Climate Dynamics:

Concepts, Scaling and Multiple Equilibria

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Contents

General framework: Climate dynamics

Over the last century, humans have altered the composition of the Earth's atmosphere and surface to the extent that these factors measurably affect current climate conditions. Paleoclimate reconstructions, in particular from ice cores have also shown that climate can change over relatively short periods such as a few years to decades. The objective of the book is to examine fundamental concepts used to understand climate dynamics. Here, we will approach climate dynamics from a fluid dynamics and complex systems point of view. The script has several parts, an application follows after every theoretical section. The content (part I-IV) is designed for 12 lessons for a master course at the University of Bremen (Dynamics II).

Part I deals with the general structure of fluid dynamical models. Like the ocean, the atmosphere is considered as a Newtonian Fluid. The concepts of scaling and vorticity are introduced. Ice dynamics is not explicitely considered here although it is an important part of the Earth system. One application deals with the Rayleigh-Bénard convection. In the script, a framework to analyze the stability of dynamical systems is presented. These systems provide the prototype of nonlinear dynamics, bifurcations, multiple equilibria. A bifurcation occurs when a parameter change causes the stability of an equilibrium. In his classic studies of chaotic systems, Lorenz has proposed a deterministic theory of climate change with his concept of the 'almost-intransitivity' of the highly non-linear climate systems. In the Lorenz equations exist the possibility of multiple stable solutions and internal variability, even in the absence of any variations in external forcing [Lorenz, 1976]. More complex models, e.g. Bryan [1986]; Dijkstra et al. [2004] also demonstrated this possibility.

In Part II, basic concepts of large-scale meteorology and oceanography are explored. The Coriolis effect is one of the dominating forces for the large-scale dynamics of the oceans and the atmosphere. In meteorology and ocean science, it is convenient to use a rotating frame of reference where the Earth is stationary. The resulting flow can be derived from scaling arguments. Several approximations can be done since the scales of the components in the dynamical equations differ in the orders of magnitude. One fundamental aspect of ocean dynamics are waves. A short theory is given and numerical examples are provided. Furthermore, the deep ocean circulation is studied in a conceptual box model. Here, we introduce an interhemispheric box model of the deep ocean circulation to study the feedbacks in the climate system. Finally, some of the waves in the climate system are introduced.

Part III deals with the stochastic climate model. This part of the course touches also statistical mechanics and applications. Several additional sections are included for those who have some more time.

The numerical examples in part IV chapters are helpful for the students who are already familiar with programming (they can improve the code and follow the main ideas of the code etc.), for those who are not familiar they should take it as a starting point for more research. Several task do not require that the complete code is understood, but one can change initial conditions or parameters in the problems.

Part I

First part: Fluid Dynamics and Dynamical Systems

Part II

Second part: Dynamics of the Climate System

Chapter 6

Application: Climate-Box-Model

6.1 Model description

Here we introduce an interhemispheric box model of the deep ocean circulation to study the feedbacks in the climate system. Like in the model of Rooth [1982] the Atlantic Ocean is described over both hemispheres. The box model consists of four oceanic and three atmospheric boxes, as indicated in Fig. 6.1. The ocean boxes represent the Atlantic Ocean from $80°N$ to $60°S$. The indices of the temperatures T, the salinities S, the surface heat fluxes H, the atmospheric heat fluxes F, the radiation terms R as well as later on the volumes bear on the different boxes (N for the northern, M for the tropical, D for the deep and S for the southern box).

The discrete boxes are utterly homogeneous, which implies that the temperatures and the salinities everywhere within one box are alike. The climate model is based on mass and energy considerations. Emphasis is placed on the overturning flow Φ of the ocean circulation.

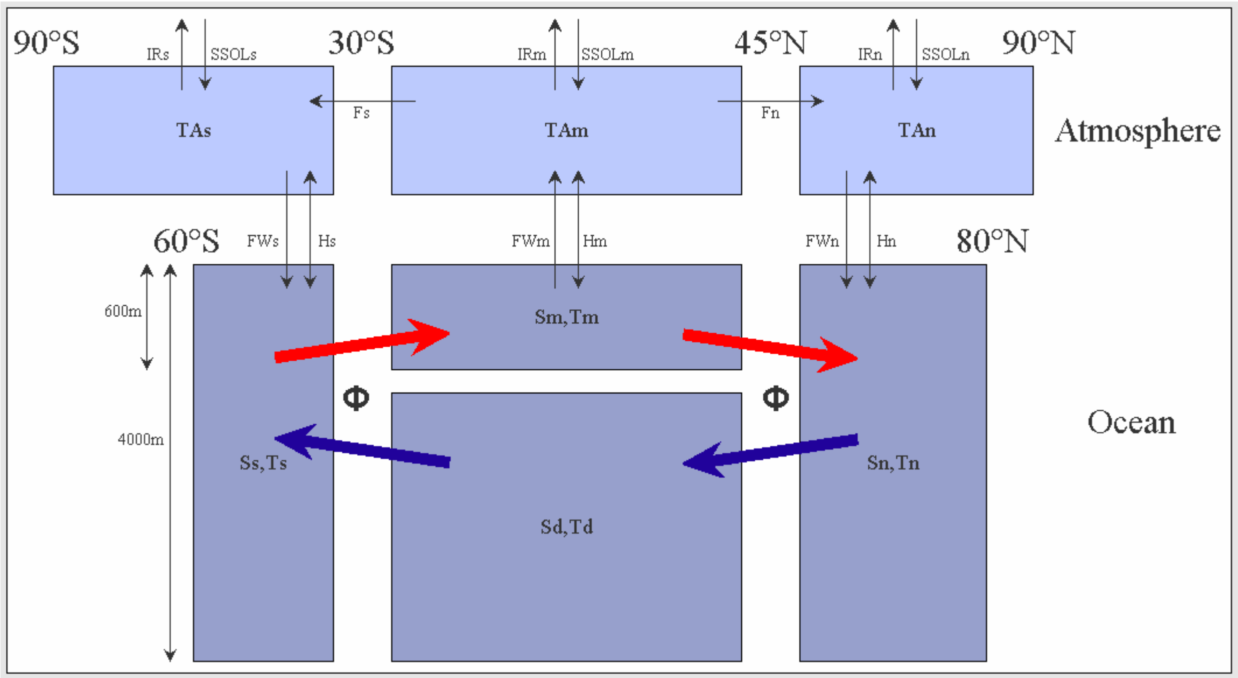


Figure 6.1: Schematic illustration of the Climate-Box-Model

The prognostic equations for the temperatures of the ocean boxes read

$$\frac{d}{dt}T_N = -(T_N - T_M)\,\frac{\Phi}{V_N} + \frac{H_N}{\rho_0 c_p dz_2}, \tag{6.1}$$

$$\frac{d}{dt}T_M = -(T_M - T_S)\,\frac{\Phi}{V_M} + \frac{H_M}{\rho_0 c_p dz_1}, \tag{6.2}$$

$$\frac{d}{dt}T_S = -(T_S - T_D)\,\frac{\Phi}{V_S} + \frac{H_S}{\rho_0 c_p dz_2} \quad \text{and} \tag{6.3}$$

$$\frac{d}{dt}T_D = -(T_D - T_N)\,\frac{\Phi}{V_D} \tag{6.4}$$

where ρ_0 denotes a reference density for saltwater and c_p the specific heat capacity of water. The factors dz_i and V_i indicate the depths and volumes of the discrete ocean boxes, respectively. The first terms in the equations are proportional to the overturning flow Φ and represent the advective transport between the boxes. The second terms (except for the deep box) represent the surface heat fluxes coupling the ocean and atmosphere. The overturning flow is assumed to be proportional to the density gradients of the oceans boxes after Stommel [1961]. Like in Rahmstorf [1996] the northern and the southern box will be taken into account for this, which leads to the equation for the calculation of the overturning flow

$$\Phi = c\left[-\alpha\left(T_N - T_S\right) + \beta\left(S_N - S_S\right)\right] \tag{6.5}$$

The constants α and β represent the thermal and the haline expansion coefficients in the equation of state. c is an adjustable parameter which is set to produce present-day overturning rates. This form of the overturning is also explained in section 5.2.[1]

The surface heat fluxes can be simplified according to Haney [1971]:

$$H_i = Q_{1_i} - Q_2\left(T_i - T_{A_i}\right) \tag{6.6}$$

Analogue to (6.1) to (6.4) the prognostic differential equations for the salinities consist out of two components. One of those is again the advective part, caused by the interconnection between the boxes and the other one is the influence of the freshwater fluxes between the ocean and the atmosphere. The latter is again only for the boxes near the surface, thus the equations are

$$\frac{d}{dt}S_N = -\left(S_N - S_M\right)\frac{\Phi}{V_N} - S_{ref}\frac{(P-E)_N}{dz_N}, \tag{6.7}$$

$$\frac{d}{dt}S_M = -\left(S_M - S_S\right)\frac{\Phi}{V_M} + S_{ref}\frac{(P-E)_M}{dz_M}, \tag{6.8}$$

$$\frac{d}{dt}S_S = -\left(S_S - S_D\right)\frac{\Phi}{V_S} - S_{ref}\frac{(P-E)_S}{dz_S}, \tag{6.9}$$

$$\frac{d}{dt}S_D = -\left(S_D - S_N\right)\frac{\Phi}{V_D}. \tag{6.10}$$

[1] For other scaling laws: [Maas, 1994]. In his model, the dynamics bears similarities with the Lorenz system.

The reference salinity S_{ref} is a characteristic average value for the entire Atlantic Ocean, and the freshwater fluxes are denoted as precipitation minus evaporation (P-E). These freshwater fluxes are calculated by the divergence of the latent heat transport in the atmosphere and are assumed to be proportional to the meridional moisture gradient explained below.

The atmospheric energy-balance-model (EBM) calculates the heat fluxes between the ocean and atmosphere, as well as horizontal latent and sensible heat transports as diffusion following Chen et al. [1995]. The EBM contains sensible and latent heat transports, radiation R_i, as well as the surface heat fluxes H_i between the atmosphere and the ocean. The atmospheric temperatures T_{A_i} follow the prognostic equations

$$c_2 \frac{d}{dt} T_{A_N} = \frac{\partial \left(F_{s_N} + F_{l_N}\right)}{\partial y} + R_N - H_N, \tag{6.11}$$

$$c_2 \frac{d}{dt} T_{A_M} = \frac{\partial \left(F_{s_S} + F_{l_S}\right)}{\partial y} + R_M - H_M, \tag{6.12}$$

$$c_2 \frac{d}{dt} T_{A_S} = \frac{\partial \left(F_{s_S} + F_{l_S}\right)}{\partial y} + R_S - H_S. \tag{6.13}$$

c_2 is related to the specific heat of air. The sensible F_{s_i} and latent F_{l_i} heat transport are described in dependence of the meridional gradient of the surface temperature T_A and moisture q

$$F_s = K_s \frac{\partial T_A}{\partial y} \tag{6.14}$$

$$F_l = K_l \left(\frac{\partial q}{\partial y}\right). \tag{6.15}$$

K_s and K_l are empirical parameters, which must be adjusted to generate realistic values for sensible and latent heat transports. The radiation terms R_i in (6.11) to (6.13) consist of an incoming solar shortwave S_i and an outgoing infrared longwave I_i part. The extraterrestrial solar radiation is not absorbed entirely, and a latitude-dependent average albedo α_i is introduced to account for the reflectance. The outgoing infrared radiation I_i is calculated through a linear formula of Budyko

[1969]. Thus, the equation for the net radiation balance is

$$R_i = S_i - I_i = S_{sol,i}\,(1-\alpha_i) - (A + BT_{A_i})\,. \tag{6.16}$$

In this model, one can even include thw effect for changes in the the greenhouse gases (by multiplying $A + BT_{A_i}$ with a factor γ) and changes in the solar constant (by changing $S_{sol,i}$) which is left to the reader (see also the exercises).

The model calculates the freshwater fluxes from the divergence of the latent heat transport $(P - E \sim \partial F_l/\partial y)$. The integration of the system is implemented with an Euler-forward scheme. The time step is $1/100$ of a year to ensure the stability of the system according to the Courant-Friedrichs-Levy-Criterion (CFL-Criterion, Courant et al. [1928][2]).

6.2 Run the model

Here, we will use this box model using R. iFurthermore, it is recomended to use R studio, which provides a user interface for R. Perturbation experiments are done for the four ocean boxes. First the function sevenbox.r has to be defined, then the script must be run selecting the perturbations in the different boxes:

```
source('sevenbox.r')

sevenbox("N")  # for the northern box
y=sevenbox("N",perturbation=-0.1)
plot(y$t,y$phi)
```

One particular package is R Shiny which provides a Gui web application easy to use. Download sevenbox.r , ui.R , server.R , run_ui.R , ageStructureFunctions.R , sevenbox_plot_func.R .

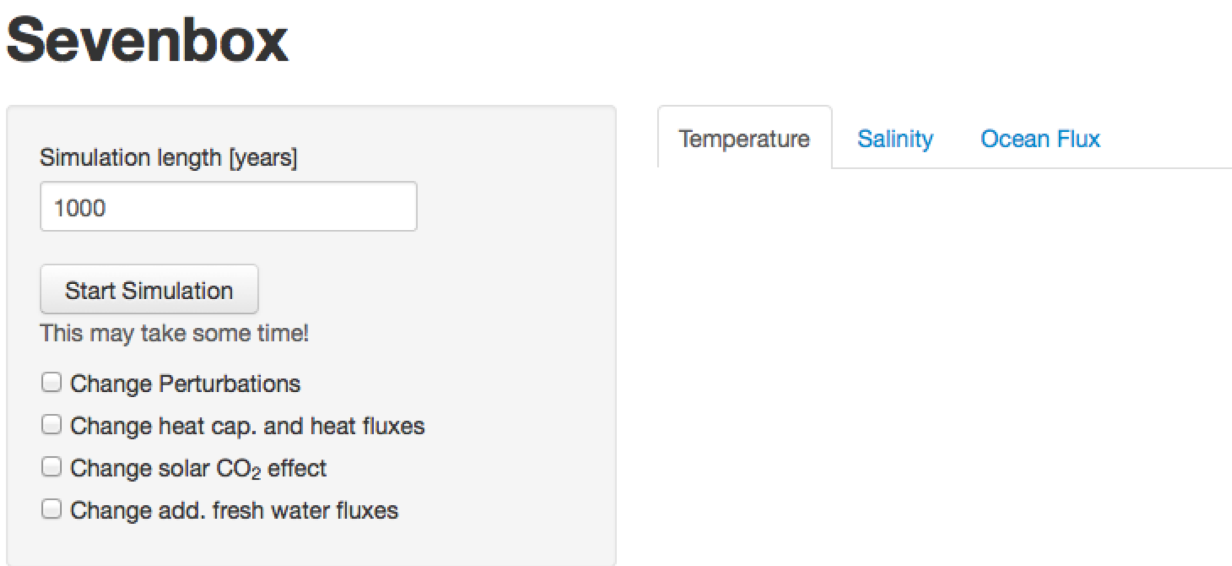
```
# go to the directory  (setwd)
source('run_ui.R')  # load Script
run_ui()   # run Script

# for multicore: provide the numbers of processors, e.g.
run_ui(8)
```

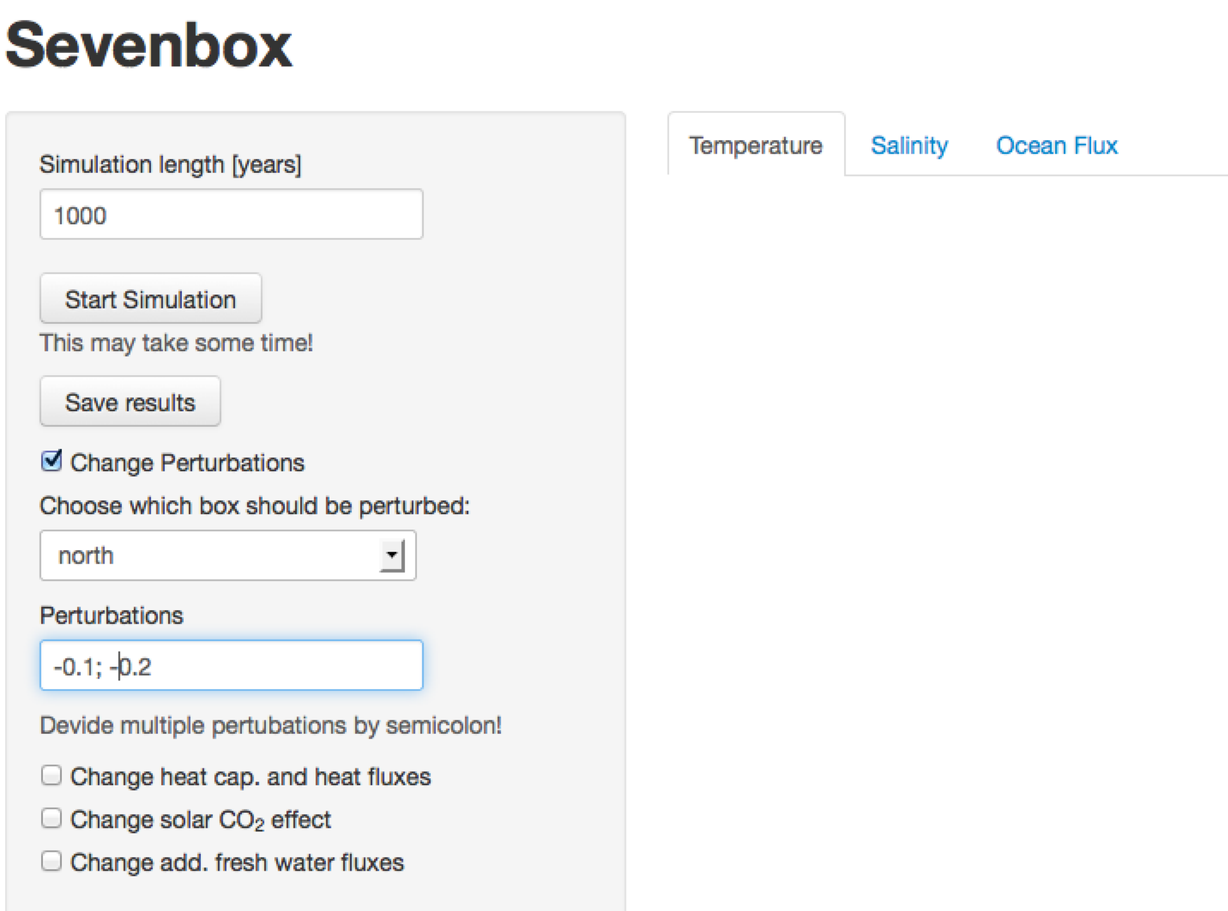
[2] For an English translation, refer to Courant et al. [1967].

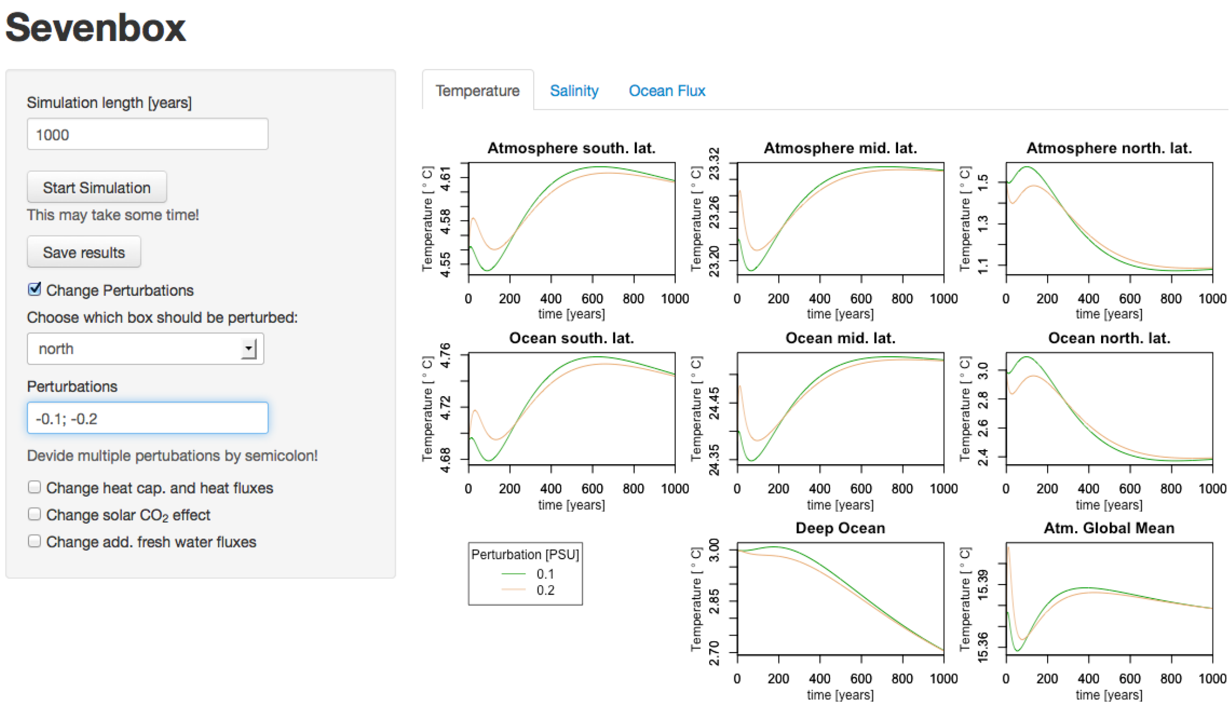
```
#  or do the following:
library(shiny)
runApp('Boxmodel_GUI')
# if you put everything into the directory Boxmodel_GUI
```

The code creates png files of model output. The coding follows the names in Fig. 6.1, and the temperature (6.1, 6.2, 6.3 , 6.4) and salinity (6.7, 6.8, 6.9 , 6.10) budgets, respectively.

Furthermore, the interhemispheric Box model on the web is available through https://paleosrv2.awi.de/. The username is *student* and the Password is *EbJir5ow* ! Fig. 6.2 illustrates how the model works.


Sevenbox
Simulation length [years]
1000
Start Simulation
This may take some time!
Change Perturbations
Change heat cap. and heat fluxes
Change solar CO_2 effect
Change add. fresh water fluxes
Temperature
Salinity
Ocean Flux


Sevenbox
Simulation length [years]
1000
Start Simulation
This may take some time!
Save results
Change Perturbations
Choose which box should be perturbed:
north
Perturbations
-0.1; -0.2
Devide multiple pertubations by semicolon!
Change heat cap. and heat fluxes
Change solar CO_2 effect
Change add. fresh water fluxes
Temperature
Salinity
Ocean Flux


Sevenbox
Simulation length [years]
1000
Start Simulation
This may take some time!
Save results
Change Perturbations
Choose which box should be perturbed:
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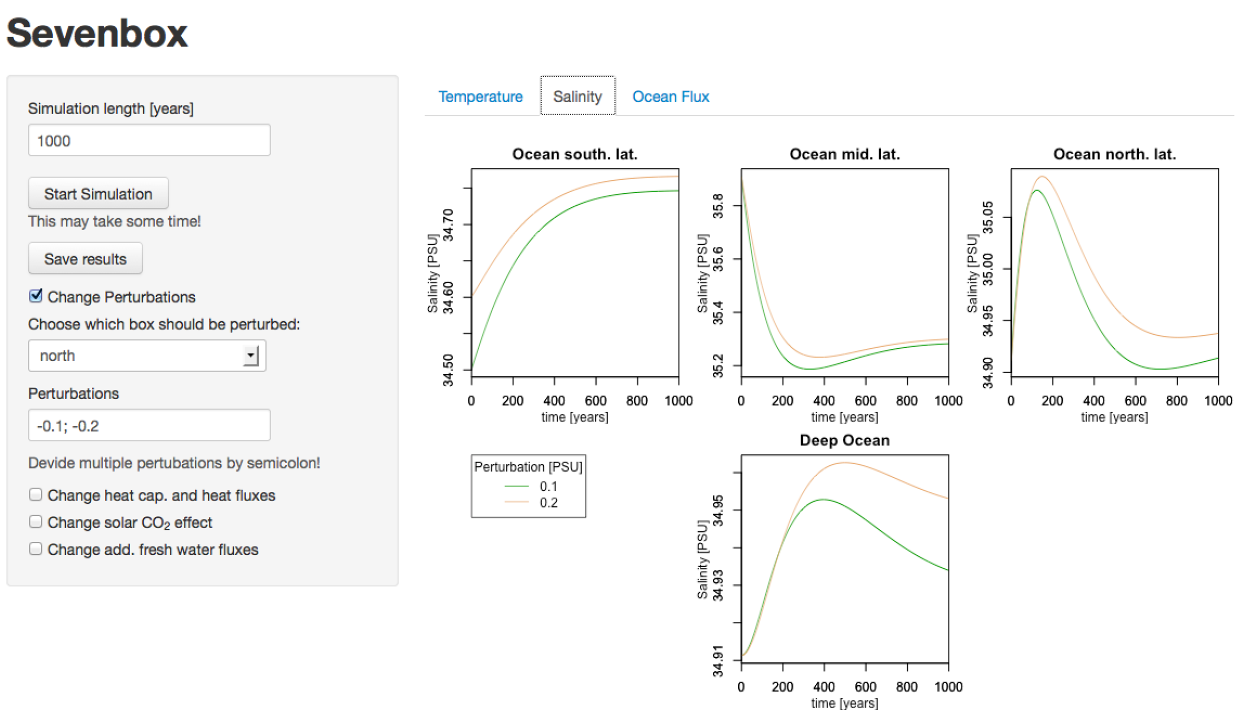
Perturbations
-0.1; -0.2
Devide multiple pertubations by semicolon!
Change heat cap. and heat fluxes
Change solar CO2 effect
Change add. fresh water fluxes
Temperature
Salinity
Ocean Flux
Atmosphere south. lat.
Atmosphere mid. lat.
Atmosphere north. lat.
Ocean south. lat.
Ocean mid. lat.
Ocean north. lat.
Deep Ocean
Atm. Global Mean
Temperature [° C]
time [years]
Perturbation [PSU]
0.1
0.2


Sevenbox
Simulation length [years]
1000
Start Simulation
This may take some time!
Save results
Change Perturbations
Choose which box should be perturbed:
north
Perturbations
-0.1; -0.2
Devide multiple pertubations by semicolon!
Change heat cap. and heat fluxes
Change solar CO2 effect
Change add. fresh water fluxes
Temperature
Salinity
Ocean Flux
Ocean south. lat.
Ocean mid. lat.
Ocean north. lat.
Deep Ocean
Salinity [PSU]
time [years]
Perturbation [PSU]
0.1
0.2

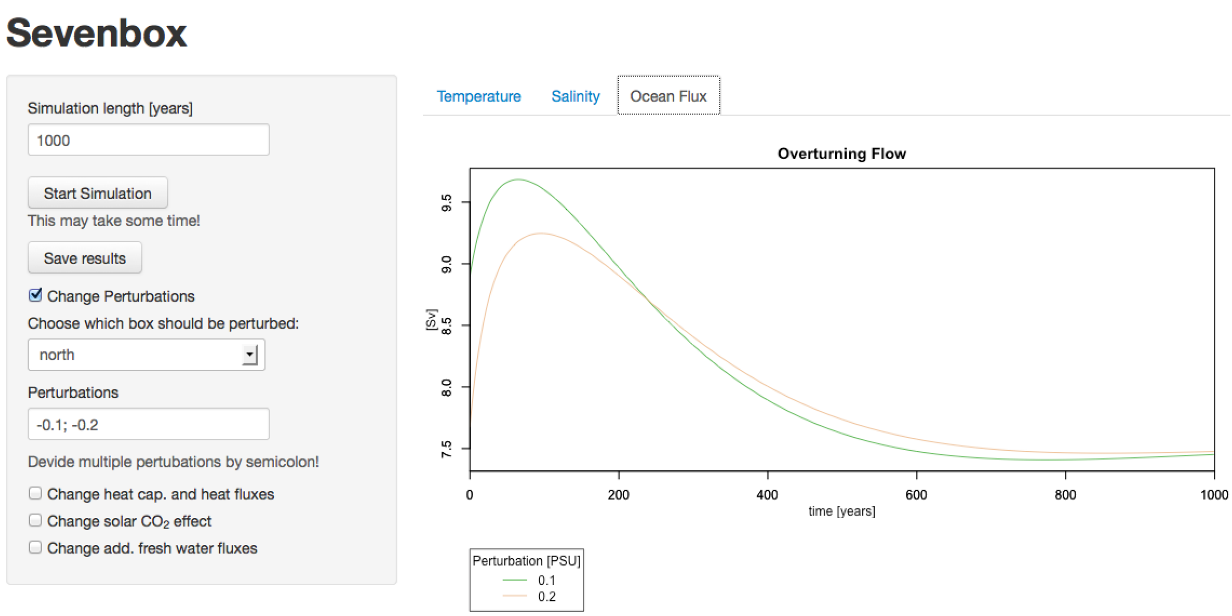


Figure 6.2: Output of the climate box model illustrating the procedure.

6.3 Model scenarios

Paleoclimatic evidence suggests (e.g. Dansgaard et al. [1993]) that some past climate shifts were associated with changes in North Atlantic Deep Water (NADW) formation. Deep water formation in the Greenland-Iceland-Norwegian Sea and in Labrador Sea drive the large-scale ocean circulation imposing strong northward heat transport. This makes the northern North Atlantic about 4 K warmer than corresponding latitudes in the Pacific and is responsible for the mild climate of Western Europe. Variations in NADW circulation therefore have the potential to cause significant climate change in the North Atlantic region.

Numerical simulations by Manabe and Stouffer [1993] showed, for the North Atlantic, that between two and four times the preindustrial CO_2 concentration, a threshold value is passed and the thermohaline circulation ceases completely. One other example of early Holocene rapid climate change is the '8200 yr BP' cooling event recorded in the North Atlantic region possibly induced by freshwater. One possible explanation for this dramatic regional cooling is a shutdown in the formation of deep water in the northern North Atlantic due to freshwater input caused by catastrophic drainage of Laurentide lakes (e.g., Barber et al. [1999]; Lohmann [2003]). After the end of the last glacial, freshwater entered into the Atlantic Ocean (Fig. **??**) and may have affected the ocean circulation.

Exercise 46 – Investigations with the box-model

1. In the regions of deep water formation in the North Atlantic, relatively small amounts of fresh water added to the surface can stabilize the water column to the extent that convection can be prevented from occurring. Such interruption decreases the poleward oceanic mass transport Φ. Furthermore, this perturbation of the meridional transport can be amplified by positive feedbacks: a weaker northward salt transport brings less dense water to high latitudes, which further reduces the high-latitude density. Discuss the case where the initial conditions in salinity at different latitudes is changed. Show this scenario in the box model!

2. Comment on the scenario of climate change as shown in the cinema movie The Day After Tomorrow: link to the website or go to the trailer.

3. Which feedbacks are acting for global warming? You can change the long wave radiation. A doubling of pCO_2 is equivalent to an additional forcing of 4 Wm^{-2}. For this you have to modify the net radiation balance (6.16) through reduction in the outgoing longwave radiation (parameter γ). Additional radiative forcing may come from increased tracer gas concentrations in the atmosphere. Please evaluate the hydrological cycle and atmospheric heat transports! What is the change in the overturning rate?

4. Change the ocean heat capacity by a factor of 10 and describe the change in the response to warming induced by 90

5. The initial values of the model represent averages for present-day climate conditions. Determine the effect of the parameter c in the numerical example (representing a different long wave radiation) Can you derive a glacial climate? The glacial climate was 3 K colder in the tropics.

6. Calculate the ocean heat transport in the model and compare it with the following estimate!

$$
\begin{aligned}
H &= \int_{bottom}^{top} \rho_0 v T dz && (6.17) \\
&= -c_p \int_{bottom}^{top} \frac{\partial \Phi}{\partial z} \quad T dz && (6.18) \\
&= c_p \int_{bottom}^{top} \Phi \quad \frac{\partial T}{\partial z} dz && (6.19) \\
&= c_p \int_{T(bottom)}^{T(top)} \quad \Phi dT && (6.20)
\end{aligned}
$$

where $\Phi = \rho_0 \Phi_{MOC}$ with Φ_{MOC} being the volume transport. Therefore, the heat transport can be estimated in terms of the mass transport in temperature layers:

$$
H = c_p \underbrace{(T(top) - T(bottom))}_{15K} \quad \underbrace{\Phi_{max}}_{20 \cdot 10^9 kg/s} \tag{6.21}
$$

which is about $1.2\ PW (PW = 10^{15} W)$.

7. Question for specialists: The coupled model shall be used to investigate the sensitivity of the system with respect to stochastic weather perturbations reflecting unresolved effects of the atmospheric transports modeled as white noise. How will the atmospheric noise influence the stability of the system?

Part III

Third part: Stochastic climate model and Mesoscopic Dynamics

Part IV

Fourth part: Programming and tools

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