossby, C.-G. (1939). "relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action". *Journal of Marine Research*, 2 (1):38–55.

Saltzman, B. (1962). Finite amplitude free convection as an initial value problem – i. *Journal of the Atmoshpheric Sciences*, 19:329–341.

Shannon, C. E. (1948). A Mathematical Theory of Communication. *Bell System Technical Journal*, 27 (3):379–423.

Stommel, H. (1961). Thermohaline convection with two stable regimes of flow. *Tellus*, 13:224–230.

Strogatz, S. (2000). *Non-linear Dynamics and Chaos: With applications to Physics, Biology, Chemistry and Engineering*. Perseus Books.

Taylor, G. (1917). Motion of solids in fluids when the flow is not irrotational. *Proc. R. Soc. Lond. A*, 93:92–113.

Townsend, S., Lenosky, T., Muller, D., Nichols, C., and Elser, V. (1992). Negatively curved graphitic sheet model of amorphous carbon. *Physical Review Letters*, 69(6):921–924.

Tritton, D. J. (1988). *Physical Fluid Dynamics*. Oxford University Press, Science Publication. ISBN 978-0-19-854493-7.

Uhlenbeck, G. E. and Ornstein, L. S. (1930). On the theory of the brownian motion. *Physical review*, 36(5):823.

van Kampen, N. G. (1981). *Stochastic processes in physics and chemistry*. North Holland. IISBN 978-0-444-52965-7.

Wüst, G. (1935). Schichtung und Zirkulation des Atlantischen Ozeans. Das Bodenwasser und die Stratosphäre. *Wiss. Ergebn. Dtsch. Atlant. Exped. 'Meteor' 1925-1927*, 6(1):1–288.

Zwanzig, R. (1960). Ensemble method in the theory of irreversibility. *The Journal of Chemical Physics*, 33:1338.

Zwanzig, R. (1980). Problems in nonlinear transport theory. In *Systems far from equilibrium*, pages 198–225. Springer.

Zwanzig, R. (2001). *Nonequilibrium statistical mechanics*. Oxford University Press, USA.