Chapter 7

Waves in the climate system

7.1 Shallow water dynamics

One of the most understood dynamics are the tidal equation or shallow water dynamics (e.g. Gill [1982]). The equations are derived from depth-integrating the Navier-Stokes equations, in the case where the horizontal length scale is much greater than the vertical length scale.Under this condition, conservation of mass implies that the vertical velocity of the fluid is small. The variables u and v denote zonal and meridional perturbation flow velocity, and η the height perturbation. The pressure in the vertically homogenous ocean is $p = g\rho(H + \eta)$. The dynamics is as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} - fv = -g\frac{\partial \eta}{\partial x} \tag{7.1}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + fu = -g\frac{\partial \eta}{\partial y} \tag{7.2}$$

where $x = R\lambda, y = R\cos\varphi$ denote eastward distance and distance from the equator, respectively. The equation for the conservation of mass

$$\frac{\partial}{\partial t}\Big(\rho\,(H+\eta)\Big) + \frac{\partial}{\partial x}\Big(u\rho\,(H+\eta)\Big) + \frac{\partial}{\partial y}\Big(v\rho\,(H+\eta)\Big) \;=\; 0$$

and since the density is constant it reads

$$\frac{\partial}{\partial t}\eta \;+\; u\frac{\partial}{\partial x}\eta + v\frac{\partial}{\partial y}\eta \;+\frac{\partial}{\partial x}(Hu) \;+\; \frac{\partial}{\partial y}(Hv) \;=\; 0 \qquad . \tag{7.3}$$

Lagrangian invariant of the shallow water dynamics

The dynamical system (7.1,7.2,7.3) has the Lagrangian invariant

$$D_t\Big(\frac{\nabla^2\psi + f}{H+\eta}\Big) = D_t q = 0 \tag{7.4}$$

where $\nabla^2\psi \;=\; \partial_x v \;-\; \partial_y u$ is the relative vorticity and ψ the streamfunction. The dynamical system (7.1,7.2,7.3) has integral invariants in domains ξ where the fluxes are zero or cancel, e.g. in periodic domains. One such invariant is the energy

$$E = \frac{1}{2}\int\Big((H+\eta)(u^2+v^2) + g\eta^2\Big)d\xi \tag{7.5}$$

and for any scalar functions f(q) of potential vorticity q, another class of integral invariants has the form

$$S = \frac{1}{2}\int (H+\eta) f(q) d\xi \tag{7.6}$$

When function f is the square function $\sim q^2$, this invariant is called potential enstrophy.

Shallow water dynamics: linear model

We now simplify the system to a linear model. Ignoring bulk advection (u and v are small) in (7.1,7.2,7.3), and assuming the wave height is a small proportion of the mean height $(\eta << H)$, we have:

$$\begin{aligned}
\partial_t u &= f\ v - g\,\partial_x \eta && (7.7)\\
\partial_t v &= -f\ u - g\,\partial_y \eta && (7.8)\\
\partial_t \eta &= -\partial_x(Hu) - \partial_y(Hv) \quad . && (7.9)
\end{aligned}$$

Skew-Hermetian propertiy of the linear shallow water dynamics

The dynamical system (7.7,7.8,7.9) can be rewritten in a more compact form (using the non-dimensional values).

$$\partial_t W + \mathsf{L} W = 0 \tag{7.10}$$

With $W = (u, v, \eta)$ and the operaor

$$\mathsf{L} = \begin{pmatrix} 0 & -f & \partial_x \\ f & 0 & \partial_y \\ \partial_x & \partial_y & 0 \end{pmatrix} \quad . \tag{7.11}$$

The x and t dependences can be separated in form of zonally propagating waves $\exp(ikx - i\omega t)$. W can therfore be writen as

$$W(x, y, t) = \begin{pmatrix} \hat{u}(y) \\ \hat{v}(y) \\ \hat{\eta}(y) \end{pmatrix} \exp(ikx - i\omega t) = \hat{W}\ \exp(ikx - i\omega t) \tag{7.12}$$

This leads to an eigenvalue problem

$$-i\omega\hat{\boldsymbol{W}}(k,y) + \hat{\mathbf{L}}\hat{\boldsymbol{W}}(k,y) = 0 \tag{7.13}$$

where

$$\hat{\mathbf{L}} = \begin{pmatrix} 0 & -f & ik \\ f & 0 & \partial_y \\ ik & \partial_y & 0 \end{pmatrix} \quad . \tag{7.14}$$

The adjoint of $\hat{\mathbf{L}}$ with respect to the inner product is the operator $\hat{\mathbf{L}}^+$ (transpose and conjugate):

$$\hat{\mathbf{L}}^+ = \begin{pmatrix} 0 & f & -ik \\ -f & 0 & \partial_y \\ -ik & \partial_y & 0 \end{pmatrix} \quad . \tag{7.15}$$

The operator $\hat{\mathbf{L}}$ is skew-Hermetian, as the adjoint of $\hat{\mathbf{L}}$ is $\hat{\mathbf{L}}^+ = -\hat{\mathbf{L}}$ (7.15). For two arbitrary vector functions $\boldsymbol{W_1}, \boldsymbol{W_2}$ one can define a scalar product:

$$(\hat{\mathbf{L}}\boldsymbol{W_1}, \boldsymbol{W_2}) = \int_{-\infty}^{\infty} \hat{\mathbf{L}}\boldsymbol{W_1} \cdot \boldsymbol{W_2^*} dy = (\boldsymbol{W_1}, -\hat{\mathbf{L}}\boldsymbol{W_2}) = (\boldsymbol{W_1}, \hat{\mathbf{L}}^+\boldsymbol{W_2}) \tag{7.16}$$

with the symbol * being the conjugate.

The skew-Hermetian property dictates that the eigenvalues of $\hat{\mathbf{L}}$ are purely imaginary, so that we have a mathematical basis for looking for wave-like solutions. In addition, the eigenfunctions form a complete orthogonal set for the functions $\boldsymbol{W}$ satisfying $(\boldsymbol{W}, \boldsymbol{W}) < \infty$. This is because $\hat{\mathbf{L}}$ is normal:

$$\hat{\mathbf{L}}\hat{\mathbf{L}}^+ = \hat{\mathbf{L}}^+\hat{\mathbf{L}} \quad . \tag{7.17}$$

Furthermore, $\hat{\mathsf{L}}$ in (7.14) belongs to the unitary group $U(3)$, forming a compact connected Lie group and has the special property $\det(\mathsf{L}) = 0$.

These considerations provide the mathematical framework for wave studies. Analytical work is presented in section 7.5 in the case of equatorial wave dynamics. The dynamical system (7.7,7.8,7.9) contains already the zoo of waves. Here, we give a short description. In the exercises, these waves are numerically solved.

Exercise 47 – Energy conservation

Show that the dynamical system (7.1,7.2,7.3) has integral invariants in domains ξ where the fluxes are zero or cancel, e.g. in periodic domains. One such invariant is the energy

$$E = \frac{1}{2}\int\Big((H+\eta)(u^2+v^2)+g\eta^2\Big)d\xi \tag{7.18}$$

and for any scalar functions f(q) of potential vorticity q, another class of integral invariants has the form

$$S = \frac{1}{2}\int(H+\eta)f(q)d\xi \tag{7.19}$$

7.2 Planetary waves on the computer

Rossby (or planetary) waves are giant meanders in high-altitude winds that are a major influence on weather. They are easy to observe as (usually 4-6) large-scale meanders of the jet stream. When these loops become very pronounced, they detach the masses of cold, or warm, air that become cyclones and anticyclones and are responsible for day-to-day weather patterns at mid-latitudes.

Each large meander, or wave, within the jet stream is known as a Rossby wave (planetary wave). Rossby waves are caused by changes in the Coriolis effect with latitude. Shortwave troughs, are smaller scale waves superimposed on the Rossby waves, with a scale of 1,000 to 4,000 kilometres long, that move along through the flow pattern around large scale, or longwave, ridges and troughs within Rossby waves (Fig. 7.1).

In planetary atmospheres, they are due to the variation in the Coriolis effect with latitude. The waves were first identified in the Earth's atmosphere by Rossby [1939]. The terms "barotropic" and "baroclinic" Rossby waves are used to distinguish their vertical structure. Barotropic Rossby waves do not vary in the vertical, and have the fastest propagation speeds. The baroclinic wave modes are slower, with speeds of only a few centimetres per second or less (atmosphere).

Oceanic Rossby waves are thought to communicate climatic changes due to variability in forcing, due to both the wind and buoyancy. Both barotropic and baroclinic waves cause variations of the sea surface height, although the length of the waves made them difficult to detect until the

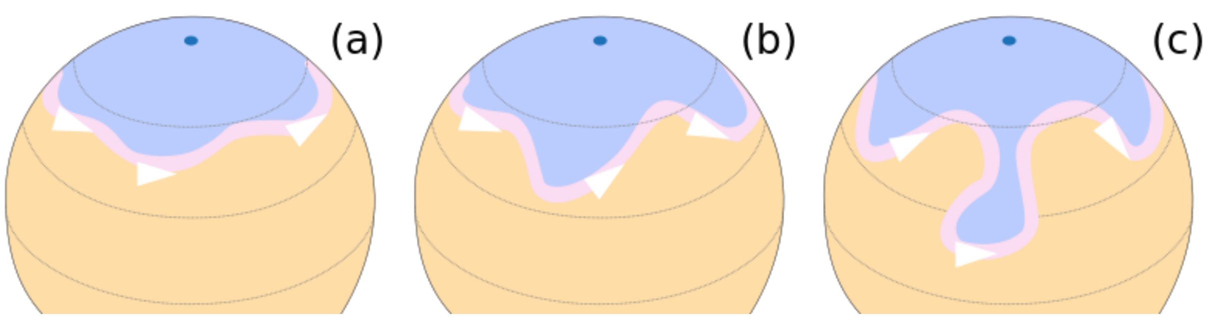

Figure 7.1: Meanders (Rossby Waves) of the Northern Hemisphere's polar jet stream developing (a), (b); then finally detaching a "drop" of cold air (c). Orange: warmer masses of air; pink: jet stream.

advent of satellite altimetry [Chelton and Schlax, 1996]. Baroclinic waves also generate significant displacements of the oceanic thermocline, often of tens of meters. Satellite observations have revealed the stately progression of Rossby waves across all the ocean basins, particularly at low- and mid-latitudes. These waves can take months or even years to cross a basin like the Pacific.

The first order equations of motion into an appropriate wave equation is cumbersome, namely because the two-dimensional geometry of the spherical surface is non-Euclidean (the Coriolis effect depends on the latitude). It can be shown [Müller et al., 1994; Müller and O'Brien, 1995; Müller and Maier-Reimer, 2000; Gerkema et al., 2008] that tidal theory differs from the plain waves because it accounts consistently for the globe's sphericity. If Cartesian coordinates are chosen with $f = \beta y$ then the dynamics reduces to the Matsuno equation as discussed in section 7.

Their emergence is due to shear in rotating fluids, so that the Coriolis force changes along the sheared coordinate.[1]

Exercise 48 – Numerical solution of shallow-water gravity waves

- open shallow1D.R
- Identify the lines of the code in which the momentum equation and in which the continuum equation are solved.
- Run the program. Which type of waves do you see?
- Change the constants of water depth H, gravity g, describe your observations!
- Can you roughly estimate the phase speed of the waves?

```
#shallow1D.R
ni<-200 #number of grid cells
```

[1] The dynamics in an inertial reference frame, e.g. with a coordinate system fixed at the Sun, would not have a Coriolis force, but would certainly observe Rossby wave propagation. In the inertial system, the near-equatorial motion is seen to be faster than off the equator. Zero vorticity in the rotating Earth's coordinate system corresponds to a basic flow with non-zero vorticity flow (zonal velocity $U = R\Omega \cos\varphi$) (φ: latitude) in the inertial reference frame [Müller and Maier-Reimer, 2000]. Linearizing the dynamics in the non-rotating system around the basic state U yields exactly Matsuno's wave equations taking the partial substantial derivative with advection U. Therefore, the effect of Earth's rotation is formally equivalent to a shear flow system. The mean flow energy is supplied by the Earth's rotation.

```
nt<-20000 #number of time steps

ia.0<-1:ni
ia.m1<-c(ni,1:(ni-1))
ia.p1<-c(2:ni,1)

g<-0.1 #9.81 m/s^2
dx<-1e5 #gridcell 10km
dt<-100 #timstep 1 second
H<-1e3 #1km depth

u<-rep(0,ni) #speed at each point
h<-rep(0,ni) #pertubation at each point
u.new<-vector()
h.new<-vector()

#h[31:50,1]<--0.5  #one pertubation in the middle
#h[51:70,1]<-0.5   #one pertubation in the middle
h[50:90]<-sin(0:40/2*pi/20)

#1st step euler forward
#momentum equation:
u.new[ia.0]<-u[ia.0]-g*dt/2/dx*(h[ia.p1]-h[ia.m1])
#Continuity eq. horizontal divergence:
h.new[ia.0]<-h[ia.0]-H*dt/2*((u[ia.p1]-u[ia.m1])/dx)

#from step 3 on use Leapfrog
for (n in 2:(nt-1))
{
  u.old<-u
  h.old<-h
  h<-h.new
  u<-u.new

  u.new[ia.0]<-u.old[ia.0]-g*dt/dx*(h[ia.p1]-h[ia.m1])
  h.new[ia.0]<-h.old[ia.0]-H*dt*((u[ia.p1]-u[ia.m1])/dx)

  # modulo operator, smoothing every 10 time steps
  if ((n%%10)==0)
  { u.new[ia.0]<-(u.new[ia.0]+u[ia.0])/2
            h.new[ia.0]<-(h.new[ia.0]+h[ia.0])/2
          }

  # modulo operator: plotting
  if ((n%%101)==0)
  {par(ask = TRUE)  # to make a break
  plot(h,type="l",lwd=2,ylim=c(-1,1))
  }
}

############
```

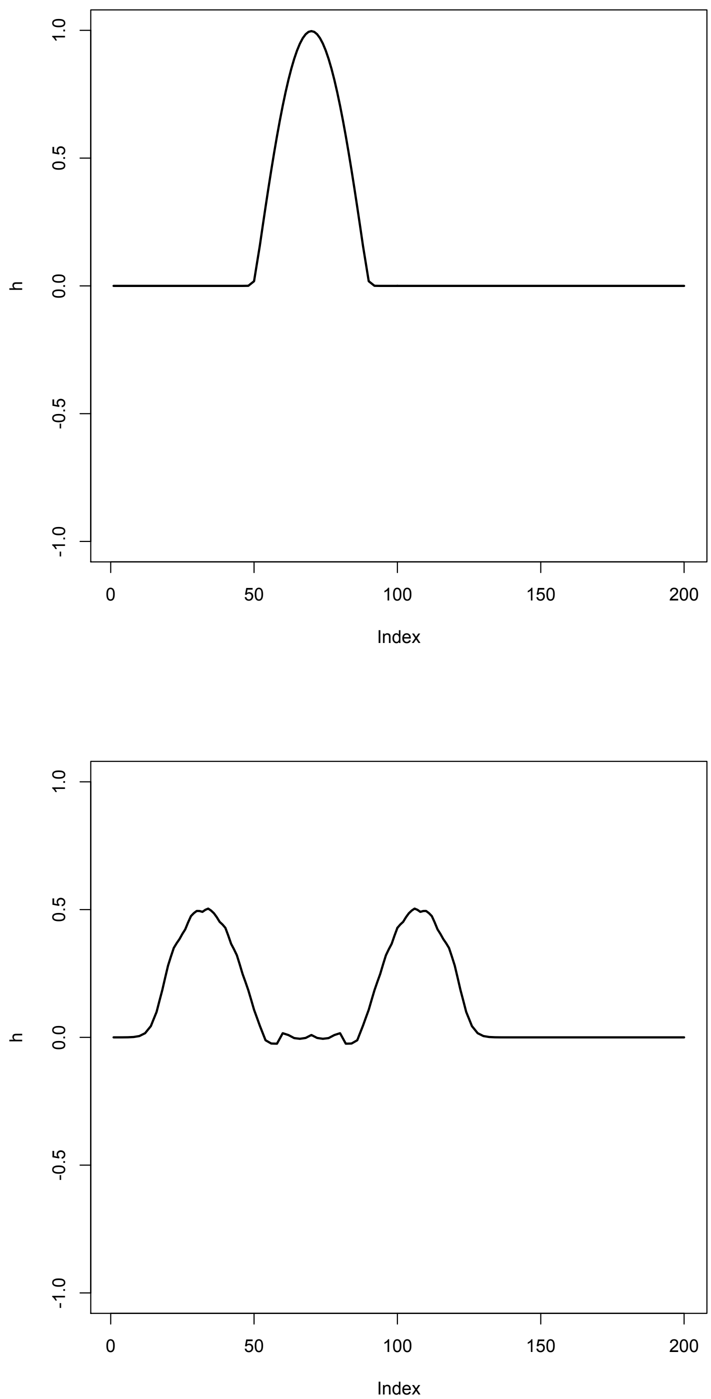

Figure 7.2: Numerical solution of 1D shallow water equation in exercise 48. Upper panel: initial condition. Lower panel: time snap shot.

Exercise 49 – Numerical solution of the shallow water equation

Study the wave dynamics on a water Earth, for simplicity the metric terms are neglected.

- open shallow2D_rossby.R
- Identify the lines of the code in which the momentum equation and in which the continuum equation are solved.
- Run the program: Which type of waves do you see?
- Change the constants of water depth H, gravity g, describe your observations!
- Can you roughly estimate the phase speed of the waves?

```
#
# shallow2D_rossby.R
#
#This is just a definition of a function to plot vectorplots
par.uin<-function()
 {  u <- par("usr")
    p <- par("pin")
    c(p[1]/(u[2] - u[1]), p[2]/(u[4] - u[3]))
 }

quiver<-function(lon,lat,u,v,scale=1,length=0.05,maxspeed=200, ...)
  {  ypos <- lat[col(u)]
    xpos <- lon[row(u)]
    speed <- sqrt(u*u+v*v)
    u <- u*scale/maxspeed
    v <- v*scale/maxspeed
    matplot(xpos,ypos,type="p",cex=0,xlab="lon",ylab="lat", ...)
    arrows(xpos,ypos,xpos+u,ypos+v,length=length*min(par.uin()))
  }

#Program starts here
#Shallow water 2D,cyclic boundary conditions + Coriolis term

nn<- 50
ni<- 2*nn+1   #number of gridcells in one direction
nt<-10000 #number of timesteps

#The physical constants
g<-0.1 #low gravity, 0.1 m/s^2
dx<-1e5 #gridcell 10km
#dx=400e3 # 400 km
dy<-dx/2  # double resolution in meridional direction
dt<-1000 #timstep 1000 second
H<-1e3 #1km depth
Omega<-1e-4

#define three index vectors.. the middle one,
```

```
#one shifted one cell to the left, and one to the right
#(including the periodic boundary conditions)
ia.0<-1:ni
ia.m1<-c(ni,1:(ni-1))
ia.p1<-c(2:ni,1)
u<-matrix(0,ni,ni) #speed at each point
v<-matrix(0,ni,ni) #speed at each point
h<-matrix(0,ni,ni) #pertubation at each point
f<-matrix(0,ni,ni) #pertubation at each point

lat<-c(-nn:nn)*90/nn
weight<-sin(lat*pi/180)
lon<-c(-nn:nn)*180/nn

f<-rep(weight*2*Omega,each=ni)  # Coriolis parameter
dim(f)<-c(ni,ni)
filled.contour(f)

u.new<-u
h.new<-h
v.new<-v

#Inital condition: One smooth blobs at each side of the "equator"(sin)
idit=nn/5*2
inix=ni-idit-1
iniy=ni-2*idit-1
endx=ni-1
endy=ni-1
endy2=2*idit+1
h[inix:endx,iniy:endy]<-sin(0:20/2*pi/10)*t(sin(0:40/2*pi/20))
h[inix:endx,1:endy2]<-sin(0:20/2*pi/10)*t(sin(0:40/2*pi/20))

#equator to study the Kelvin wave:
ii=idit+1
iy=nn-10
iy2=nn+10
h[1:ii,iy:iy2]<- -sin(0:20/2*pi/10)*t(sin(0:20/2*pi/10))

#Inital condition: One smooth blobs at each side of the "equator"(sin)
#h[60:80,60:80]<-sin(0:20/2*pi/10)*t(sin(0:20/2*pi/10))
#h[30:50,80:100]<-sin(0:20/2*pi/10)*t(sin(0:20/2*pi/10))

#1st step euler forward
u.new[ia.0,ia.0]<-u[ia.0,ia.0]-g*dt/2/dx*(h[ia.p1,ia.0]-h[ia.m1,ia.0])
v.new[ia.0,ia.0]<-v[ia.0,ia.0]-g*dt/2/dy*(h[ia.0,ia.p1]-h[ia.0,ia.m1])
h.new[ia.0,ia.0]<-h[ia.0,ia.0]
        -H*dt/2*((u[ia.p1,ia.0]-u[ia.m1,ia.0])/dx
        + (v[ia.0,ia.p1]-v[ia.0,ia.m1])/dy)

#Divide the screen in two parts
# par(mfcol=c(1,2))
#par(mfcol=c(2,1))
```

```
#Leapfrog from the third step on
for (n in 3:(nt-1))
{
  u.old<-u
  v.old<-v
  h.old<-h
  h<-h.new
  u<-u.new
  v<-v.new
  u.new[ia.0,ia.0]<-u.old[ia.0,ia.0]
      -g*dt/dx*(h[ia.p1,ia.0]-h[ia.m1,ia.0])+dt*f*v
  v.new[ia.0,ia.0]<-v.old[ia.0,ia.0]
      -g*dt/dy*(h[ia.0,ia.p1]-h[ia.0,ia.m1])-dt*f*u
  h.new[ia.0,ia.0]<-h.old[ia.0,ia.0]
      -H*dt*((u[ia.p1,ia.0]-u[ia.m1,ia.0])/dx
      + (v[ia.0,ia.p1]-v[ia.0,ia.m1])/dy)

#plot every 50th image
  if ((n %% 50) == 0) {
        #quiver(lon,lat,u,v,scale=200,maxspeed=1.5,length=3)
        #image(lon,lat,h,zlim=c(-1,1),col=rainbow(200)) # color coated
        persp(h/3, theta = 0, phi = 40, scale = FALSE, ltheta = -120,
      shade = 0.6, border = NA, box = FALSE,zlim=c(-0.3,0.3))
      }
  }
```

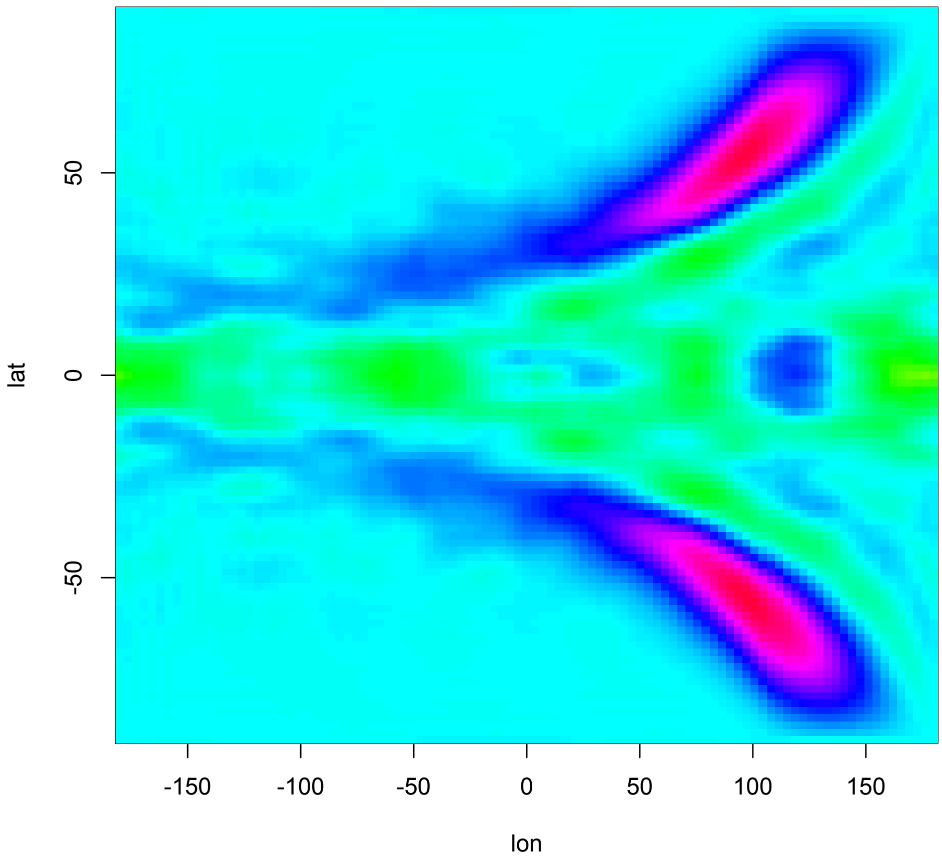

Figure 7.3: Global Rossby and Kelvin wave signatures in the exercise 49.

7.3 Plain waves

The analysis of the spherical version of the tidal problem is complicated because the Coriolis effect depends on the latitude and in general we do not have plain waves with sinus and cosinus base functions.[2] However, because of its simplicity, we will study the plain wave theory here. In this approach, the Coriolis parameters f and β are taken as **fixed parameters** in the equations. Then, the wave equations can be reduced to plain waves with eigenfunctions $\sim \exp(ikx + ily - i\omega t)$.

[2] This approximation may be questioned because the trapped character of the Rossby waves is no included, which is however, observed and simulated (Fig. 7.3). This shows a general problem in perturbation theory: The concept of manipulations in the differential equations (e.g., by neglecting terms) is not entirely free from ambiguities, and may lead to a undesirable transition in the solutions of the system. The type of solutions shall be of the form of the observed (macroscopic) functions and a proper framework of approximations is required (section 8.4).

7.3.1 Inertial Waves

From the equations (7.7,7.8,7.9), we drop the term $\partial_x \eta$, $\partial_y \eta$, and $f = f_0 = const.$ (no pressure gradients and constant f). Then, air or water mass moving with speed v subject only to the Coriolis force travels in a circular trajectory called an 'inertial circle'. Since the force is directed at right angles to the motion of the particle, it will move with a constant speed, and perform a complete circle with frequency f. The magnitude of the Coriolis force also determines the radius of this circle:

$$R = v/f \, . \tag{7.20}$$

On the Earth, a typical mid-latitude value for f is $10^{-4}s^{-1}$; hence for a typical atmospheric speed of 10 m/s the radius is 100 km, with a period of about 14 hours. In the ocean, where a typical speed is closer to 10 cm/s, the radius of an inertial circle is 1 km. These inertial circles are clockwise in the Northern Hemisphere (where trajectories are bent to the right) and anti-clockwise in the Southern Hemisphere. If the rotating system is a parabolic turntable, then f is constant and the trajectories are exact circles. On a rotating planet, f varies with latitude and the paths of particles do not form exact circles. Since the parameter f varies as the sine of the latitude, the radius of the oscillations associated with a given speed are smallest at the poles and increase toward the equator (Fig. 7.4).

$$\frac{\partial u}{\partial t} - f_0 v = 0 \tag{7.21}$$

$$\frac{\partial v}{\partial t} + f_0 u = 0 \tag{7.22}$$

yields

$$\frac{\partial^2 u}{\partial t^2} = -f_0^2 u \quad . \tag{7.23}$$

Figure 7.4: Schematic representation of inertial circles of air masses in the absence of other forces, calculated for a wind speed of approximately 50 to 70 m/s. Note that the rotation is exactly opposite of that normally experienced with air masses in weather systems around depressions.

The solution is

$$u(t) = u(0)\sin(f_0 t) \tag{7.24}$$

$$v(t) = u(0)\cos(f_0 t) \tag{7.25}$$

which is known as intertial movement and can be observed in drifting buoys (upper panel Fig. 7.5). The water parcels move around a circle of radius of $u(0)/f_0$ in a clockwise direction (anticyclonically) with a period $2\pi/f_0$.

Exercise 50 – Inertial waves

- Derive the solution of (7.21, 7.22). Since the force is directed at right angles to the motion of the particle, it will move with a constant speed, and perform a complete circle with frequency f. Show that the magnitude of the Coriolis force determines a radius R of this circle. Hint: A typical mid-latitude value for f is $10^{-4}s^{-1}$; a typical atmospheric speed of $10m/s$, in the ocean a typical speed is closer to $10cm/s$.

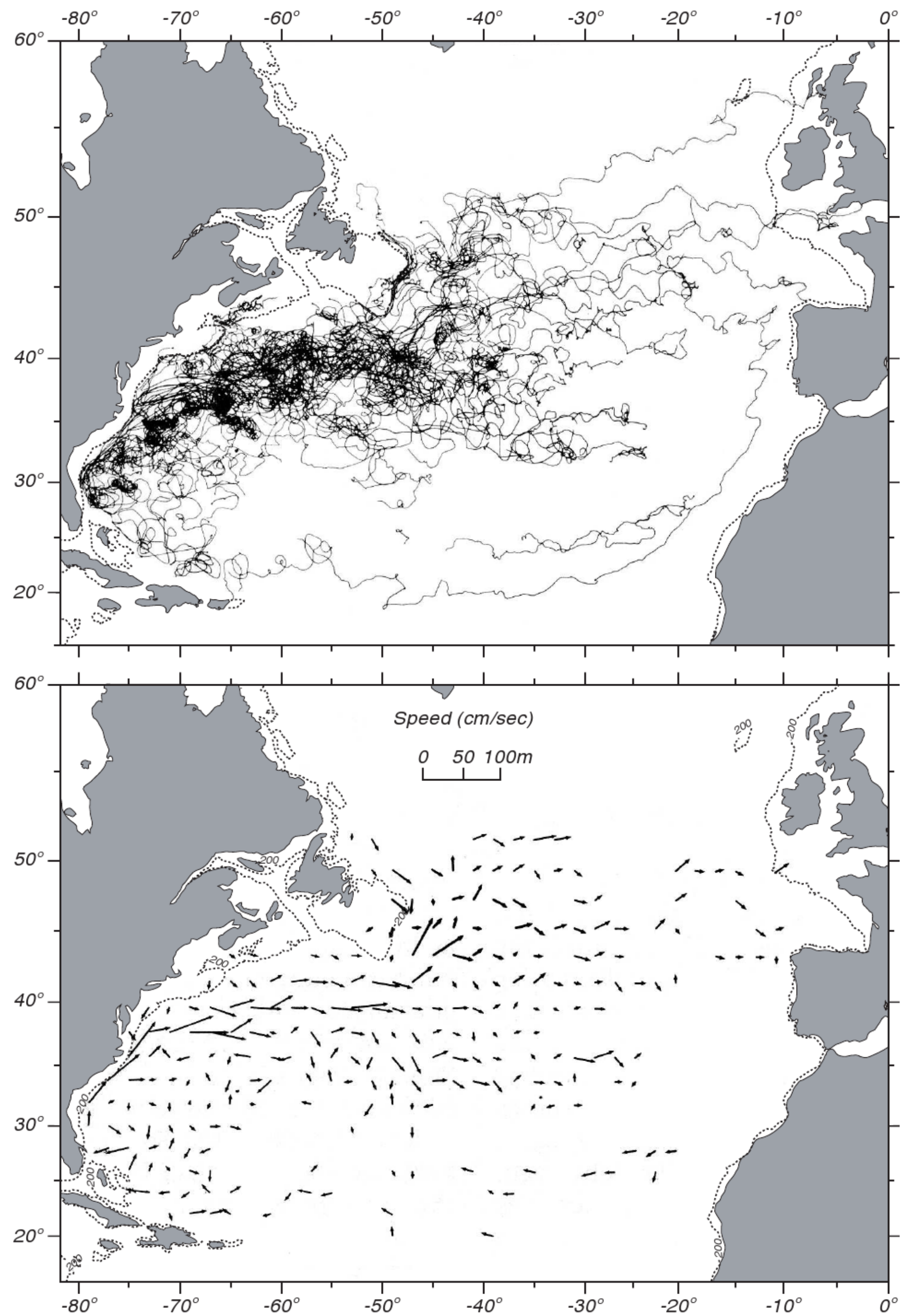

Figure 7.5: **Top:** Tracks of 110 drifting buoys deployed in the western north Atlantic. **Bottom:** Mean velocity of currents in $2° \times 2°$ boxes calculated from tracks above. Boxes with fewer than 40 observations were omitted. Length of arrow is proportional to speed. Maximum values are near $0.6ms$ in the Gulf Stream near $37°$N $71°$W. After Richardson (1981).

- Provide the solution for the coordinates $x(t), y(t)$.
- Show that the dynamics in the inertial coordinate system reduces to

$$u_{in}(t) = 0 \tag{7.26}$$

$$v_{in}(t) = u(0)\cos\Omega t \tag{7.27}$$

The trajectory in the inertial frame is a straight line. The length of the line is twice the diameter of the inertial circle and the frequency of the oscillation is one-half that observed in the rotating frame.

7.3.2 Gravity Waves

Shallow-water gravity waves are defined through their dynamics without the effect of the Earth's rotation, i.e. $f = 0$:

$$\frac{\partial^2 \eta}{\partial t^2} = gH \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \eta \tag{7.28}$$

With the ansatz

$$\eta = \exp(ikx + ily - i\omega t) \tag{7.29}$$

ω is given by

$$\omega(k, l) = \pm\sqrt{gH\,(k^2 + l^2)}, \tag{7.30}$$

where k and l are the zonal and meridional wavenumbers. Since there is no preferred direction in the (x,y) coordinate, we simply drop the y-dependence and introduce the phase speed

$$c = \omega/k = \pm\sqrt{gH} \quad . \tag{7.31}$$

In the limit $\beta \to 0$ i.e. $f = f_0 = const.$, the dynamics consists of gravity waves with

$$\omega^2 = f_0^2 + (ck)^2 \tag{7.32}$$

Output from a shallow water equation model of water in a bathtub. The water experiences five splashes which generate surface gravity waves that propagate away from the splash locations and reflect off the bathtub walls. (https://en.wikipedia.org/wiki/Shallow_water_equations#/media/File:Shallow_water_waves.gif)

Exercise 51 – Baroclinic shallow-water gravity waves

In case we have a layed ocean, we consider the so-called baroclinic dynamics with the modified gravity $g' = \frac{\rho_1 - \rho_2}{\rho_1}$ using the densities $\rho_{1,2}$. Task: Derive the baroclinic dynamics using the shallow water equations for 2 different layers and subtract the equations from each other!

Exercise 52 – Shallow-water waves

We consider tidal equation on the $\beta-$plane. This fluid dynamical system is described as

$$\partial_t u = f\,v - g\,\partial_x \eta \tag{7.33}$$

$$\partial_t v = -f\,u - g\,\partial_y \eta \tag{7.34}$$

$$\partial_t \eta = -\partial_x (Hu) - \partial_y (Hv) \quad . \tag{7.35}$$

The variables u and v denote zonal and meridional perturbation flow velocity, and η the height perturbation.

- Derive the dispersion relationships $\omega(k)$ for the cases:

  a) In the limit $\beta \to 0$, i.e. $f \to f_0$

  b) $c \to \infty$.

  c) For infinite Rossby radius $a = \sqrt{c/(2\beta)}$.

  d) When filtering out gravity waves by eliminating the time derivative in (7.35), (u, v) in (7.33, 7.34) can be taken as plane waves proportional to $\exp(ikx + ily)$, where l denotes the meridional wave number. Derive the dispersion relationships $\omega(k)$ for the so-called non-divergent Rossby waves.
- Provide typical values of $\omega(k)$ for M,N=1,2,3 and the atmosphere and ocean.

7.3.3 Extratropical Rossby Waves

From the equations (7.7,7.8,7.9), we drop the term $\partial_t \eta$ and introduce the stream function ψ through

$$u = \frac{\partial \psi}{\partial y} \quad ; \quad v = -\frac{\partial \psi}{\partial x} \tag{7.36}$$

such that (7.9) is fulfilled. Taking $\frac{\partial}{\partial y}$ of (7.7) and subtract $\frac{\partial}{\partial x}$ of (7.8) elimintates the η term as in section 1.3:

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi = -\beta \frac{\partial \psi}{\partial x} \tag{7.37}$$

With the ansatz

$$\psi = \exp(ikx + ily - i\omega t) \tag{7.38}$$

and assumption that β is just a parameter, ω is given by

$$\omega(k,l) = -\frac{\beta k}{k^2 + l^2}, \tag{7.39}$$

where k and l are the zonal and meridional wavenumbers. Again, β is used as a parameter (also called Rossby parameter) and is not expressed in terms of y:

$$\beta = \frac{df}{dy} = \frac{1}{R}\frac{d}{d\varphi}(2\Omega \sin\varphi) = \frac{2\Omega\cos\varphi}{R} \tag{7.40}$$

where φ is the latitude, Ω is the angular speed of the Earth's rotation, and R is the mean radius of the Earth. The wave speed $c = \omega/k = -\beta\left(k^2 + l^2\right)^{-1}$. The feature that the phase speed is faster at low latitudes can be also seen in Fig. 7.3 using the full dynamics.

More information about Rossby waves: https://youtu.be/6UCiRIc0nK0

Rossby waves and extreme weather: https://youtu.be/MzW5Isbv2A0

Exercise 53 – Rossby waves

Consider the vorticity equation

$$\frac{D}{Dt}[(\zeta + f)/h] = 0 \tag{7.41}$$

with h =const., u and v are the velocity components.

1. Assume a mean flow with constant zonal velocity U

$$u = U = const > 0 \tag{7.42}$$

   and a varying north-south component

$$v = v(x, t) \tag{7.43}$$

   which gives the total motion a wave-like form around a reference latitude where the wave is trapped. Derive the associated vorticity equation and linearize the vorticity equation by dropping all non-linear terms!

2. With the ansatz

$$v(x, t) = A\cos[(kx - \omega t)] \tag{7.44}$$

   determine the disperion relation $\omega(k)$, group velocity $\frac{\partial\omega}{\partial k}$, and the phase velocity $c = \omega/k$.

3. Derive the wavelength $L = 2\pi/k$ of the stationary wave given by $c = 0$.
4. A typical wavelength is 6000 km, a typical U is 15 m/s. Does the wave propagate from east to west or opposite?

7.4 Kelvin waves

7.4.1 Coastal Kelvin waves

A Kelvin wave is a wave in the ocean or atmosphere that balances the Coriolis force against a topographic boundary such as a coastline. If one assumes that the Coriolis coefficient f is constant along the right boundary conditions, $u = 0$, and the zonal wind speed is set equal to zero, then the equations become the following:

$$\frac{\partial \eta}{\partial t} = -H\frac{\partial v}{\partial y} \tag{7.45}$$

$$\frac{\partial v}{\partial t} = -g\frac{\partial \eta}{\partial y} \tag{7.46}$$

and therefore

$$\frac{\partial^2 \eta}{\partial t^2} = gH \quad \frac{\partial^2}{\partial y^2}\eta \tag{7.47}$$

The solution to these equations yields the following phase speed: $c^2 = gH$ and $\omega = \pm cl$, which is the same speed as for shallow-water gravity waves without the effect of Earth's rotation. We see that η and v have also an x-dependence

$$\eta(x,y,t) = \tilde{\eta}(x)\exp(ily - i\omega t) \tag{7.48}$$

$$v(x,y,t) = \tilde{v}(x)\exp(ily - i\omega t) \quad . \tag{7.49}$$

Using (7.46), we obtain

$$-i\omega\,\tilde{v}(x) = -gil\,\tilde{\eta}(x) \quad \text{and therefore } \tilde{v}(x) = \frac{g}{\omega}l\,\tilde{\eta}(x) = \pm\frac{g}{c}\,\tilde{\eta}(x) \tag{7.50}$$

$$\text{From the u-momentum equation} \quad \frac{\partial \eta}{\partial x} = \frac{f}{g}v \tag{7.51}$$

we obtain therefore $$\frac{\partial \tilde{\eta}}{\partial x} = \pm \frac{f}{c} \tilde{\eta} \tag{7.52}$$

where only the minus sign provides a useful solution (not blowing up). The solution has an exponential decay of $\tilde{\eta}(x) = \exp(-x/L_r)$ on the scale of the Rossby radius $L_r = c/f$. The wave has a trapped character along the boundary. It is important to note that for an observer traveling with the wave, the coastal boundary (maximum amplitude) is always to the right in the Northern Hemisphere and to the left in the Southern Hemisphere, i.e. these waves move equatorward/southward on a western boundary and poleward/northward on an eastern boundary. Thus, the waves move cyclonically around an ocean basin.

On the black board: A Coastal Kelvin Wave moving northward along the coast is deflected to the right, but the coast prevents the wave from turning right and instead causes water to pile up on the coast. The pile of water creates a pressure gradient directed offshore and a geostrophic current directed northward.

On the northern hemisphere: The Kelvin wave always travels with the wall on its right side (anti-clockwise). The wave amplitude decreases exponentially away from the wall. The wave is trapped along the wall by rotation. Rotation does not affect the particle motion and wave propagation; only traps the wave to the coastline.

7.4.2 Equatorial Kelvin waves

Analogous we have Equatorial Kelvin waves: assume $v = 0$, then the equations become the following:

$$\frac{\partial \eta}{\partial t} = -H \frac{\partial u}{\partial y} \tag{7.53}$$

$$\frac{\partial u}{\partial t} = -g \frac{\partial \eta}{\partial y} \tag{7.54}$$

and therefore again

$$\frac{\partial^2 \eta}{\partial t^2} = gH \quad \frac{\partial^2}{\partial y^2}\eta \tag{7.55}$$

The solution to these equations yields the phase speed: $c^2 = gH$ and $\omega = ck,$ which is the same speed as for shallow-water gravity waves without the effect of Earth's rotation. We see that η and u have also an x-dependence

$$\eta(x,y,t) \;=\; \tilde{\eta}(y)\exp(ikx - i\omega t) \tag{7.56}$$

$$u(x,y,t) \;=\; \tilde{u}(y)\exp(ikx - i\omega t) \quad . \tag{7.57}$$

Using (7.54), we obtain

$$-i\omega\,\tilde{u}(y) = -gik\,\tilde{\eta}(y) \quad \text{and therefore } \tilde{u}(y) = \frac{g}{\omega}k\,\tilde{\eta}(y) = \frac{g}{c}\,\tilde{\eta}(y) \tag{7.58}$$

$$\text{From the v-momentum equation} \quad \frac{\partial \eta}{\partial y} = \frac{\beta y}{g}u \tag{7.59}$$

$$\text{we obtain therefore} \quad \frac{\partial \tilde{\eta}}{\partial y} = -\frac{\beta y}{c}\tilde{\eta}\,. \tag{7.60}$$

The solution is $\tilde{\eta}(x) = \exp(-\beta y^2/c)$ with the scale of the Rossby radius $L_r = \sqrt{c/\beta}$. The wave has a trapped character along the equator.

A feature of a Kelvin wave is that it is non-dispersive, i.e., the phase speed of the wave crests is equal to the group speed of the wave energy for all frequencies. This means that it retains its shape in the alongshore direction over time. In the ocean these waves propagate along coastal boundaries (and hence become trapped in the vicinity of the coast itself) on a scale of about 30 km.

Equatorial Kelvin waves are a special type of Kelvin wave that balances the Coriolis Force in the northern hemisphere against its southern hemisphere counterpart. This wave always propagates eastward and only exists on the equator. Equatorial Kelvin Waves propagating in the thermocline have wave speeds slow enough to give a Rossby Radius of Deformation that is on the order of 250 km and thus they appear to be trapped close to the equator.

7.5 Equatorial waves: Theory of Matsuno

We consider the equations (7.7,7.8,7.9) on the equatorial $\beta-$plane. In the equatorial region, the fluid dynamical system is described as

$$\partial_t u = \beta y \, v - g \, \partial_x \eta \tag{7.61}$$

$$\partial_t v = -\beta y \, u - g \, \partial_y \eta \tag{7.62}$$

$$\partial_t \eta = -\partial_x (Hu) - \partial_y (Hv) \quad . \tag{7.63}$$

We non-dimensionalize the system through the parameters listed in Table 7.1. In the non-dimensional form (and dropping the stars in Table 7.1), the system reads then

$$\partial_t u = y \, v - \partial_x \eta \tag{7.64}$$

$$\partial_t v = -y \, u - \partial_y \eta \tag{7.65}$$

$$\partial_t \eta = -\partial_x u - \partial_y v \quad . \tag{7.66}$$

Introducing the new variables

$$q = \eta + u \tag{7.67}$$

$$r = \eta - u \tag{7.68}$$

Parameter	description	formula	typical values
H	equivalent height		
g	reduced gravity		
R	Earth's radius		$6.371 \cdot 10^6\, m$
Ω	Earth's rotation rate	$2\pi\, day^{-1}$	$7.272 \cdot 10^{-5}\, s^{-1}$
M	zonal wave number		$0, \pm 1, \pm 2, \ldots$
N	mode number		$0, 1, 2, \ldots$
φ	latitude		
f	Coriolis parameter	$2\Omega \sin\varphi$	
β	β-term	$2\Omega/R$	$2.0 \cdot 10^{-11}\, m^{-1}s^{-1}$
c	barotropic phase speed of pure gravity wave	$\sqrt{gH}$	atmosphere: $2000\, ms^{-1}$ ocean: $200\, ms^{-1}$
c	baroclinic phase speed of pure gravity wave	$\sqrt{gH}$	atmosphere: $20 - 80\, ms^{-1}$ ocean: $2\, ms^{-1}$
a	meridional wave guide (Rossby radius)	$\sqrt{\frac{c}{2\beta}}$	atmosphere: $6.6 \cdot 10^5\, m$ ocean: $6.6 \cdot 10^4\, m$
t^*	time	$t\,\sqrt{2\beta c}$	
x^*	eastward distance	x/a	
y^*	meridional distance	y/a	
ω^*	frequency	$\omega/\sqrt{2\beta c}$	
k^*	zonal wave vector	Ma/R	

Table 7.1: List of parameters for the Matsuno equations.

yields

$$\partial_t q = -\partial_x q - \left[\partial_y - \frac{y}{2}\right] v \tag{7.69}$$
$$\partial_t v = -\frac{1}{2}\left[\partial_y + \frac{y}{2}\right] q - \frac{1}{2}\left[\partial_y - \frac{y}{2}\right] r \tag{7.70}$$
$$\partial_t r = +\partial_x r - \left[\partial_y + \frac{y}{2}\right] v \quad . \tag{7.71}$$

The dynamics (7.69,7.70,7.71) describe wave propagation in an inhomogenous and anisotropic medium. Zonal wave dynamics differ significantly from those in meridional direction. The primary source of inhomogenity is due to the Coriolis force. The x and t dependences can be separated in form of zonally propagating waves $\exp(ikx - i\omega t)$. The eigenfunctions in $y-$direction are related to parabolic cylinder functions $\Big($or Hermite polynomials with weight $\exp(-y^2)\Big)$. The Hermite polynomials are defined as

$$He_n(y) = (-1)^n e^{y^2/2} \frac{d^n}{dy^n} e^{-y^2/2} \tag{7.72}$$

The first Hermite polynomials are

$$He_0(y) = 1 \tag{7.73}$$
$$He_1(y) = x \tag{7.74}$$
$$He_2(y) = y^2 - 1 \tag{7.75}$$
$$He_3(y) = y^3 - 3y \tag{7.76}$$
$$He_4(y) = y^4 - 6y^2 + 3 \tag{7.77}$$

To display the Hermite polynomials:

```
# for a read.me: http://cran.r-project.org/doc/manuals/R-intro.pdf
# generate a list of normalized Hermite polynomials of orders 0 to 10
install.packages("orthopolynom")
normalized.p.list <- hermite.he.polynomials(5, normalized=TRUE ) # a list
print(normalized.p.list)  # display the polynomials
```

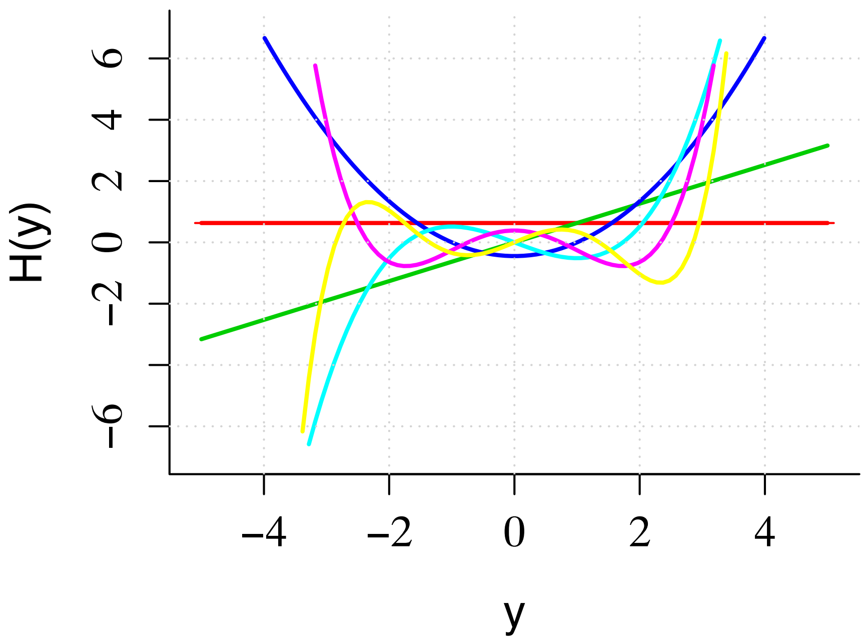

Figure 7.6: Hermite polynomials to degree 5.

```
H=normalized.p.list
ticks=seq(from=-5, to=5, by=10)
plot(H[[1]], xlim=c(-5,5),ylim=c(-7,7),col="red",ylab="H(y)",xlab="y")
for (i in 1:6) {lines(H[[i]],xlim=c(-5,5),ylim=c(-7,7),col =i+1, lwd = 2)}
grid(nx=NULL,col="lightgray",lty ="dotted",lwd=2,equilogs=TRUE)
dev.copy(png,'Hermite.png')
dev.off()
```

They satisfy following recursion relationship:

$$\left[\partial_y + \frac{y}{2}\right] \; D_N = \; N \; D_{N-1} \quad ; \qquad \left[\partial_y - \frac{y}{2}\right] \quad D_N = \; - D_{N+1} \qquad . \tag{7.78}$$

The operators $[\partial_y \pm \frac{y}{2}]$ annihilate or excite one quantum of mode index number N and are called lowering and raising ladder operators in quantum mechanics. A basic feature of $D_N \sim \exp(-y^2)$ is that significant wave amplitudes are trapped in a wave guide centered at the latitude φ_0, similar to the equator-centered Yoshida guide [Gill, 1982].

The Fourier modes $\hat{\xi}_N(t) := (\hat{q}_{N-1}, \hat{v}_N, \hat{r}_{N+1})$ correspond to order $N > 0$ and wave vector k. The prognostic equations for the Fourier modes are first order in time

$$\frac{d}{dt}\,\hat{\xi}_N \quad = \quad A_N(k) \;\; \hat{\xi}_N \qquad . \tag{7.79}$$

and are described by 3×3 matrices $A_N(k)$

$$A_N(k) \;=\; \begin{pmatrix} -ik & 1 & 0 \\ -N/2 & 0 & 1/2 \\ 0 & -(N+1) & ik \end{pmatrix} \qquad . \tag{7.80}$$

Matrix $A_N(k)$ describes the dynamics of one Rossby and two gravity waves with eigenfrequencies ω (eigenvalue of $A = i\omega$) satisfying

$$\omega^3 - \omega\left(\frac{2N+1}{2} + k^2\right) - \frac{k}{2} \;=\; 0 \quad . \tag{7.81}$$

The sum of the eigenfrequencies in (7.81) is zero due to $trace\,(A_N) = 0$ and

$$\sum_{l=1}^{3}\omega_l \;=\; \lim_{T\to\infty}\frac{1}{T}\int_0^T dt \;\; trace\,(A_N) \;=\; 0 \quad . \tag{7.82}$$

For $N = 0,$ the system matrix A_0 is specified to be

$$A_0(k) \;=\; \begin{pmatrix} \boxed{ik} & 0 & 0 \\ 0 & 0 & 1/2 \\ 0 & -1 & ik \end{pmatrix} \qquad . \tag{7.83}$$

The different signs of the $\cdot_{11}-$elements in (7.80) and (7.83) originate from the requirement that the corresponding eigenmode $q_{N=0}$ in (7.83) is integrable [Gill, 1982]. This mode with $v = r = 0$ is called equatorial Kelvin wave which propagates eastward without dispersion:

$$\omega = k \quad . \tag{7.84}$$

The dynamics of the Kelvin wave is decoupled from the Yanai wave dynamics described by the second and third eigenvectors of matrix (7.83). The Yanai wave, also known as mixed planetary-gravity wave in the literature [Gill, 1982], has a quadratic relation

$$\omega^2 \;-\; k\omega \;-1/2 \;=\; 0 \quad . \tag{7.85}$$

Dispersion curves for the Rossby/gravity (7.81), Kelvin (7.84), and Yanai (7.85) waves are shown in Fig. 7.7 as a function on zonal wave vector $k \;=\; Ma/R$ and mode number $N.$ The figure depicts eastward propagating Kelvin and westward propagating Rossby modes. Gravity waves can propagate east- and westward. The Yanai wave behaves as a gravity wave for $k \;\geq\; 0$ and as a Rossby wave for $k \;<\; 0\,.$ Note that (7.81) is invariant under $\omega \;\rightarrow\; -\omega, k \;\rightarrow\; -k$, which is a consequence of (7.82). Dispersion diagrams like Fig. 7.7 can be found in standard text books of geophysical fluid dynamics showing the upper [Gill, 1982] or right [Holton, 2004] part of Fig. 7.7, respectively.

The equatorial zone essentially acts as a waveguide, causing disturbances to be trapped in the vicinity of the equator. For the first baroclinic mode in the ocean, a typical phase speed would be about $2.8m/s,$ causing an equatorial Kelvin wave to take 2 months to cross the Pacific Ocean

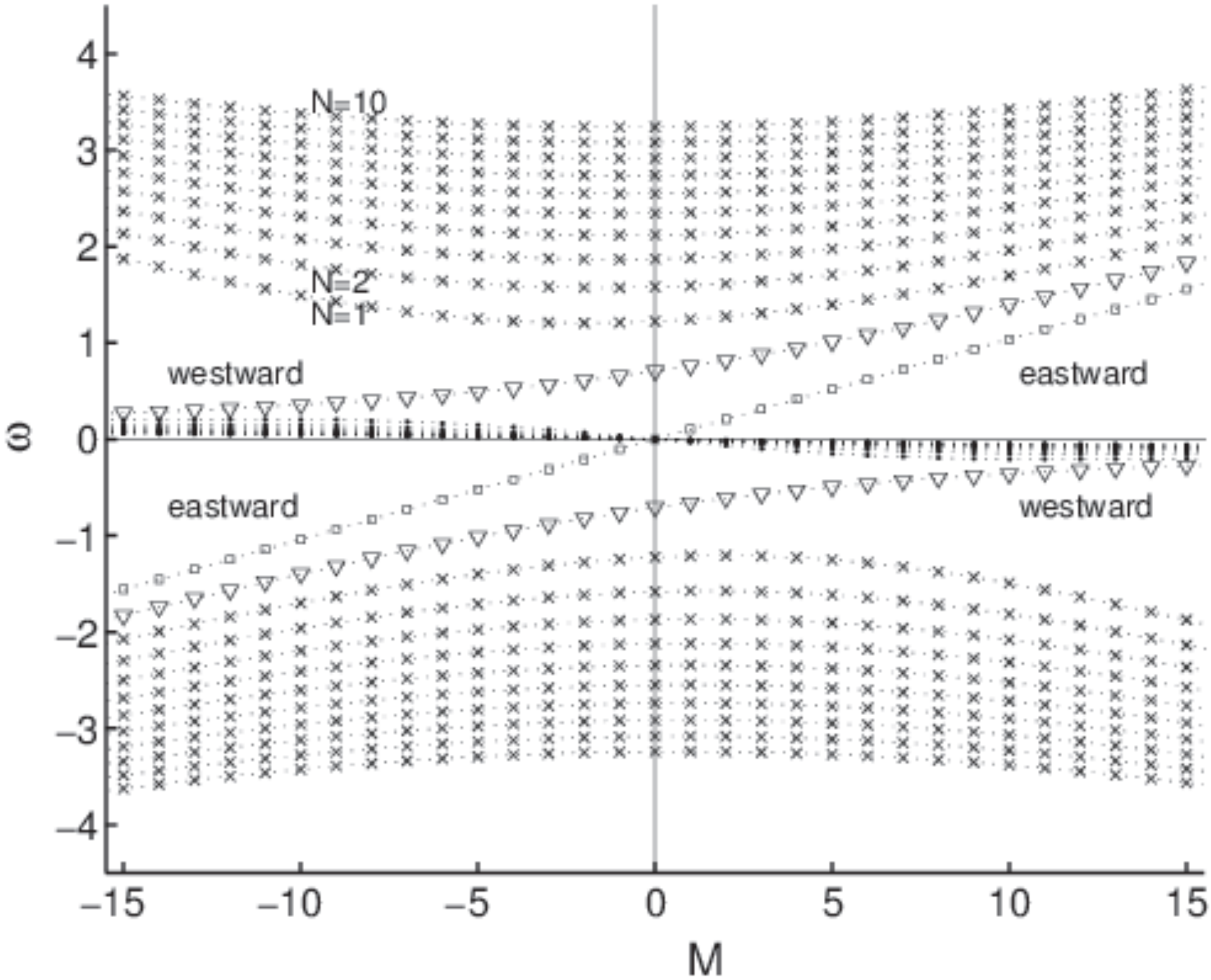

Figure 7.7: Dispersion relation for equatorial waves. Curves show dependence of frequency on zonal wave number M for mode numbers $N \leq 10$. Kelvin waves propagate eastward, Rossby waves (•) westward, while gravity waves (×) exist for both directions. Yanai waves (▽) behave Rossby-like for $M < 0$ and gravity-like for $M \geq 0$.

between New Guinea and South America; for higher ocean and atmospheric modes, the phase speeds are comparable to fluid flow speeds. Why is the Kelvin wave trapped? When the motion at the equator is to the east, any deviation toward the north is brought back toward the equator because the Coriolis force acts to the right of the direction of motion in the Northern Hemisphere, and any deviation to the south is brought back toward the equator because the Coriolis force acts to the left of the direction of motion in the Southern Hemisphere. Note that for motion toward the west, the Coriolis force would not restore a northward or southward deviation back toward the equator; thus, equatorial Kelvin waves are only possible for eastward motion (as noted above). Both atmospheric and oceanic equatorial Kelvin waves play an important role in the dynamics of El Niño-Southern Oscillation, by transmitting changes in conditions in the Western Pacific to the Eastern Pacific [Gill, 1982]. This can be also studied in exercise 49.

It is instructive to look for approximations in tidal theory. One can simplify the solution of the Matsuno theory, or simplify the equations (7.7,7.8,7.9) which will be done in section 7.3.

When filtering out gravity waves by eliminating the time derivative in (7.63), (u, v) in (7.61, 7.62) is equivalent to $c \to \infty$. The evolution equation reduces to

$$\partial_t(\partial_y u - \partial_x v) = \beta u \tag{7.86}$$

with plane waves proportional to $\exp(ikx + ily)$. Then, non-divergent Rossby waves with $\omega = -\beta k/(k^2 + l^2)$ are retained only. The trapped character of the waves vanishes with infinite Rossby radius $a = \sqrt{c/(2\beta)}$, a measure of the wave guide geography.

Exercise 54 – Shallow-water dynamics: eigenfunctions

- Show that the eigenfunctions in $y-$direction, which are are related to parabolic cylinder functions $\Big($or Hermite polynomials with weight $\exp(-y^2)\Big)$, satisfy following recursion relationship:

$$\left[\partial_y + \frac{y}{2}\right] D_N = N\ D_{N-1} \quad ; \quad \left[\partial_y - \frac{y}{2}\right] D_N = -D_{N+1} \quad . \tag{7.87}$$

  The operators $[\partial_y \pm \frac{y}{2}]$ annihilate or excite one quantum of mode index number N and are called lowering and raising ladder operators in quantum mechanics.

$$D_N(y) = \frac{1}{\pi}\int_0^\pi \sin(N\Theta - y\sin\Theta))$$

- Show that the functions are orthogonal, i.e.

$$\int_{-\infty}^{\infty} dy D_N(y) D_M(y) = \delta_{NM}\, N! \sqrt{2\pi}$$

- The dynamics in an inertial reference frame, e.g. with a coordinate system fixed at the Sun, would not have a Coriolis force (and thus $f = 0$), but would certainly observe Rossby wave propagation. How can this be reconciled?

  (Hint: In the inertial system, the near-equatorial motion is seen to be faster than off the equator. Zero vorticity in the rotating Earth's coordinate system corresponds to a basic flow $U = R\Omega\, \cos\varphi$ with non-zero vorticity flow.)

Exercise 55 – Shallow-water dynamics: A different approach

We may seek travelling-wave solutions of the form

$$\Big\{u, v, \eta\Big\} = \Big\{\hat{u}(y), \hat{v}(y), \hat{\eta}(y)\Big\} e^{i(kx-\omega t)} \tag{7.88}$$

Please check the following arguments.

- Subsitituting this exponential form into the three equations (7.61,7.62,7.63), and eliminating u, and η leaves us with an eigenvalue equation for $\hat{v}(y)$

$$-\frac{\partial^2 \hat{v}}{\partial y^2} + \left(\frac{\beta^2}{c^2}\right)\hat{v} = \left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega}\right)\hat{v}. \tag{7.89}$$

- Recognizing this as the Schrödinger equation of a quantum harmonic oscillator of frequency β/c, we know that we must have

$$\left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega}\right) = \frac{\beta}{c}(2n+1), \quad n \geq 0 \tag{7.90}$$

  for the solutions to tend to zero away from the equator. For each integer n, this last equation provides a dispersion relation linking the wavenumber k to the angular frequency ω.

- In the special special case $n = 0$ the dispersion equation reduces to

$$(\omega + ck)(\omega^2 - ck\omega - c\beta) = 0, \tag{7.91}$$

but the root $\omega = -ck$ has to be discarded because we had to divide by this factor in eliminating u, η.

- The remaining pair of roots correspond to the Yanai or mixed Rossby-gravity mode whose group velocity is always to the east and interpolates between two types of $n > 0$ modes: the higher frequency Poincare gravity waves whose group velocity can be to the east or to the west, and the low-frequency equatorial Rossby waves whose dispersion relation can be approximated as

$$\omega = \frac{-\beta k}{k^2 + \beta(2n+1)/c} \quad . \tag{7.92}$$

7.6 General form of wave equations*

The general form of the wave equation is:

$$\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} = \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} + \frac{\partial^2 q}{\partial z^2} \tag{7.93}$$

where q is the disturbance and c the *propagation velocity*. In general holds: $c = \nu\lambda$. By definition holds: $k\lambda = 2\pi$ and $\omega = 2\pi\nu$. Therefore,

$$c = \nu\lambda = 2\pi\nu/k = \omega/k \quad . \tag{7.94}$$

In principle, there are two types of waves:

1. Longitudinal waves: for these holds $\vec{k} \parallel \vec{c} \parallel \vec{q}$. In a longitudinal wave the particle displacement is parallel to the direction of wave propagation. The animation (http://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html) shows a one-dimensional longitudinal plane wave propagating down a tube. The particles do not move down the tube with the wave; they simply oscillate back and forth about their individual equilibrium positions. Pick a single particle and watch its motion. The wave is seen as the motion of the compressed region (ie, it is a pressure wave), which moves from left to right. The second animation shows the difference between the oscillatory motion of individual particles and the propagation of the wave through the medium. The animation also identifies the regions of compression and rarefaction.

2. Transversal waves: for these holds $\vec{k} \parallel \vec{c} \perp \vec{q}$. In a transverse wave the particle displacement is perpendicular to the direction of wave propagation. The animation (http://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html) below shows a one-dimensional transverse plane wave propagating from left to right. The particles do not move along with the wave; they simply oscillate up and down about their individual equilibrium positions as the wave passes by. Pick a single particle and watch its motion. The S waves

(Secondary waves) in an earthquake are examples of Transverse waves. S waves propagate with a velocity slower than P waves, arriving several seconds later.

3. Water waves: Water waves are an example of waves that involve a combination of both longitudinal and transverse motions. As a wave travels through the waver, the particles travel in clockwise circles. The radius of the circles decreases as the depth into the water increases. The animation (http://www.acs.psu.edu/drussell/Demos/waves/wavemotion.html) below shows a water wave travelling from left to right in a region where the depth of the water is greater than the wavelength of the waves. I have identified two particles in yellow to show that each particle indeed travels in a clockwise circle as the wave passes.

The *phase velocity* is given by

$$c_{\rm ph} = \omega/k \quad . \tag{7.95}$$

The *group velocity* is given by:

$$c_{\rm g} = \frac{d\omega}{dk} = c_{\rm ph} + k\frac{dc_{\rm ph}}{dk} \tag{7.96}$$

If $c_{\rm ph}$ does not depend on ω holds: $c_{\rm ph} = c_{\rm g}$. In a dispersive medium it is possible that $c_{\rm g} > c_{\rm ph}$ or $c_{\rm g} < c_{\rm ph}$. If one wants to transfer information with a wave, e.g. by modulation of an electromagnetic wave, the information travels with the velocity at with a change in the electromagnetic field propagates. This velocity is often almost equal to the group velocity.

For some media, the propagation velocity follows from:

- Pressure waves in a liquid or gas: $c = \sqrt{\kappa/\varrho}$, where κ is the modulus of compression.

- For pressure waves in a gas also holds: $c = \sqrt{\gamma p/\varrho} = \sqrt{\gamma RT/M}$.

Plane waves

The equation for a harmonic traveling plane wave is

$$q(\vec{x},t) = \hat{q}\cos\Big(\vec{k}\cdot\vec{x} \pm \omega t + \varphi\Big) \quad .$$

When the situation is spherical or cylindrical symmetric, the the homogeneous wave equation can be solved. When the situation is spherical symmetric, the homogeneous wave equation is given by:

$$\frac{1}{c^2}\frac{\partial^2(rq)}{\partial t^2} - \frac{\partial^2(rq)}{\partial r^2} = 0$$

with general solution:

$$q(r,t) = C_1\frac{f(r-ct)}{r} + C_2\frac{g(r+ct)}{r}$$

When the situation has a cylindrical symmetry, the homogeneous wave equation becomes:

$$\frac{1}{c^2}\frac{\partial^2 q}{\partial t^2} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial q}{\partial r}\right) = 0$$

This is a Bessel equation, with solutions which can be written as Hankel functions. For sufficient large values of r these are approximated by:

$$q(r,t) = \frac{\hat{q}}{\sqrt{r}}\cos(k(r \pm vt))$$

If an observer is moving w.r.t. the wave with a velocity $c_{\rm obs}$, she/he will observe a change in frequency: the *Doppler effect*. This is given by: $\dfrac{\nu}{\nu_0} = \dfrac{c_{\rm f} - c_{\rm obs}}{c_{\rm f}}$.

The general solution in one dimension

Starting point is the equation:

$$\frac{\partial^2 q(x,t)}{\partial t^2} = \sum_{m=0}^{N} \left(b_m \frac{\partial^m}{\partial x^m} \right) q(x,t)$$

where $b_m \in I\!R$. Substituting $q(x,t) = A\mathrm{e}^{i(kx-\omega t)}$ gives two solutions $\omega_j = \omega_j(k)$ as dispersion relations. The general solution is given by:

$$q(x,t) = \int_{-\infty}^{\infty} \left(a(k)\mathrm{e}^{i(kx-\omega_1(k)t)} + b(k)\mathrm{e}^{i(kx-\omega_2(k)t)} \right) dk$$

Because in general the frequencies ω_j are non-linear in k there is dispersion and the solution cannot be written any more as a sum of functions depending only on $x \pm ct$: the wave front transforms.

7.7 Spheroidal Eigenfunctions of the Tidal Equation*

Laplace's tidal equations, governing the small amplitude dynamics of a shallow fluid on a rotating sphere, are the fundamental linear problems of large-scale geophysical fluid dynamics. Originally formulated by Laplace, its general solution and, in particular, its full dispersion relation are still not known. The current understanding of the problem rests essentially on the equatorial β-plane approximation [Matsuno, 1966] and extensive numerical studies [Longuet-Higgins, 1968]. In this framework the system has been found to exhibit Rossby waves, Yanai waves, and gravity waves, including the Kelvin wave. In spite of its elegance and fundamental significance for the terrestrial climate problem, the β-plane concept is not entirely free from ambiguities. Although the Matsuno wave equation appears to be a well-posed problem, its dispersion relation admits an unphysical, westward propagating "Kelvin" mode. This has to be ruled out a posteriori [Matsuno, 1966]. Furthermore, the β-plane concept does not yield a physically meaningful nonrotating limit. And finally, the Matsuno equation is invariant under meridional translations. Yanai waves should thus be observed at all latitudes and every latitude defines the center of a waveguide. Such properties are not physically realizable. On the other hand, the analysis of the spherical version of the tidal problem is complicated. First, the manipulation of the first order equations of motion into an appropriate wave equation is cumbersome, namely because the two-dimensional (2D) geometry of the spherical surface is non-Euclidean. Second, as shown below, the basic wave operator in the various forms of the tidal problem is the spheroidal wave equation. For this equation, infinity is an irregular singular point precluding the establishment of recurrence relations similar to those for functions of the hypergeometric type [Flammer, 1957]. The transformation of differential operations in physical space into some simpler algebra in wave number space is thus impossible. Geometrical difficulties are largely simplified by a systematic application of tensor analysis in 2D Riemann space. Here, index notation will be used with indices m, n,... running from 1 to 2 and a semicolon denoting covariant differentiation. For details of the notation and the form of the geometrical tensors in spherical, geophysical coordinates see [Townsend et al., 1992]. The linearized equations of motion of a shallow fluid on a rotating sphere are shown in [Townsend et al., 1992] to assume the form

$$\partial_t r + j^n{}_{;n} = 0,$$

$$\partial_t j_n + \epsilon_{mn} f j^m + c^2 \partial_n r = 0,$$

where $j_n = R v_n$ is the effective momentum density, R the constant equilibrium mass per unit area, and $f = 2\Omega \sin\varphi$ using the latitude dependent Coriolis parameter. For the potential vorticity z, defined by

$$Rz = \epsilon^{mn} v_{n;m} - fr/R,$$

the linearized equations of motion imply the relation

$$R\partial_t z + f^n v_n = 0,$$

where f^n is the contravariant gradient of the Coriolis parameter. Furthermore, the gradient of the divergence of a vector on the spherical surface is given by

$$j^a{;}_{an} = g^{ab}(j_n{;}_{ab} - \epsilon_{na}\epsilon^{rs} j_r{;}_{sb} - G_{ambn} j^m),$$

where g^{ab} is the metric and G_{ambn} the Riemannian [Townsend et al., 1992]. Using this identity and the potential vorticity equation it is fairly straightforward to derive the system

$$R^2\left[(\partial_t^2 + f^2 - c^2\Delta)\partial_t + c^2\epsilon^{ab} f_a \partial_b\right]\partial_t z = -c^2(\Delta f)\partial_t^2 r, \tag{7.97a}$$

$$\left[(\partial_t^2 + f^2 - c^2\Delta)\partial_t + c^2\epsilon^{ab} f_b \partial_a\right] r = -2R^2 f \partial_t z, \tag{7.97b}$$

from the equations of motion, where Δ denotes the 2D Laplacian in spherical coordinates. If Cartesian coordinates (x, y) are chosen with $f = \beta y$, i.e., in particular $\Delta f = 0$, the first of these equations reduces to the Matsuno equation. To obtain (7.97a) in the spherical case, eliminate r from the equations of motion

$$(\partial_t^2 + f^2 - c^2\underline{\Delta})\partial_t j_n + c^2\epsilon_{na}\left[\partial^a(f_b j^b) + f^a j^b{;}_b\right] = 0, \tag{7.98}$$

with

$$\underline{\Delta} j_n = g^{ab} j_n{;}_{ab} - a^{-2} j_n,$$

where a denotes the Earth's radius. Using

$$f^n \underline{\Delta} j_n = \Delta(f^n j_n) - (\Delta f) j^n{;}_n$$

.

scalar multiplication of (7.98) with the contravariant gradient of the Coriolis parameter yields (7.97a). Equation (7.98) as well as the coupled nature of the system (7.97) demonstrate that the tidal equation is inherently a 20 vector wave equation. In order to evaluate the eigenfunctions of this system, the dependent variables are assumed to be proportional to

$$e^{-i(\omega t - M\lambda}F(y),$$

where λ is longitude, $y = \sin\varphi$, and M the zonal wave number. Substituting this into (7.97), the system becomes

$$(P - m)V = -2\alpha y D, \tag{7.99a}$$

$$(P + m)D = 2\alpha y V, \tag{7.99b}$$

Here $D = r/R = iv^{n};_{n}/\omega$ and $V = aRz/c = -i\alpha v_{\varphi} \times \cos\varphi/cv$, while

$$P = a^2\Delta - \alpha^2 y^2 + \nu^2$$

is the prolate spheroidal wave operator with Lamb parameter $u = 2a\Omega/c, \nu = a\omega/c$, and $m = \alpha M/\nu$. In the form (7.99) the tidal problem emerges as a system of coupled spheroidal wave equations. In special cases, exact analytical solutions can be readily obtained without considering the complete fourth order system. Elimination of j_1 between the continuity equation and the one-component of the momentum budget yields

$$\alpha D = -(\mu + y^2)^{-1}\left[(1 - y^2)\partial_y - my\right]V,$$

with $\mu = (M^2 - \nu^2)/\nu^2$, while elimination of ji between the one-component and the two-component of the momentum budget leads to

$$\alpha V = -(n^2 - y^2)^{-1}\left[(1 - y^2)\partial_y + my\right]D,$$

with $n = \nu/\alpha$. Inserting these expressions into the right hand side (rhs) of (7.99) results in

$$(P+m)V = 2(\mu + y^2)^{-1}\left[y(1-y^2)\partial_y + m\mu\right]V, \tag{7.100a}$$

$$(P-m)D = -2(n^2 - y^2)^{-1}\left[y(1-y^2)\partial_y + mn^2\right]D. \tag{7.100b}$$

Equation (7.100a) is the spherical generalization of the Matsuno equation, while (7.100b) is the form of the tidal equation studied by Longuet-Higgins. In view of the general solution, (7.100) may not be the most convenient form as it exhibits far less symmetry than the coupled system (7.99) suggests. It nevertheless lends itself readily to the evaluation of two special cases. For standing waves (M = 0) the spherical Matsuno equation (7.100a) is exactly solved by

$$V = (1-y^2)^{1/2}(AS_L^1 + BS_L^{-1}),$$

with constants A and B and prolate spheroidal wave functions $S_L^{\pm 1}(y;\alpha^2)$ of order ± 1 and degree $L \geq 1.$ The corwhere responding divergence becomes

$$D \sim \partial_y(1-y^2)^{1/2}(AS_L^1 + BS_L^{-1}). \tag{7.101}$$

A closed expression for the eigenvalues of the spheroidal wave equation does not exist and approximations depend strongly on the value of the Lamb parameter. On Earth, the value of the Lamb parameter ranges from $\alpha \approx 1$ for the atmospheric Lamb wave, over $\alpha \approx 5$ for barotropic gravity waves in the ocean to $\alpha \approx 300$ for the first baroclinic mode in the ocean. For $\alpha^2 \ll 1$, spheroidal wave functions are approximated by expansions in terms of associated Legendre polynominals, and the dispersion relation for standing waves becomes to $O(\alpha^0)$ [Flammer, 1957] [Abramowitz and Stegun, 1965].

$$\nu^2 = \Lambda^2 + \alpha^2(2\Lambda^2 - 3)/(4\Lambda^2 - 3) \tag{7.102a}$$

with $\Lambda^2 = L(L+1)$. In the nonrotating limit, this reduces to the familiar $\nu^2 = L(L+1)$. For $\alpha^2 \gg 1$, the prolate spheroidal wave function S_L^K is appropriately approximated by parabolic cylinder functions of nonnegative, integer order, and the corresponding eigenvalue to $O(\alpha^0)$ becomes [Flammer, 1957] [Abramowitz and Stegun, 1965]

$$\epsilon(L, K, \alpha) = \alpha q + K^2 - 1 + p,$$

with $q = 2N + 1$, $p = (3 - q^2)/8$, and $N = L - |K|$. In the present case of standing waves ($\epsilon = \nu^2$, $K^2 = 1$) this yields the dispersion relation to $O(\alpha^0)$

$$\nu^2 = \alpha(2N_2 + 1) + p, \tag{7.102b}$$

where the mode number $N_2 = L - 1$ measures the number of zeros of j_2 in the open interval $y \in (-1, 1)$. With the inclusion of higher orders in α and $1/\alpha$, respectively, the dispersion relation (7.102) and the corresponding expansion of the spheroidal wave function permit the construction of Fig. 1 and Fig. 7 of [Longuet-Higgins, 1968] to an arbitrary degree of accuracy from [Abramowitz and Stegun, 1965] . The asymptotic expansion of (7.100a) provides an estimate of the domain of validity of the β-plane approximation in physical and wave number space. For low frequencies $\nu^2 \ll M^2$, the first order Taylor expansion of the denominator on the rhs of (7.100a) is justified and yields

$$(\alpha^2\Delta - \delta y^2 + \nu^2 - m)V = O,$$

with $\delta = \alpha^2 - 2m/\mu$, . For large a in the low-frequency domain under consideration $\alpha^2 \gg 2m/\mu$, so that $\delta \approx \alpha^2$. Assuming $V = (1 - y^2)^{|M|/2}F$, where the absolute value of M ensures the regularity of V at the poles for negative M, and transforming to $x = y\sqrt{\alpha}$

$$\left[(\alpha - x^2)\partial_x^2 - 2(|M|+1)x\partial_x - \alpha x^2 + \alpha(2N_2+1)\right] F \approx 0$$

yields for $\alpha \gg x^2$, i.e., in the vicinity of the equator $y^2 \ll 1$

$$(\partial_x^2 - x^2 + 2N_2 + 1 - x + 2N2 + 1)F \approx O$$

the familiar β-plane version of the Matsuno equation with dispersion relation

$$\nu^3 - \left[\alpha(2N_2+1) + M^2\right]\nu - \alpha M = 0,$$

with $N_2 = L - |M|$. The β-plane approximation thus emerges asymptotically from the full tidal equation as an equatorial ($y^2 \ll 1$), baroclinic ($\alpha^2 \gg 1$) low-frequency approximation. While the original frequency restriction $\nu^2 \ll M^2$ admits Rossby waves only, the additional restriction to low latitudes also allows for low-frequency gravity waves in this approximation. The equatorial nature of these asymptotics is obviously not compatible with a "midlatitude β-plane," while the large u condition rules out a nonrotating limit. The β-plane approximation essentially neglects the coupling of Eqs. (7.99a) and (7.99b). In the full tidal vector equation it is this coupling that excludes the "wrong Kelvin wave" a priori. A second special, but exact solution can be obtained for inertial waves. At the inertial frequency $\nu = \alpha$, (7.100b) has the exact solution

$$D = (1-y^2)^{1/2}(AS_L^{M-1} + BS_L^{1-M}), \tag{7.103}$$

with $L \geq 0$ and $1 - L \leq M \leq L + 1$. The dispertion relation in this case

$$\left[\epsilon = \alpha^2 - M,\ K^2 = (M-1)^2\right]$$

can again be read from [Abramowitz and Stegun, 1965] . For small α one finds to $O(\alpha^2)$

$$\alpha^2 = \Lambda^2 + M + \alpha^2 \left[2\Lambda^2 - 2M(M-2) - 3\right] / (4\Lambda^2 - 3) \tag{7.104a}$$

admitting only $L = 0$ and $M = 0$ in the nonrotationg case, while the dispersion relation for large α becomes $O(\alpha^2)$

$$\alpha^2 = \alpha(2N_0 + 1) + M^2 - M + p \tag{7.104b}$$

where $N_0 = L - |M - 1|$ measures the number of seros of the mass perturbation r in the open interval $y \in (-1, 1)$. At $M = 0$, the relation (7.104) coincides obviously with (7.102) at $\nu = \alpha$, where $N_0(M = 0, \nu = \alpha = N_2(M = 0, \nu = \alpha)$, while in general the number of zeros of the mass perturbation r will differ from the number of zeros of j_2. The divergence (7.103) at M=0 agrees with (7.101) at $\nu = \alpha$, since for spheroidal wave functions [Flammer, 1957] :

$$M_{E/W} = \frac{1}{2} \pm \sqrt{\alpha^2 - \alpha q + p + \frac{1}{4}},$$

with

$$|M_W| = M_E - 1 < M_E. \tag{7.105}$$

The same inequality is found from (7.104a). In contrast to the β-plane approximation, modes in this frequency domain are labeled by Np, and the phase speed of eastward propagating gravity waves is smaller than the westward speed at the same frequency and mode number. At high frequencies $\nu^2 \gg \alpha^2$, the approximation of (7.100b) by

$$(\alpha^2\Delta - \delta y^2 + \nu^2 + m)D \approx 0,$$

with $\delta = \alpha^2 - 2M/n^3$, is uncritical. For large α and positive M not too large $\delta \approx \alpha^2$, and the expansion of this equation similar to the Matsuno equation yields the

approximate dispersion relation

$$\nu^3 - \left[\alpha(2N_0 + 1) + M^2\right]\nu + \alpha M = 0,$$

with $N_0 = L - |M|$. The eastward propagating "Rossby" solutions of this dispersion have to be discarded, as they do not satisfy the defining inequality. On the other hand, the gravity solutions including the Kelvin wave satisfy this inequality. As a consequence of the positive sign of the last term, westward phase speeds are larger than eastward speeds. This is in agreement with the exact solution (7.104). The wave number space of the tidal equation is thus separated at the inertial frequency $\nu = \alpha$ into a lowfrequency domain, where modes are governed by N_2 and a high-frequency domain with gravity modes controlled by N_0. This is the mode number that survives the transition to the nonrotating case. For $\Omega = 0$, D is proportional to the associated Legendre polynominal P_L^M with $N_0 = L - |M|$ zeros in the open interval $y \in (-1, 1)$, and the dispersion relation $\nu^2 = L(L+1)$ can alternatively be written

$$\nu^2(M) = N_0(N_0 + 1) + (2N_0 + 1)|M| + M^2 = \nu^2(-M)$$

. As indicated by the inequality (7.105), rotation leads primarily to the loss of this synunetry for gravity modes. At lower frequencies, additional Rossby modes emerge in rotating systems, which can no longer be accormnodated by Np. Hence, the mode number Np governs that domain of wave number space, where rotation merely modifies modes already existing in the nonrotating case. The transition between mode numbers is only possible due to the vector character of the tidal equation. On Earth, this highfrequency domain of wave number space is occupied by Lamb waves and barotropic gravity waves, which are of minor significance on larger scales. The atmosphere of Venus, on the other hand, is characterized by much lower values of the Lamb parameter, and gravity waves with $\nu \geq \alpha$ gain greater relevance for large-scale aspects of the circulation. The concept of covariant differentiation renders the derivation of wave equations from the equations of motion fairly straightforward. The prolate spheroidal wave operator assumes a central role in these vector wave equations. For the first time since Laplace's formulation of the problem, exact analytical solutions are presented in the special cases of standing waves and inertial waves. These solutions confirm corresponding numerical calculations, while the asymptotics of the spheroidal wave equation for equatorial, baroclinic, low-frequency waves yield indeed the β-plane approximation. Furthermore, the results demonstrate a fundamental separation of the wave number space of the tidal equation at the inertial frequency. At higher frequencies, the mode number N_0 controls the dispersion of gravity waves, which experience rotation effects merely as a loss of syrrunetry with respect to $M = 0$. For a corotating observer westward

propagating gravity waves at these frequencies are faster and longer than their eastward counterparts. At low frequencies the dispersion of additional Rossby modes is incorporated by transition to the mode number N_2. This transition is primarily possible due to the vector character of the tidal problem. It can be expected that in the framework of the complete analytical theory of the tidal equation the angular momentum of eigensolutions, measured by the degree L, will be essential for this separation of the wave number space.

Part III

Third part: Stochastic climate model and Mesoscopic Dynamics

Part IV

Fourth part: Programming and tools

Bibliography

Abramowitz, M. and Stegun, I. A. (1965). Handbook of mathematical functions with formulas, graph, and mathematical tables. *Applied Mathematics Series*, 55:1046.

Arnold, L. (1995). *Random dynamical systems*. Springer.

Arnold, L. (2001). *Hasselmann's program revisited: The analysis of stochasticity in deterministic climate models*, volume 49. Birkhäuser, Boston.

Baker, G. L. and Blackburn, J. A. (2005). *The pendulum: a case study in physics*, volume 8. Oxford University Press Oxford.

Barber, D., Dyke, A., Hillaire-Marcel, C., Jennings, J., Andrews, J., Kerwin, M., Bilodeau, G., McNeely, R., Southon, J., Morehead, M., and Gagnonk, J.-M. (1999). Forcing of the cold event of 8,200 years ago by catastrophic drainage of laurentide lakes. *Nature*, 400(6742):344–348.

Bhatnagar, P., Gross, E. P., and Krook, M. K. (1954). A model for collision process in gases. i. small amplitude processes in charged and neutral one-component system. *Phys. Rev*, 94:511.

Boltzmann, L. (1896). *Vorlesungen über Gastheorie : 2 Volumes (in German)*. Leipzig 1895/98 UB: O 5262-6.

Boltzmann, L. (1995). *Lectures on Gas Theory*. Dover Publ. New York. ISBN 978-0486684550.

Broecker, S. and Peng, T.-H. (1982). *Tracers in the Sea*. Columbia University.

Broecker, W. S. (1987). The biggest chill. *Natural History*, 97(2):74–82.

Broecker, W. S. et al. (1991). The great ocean conveyor. *Oceanography*, 4(2):79–89.

Brüning, R. and Lohmann, G. (1999). Charles s. peirce on creative metaphor: a case study on the conveyor belt metaphor in oceanography. *Foundations of science*, 4(4):389–403.

Bryan, F. (1986). High latitude salinity effects and inter-hemispheric thermohaline circulations. *Nature*, 323(3):301–304.

Buckingham, E. (1914). On physically similar systems; illustrations of the use of dimensional equations. *Physical Review*, 4(4):345–376.

Budyko, M. I. (1969). The effect of solar radiation variations on the climate of earth. *Tellus*, 21:611–619.

Busch, W. (1865). *Max und Moritz (in German); Max and Maurice, a Juvenile History in Seven Tricks* . Braun und Schneider, München.

Cercignani, C. (1987). *The Boltzmann equation and its applications*. Springer New York. ISBN 978-0387966373.

Cercignani, C. (1990). *Mathematical methods in kinetic theory*. Plenum, 2 edition. ISBN 978-0306434600.

Chelton, D. B. and Schlax, M. G. (1996). Global Observations of Oceanic Rossby Waves. *Science*, 272:234–238.

Chen, D., Gerdes, R., and Lohmann, G. (1995). A 1-d atmospheric energy balance model developed for ocean modelling. *Theoretical and Applied Climatology*, 51:25–38.

Chorin, A. J. and Hald, O. H. (2006). Stochastic tools in mathematics and science. surveys and tutorials in the applied mathematical sciences, vol. 1.

Chorin, A. J., Kast, A. P., and Kupferman, R. (1999). Unresolved computation and optimal predictions. *Communications on pure and applied mathematics*, 52(10):1231–1254.

Chorin, A. J., Kupferman, R., and Levy, D. (2000). Optimal prediction for hamiltonian partial differential equations. *Journal of Computational Physics*, 162(1):267–297.

Courant, R., Friedrichs, K., and Lewy, H. (1928). Über die partiellen Differenzengleichungen der mathematischen Physik. *Mathematische Annalen*, 100:32–74.

Courant, R., Friedrichs, K., and Lewy, H. (1967). On the partial difference equations of mathematical physics. *IBM J. Res. Dev.*, 11(2):215–234.

Dansgaard, W., Johnsen, S., Clausen, H., Dahl-Jensen, D., Gundestrup, N., Hammer, C., C.S. Hvidberg, J. S., Sveinbjoernsdottir, A., Jouzel, J., and Bond, G. (1993). Evidence for general instability of past climate from a 250-kyr ice-core record. *Nature*, 364:218–220.

d'Humieres, D., Bouzidi, M., and Lallemand, P. (2001). Thirteen-velocity three-dimensional lattice boltzmann model. *PRE*, 63(6, Part 2).

Dijkstra, H., Raa, L. T., and Weijer, W. (2004). A systematic approach to determine thresholds of the ocean's thermohaline circulation. *Tellus A*, 56 (4):362.

Doedel, E. J., Champneys, A. R., Fairgrieve, T. F., Kuznetsov, Y. A., Sandstede, B., and Wang, X. (1997). Continuation and bifurcation software for ordinary differential equations (with homcont). *Available by anonymous ftp from ftp cs concordia ca, directory pub/doedel/auto*.

Egger, J. (2001). Master equations for climatic parameter sets. *Climate Dynamics*, 18(1-2):169–177.

Einstein, A. (1905). Investigations on the theory of the brownian movement. *Ann. der Physik*, 17:549–560.

Einstein, A. (1926). Die Ursache der Mäanderbildung der Flußläufe und des sogenannten Baerschen Gesetzes. *Naturwissenschaften*, 14:223–224.

Evans, D. J. and Morriss, G. (2008). *Statistical mechanics of nonequilibrium liquids*. Cambridge University Press.

Fairbanks, R. G. (1989). A 17, 000-year glacio-eustatic sea level record: influence of glacial melting rates on the younger dryas event and deep-ocean circulation. *Nature*, 342(6250):637–642.

Feigenbaum, M. J. (1980). The transition to aperiodic behaviour in turbulent systems. *Commun. Math. Phys.*, 77.

Flammer, C. (1957). *Spheroidal wave functions*. Stanford University Press.

Frisch, U. (1996). *Turbulence: the legacy of A.N. Kolmogorov*. Cambridge University Press. ISBN 0-521-45103-5.

Gerkema, T., Zimmerman, J., Maas, L., and Van Haren, H. (2008). Geophysical and astrophysical fluid dynamics beyond the traditional approximation. *Reviews of Geophysics*, 46(2).

Gill, A. E. (1982). *Atmosphere-ocean dynamics*, volume 30. Academic Press. International Geophysics Series.

Givon, D., Kupferman, R., and Stuart, A. (2004). Extracting macroscopic dynamics: model problems and algorithms. *Nonlinearity*, 17(6):R55.

Gottwald, G. (2010). On recent trends in climate dynamics. *AMS Gazette*, 37(5).

Grassberger, P. and Procaccia, I. (1983). Measuring the strangeness of strange attractors. *Physica D: Nonlinear Phenomena*, 9(2):189–208.

Haken, H. (1996). Slaving principle revisited. *Physica D: Nonlinear Phenomena*, 97(1):95–103.

Haney, R. L. (1971). Surface thermal boundary conditions for ocean circulation models. *Journal of Physical Oceanography*, 1:241–248.

Hasselmann, K. (1976). Stochastic climate models. Part I. Theory. *Tellus*, 6:473–485.

He, X. and Luo, L. S. (1997). Theory of the lattice Boltzmann method: From the Boltzmann equation to the lattice Boltzmann equation. *Phys. Rev. E*, 56(6):6811–6817.

Holton, J. R. (2004). *An Introduction to Dynamic Meteorology*. Elsevier Academic Press, Burlington, MA.

Kambe, T. (2007). *Elementary Fluid Mechanics*. World Scientific Publishing.

Kuznetsov, Y. A. (1998). *Elements of applied bifurcation theory*, volume 112. Springer, New York.

Landau, L. D. and Lifshitz, E. M. (1959). *Fluid Mechanics*, volume 6 of *Course of Theoretical Physics*. Pergamon Press, Oxford.

Langevin, P. (1908). On the theory of brownian motion. *Comptes Rendues*, 146:530–533.

Leith, C. (1975). Climate response and fluctuation dissipation. *Journal of the Atmospheric Sciences*, 32(10):2022–2026.

Lohmann, G. (2003). Atmospheric and oceanic freshwater transport during weak atlantic overturning circulation. *Tellus A*, 55(5):438–449.

Longuet-Higgins, M. S. (1968). The eigenfunctions of laplace's tidal equations over a sphere. *Philosophical Transactions for the Royal Society of London. Series A, Mathematical and Physical Sciences*, pages 511–607.

Lorenz, E. (1982). Atmospheric predictability experiments with a large numerical model. *Tellus A*, 34:505–513.

Lorenz, E. N. (1960). Maximum simplification of the dynamic equations. *Tellus*, 12(3):243–254.

Lorenz, E. N. (1963). Deterministic nonperiodic flow. *Journal of the atmospheric sciences*, 20(2):130–141.

Lorenz, E. N. (1976). Nondeterministic theories of climatic change. *Quaternary Research*, 6(4):495–506.

Lorenz, E. N. (1984). Irregularity: a fundamental property of the atmosphere*. *Tellus A*, 36(2):98–110.

Lucarini, V., Blender, R., Herbert, C., Pascale, S., Ragone, F., and Wouters, J. (2014). Mathematical and physical ideas for climate science. *Rev. Gephys.*

Maas, L. R. (1994). A simple model for the three-dimensional, thermally and wind-driven ocean circulation. *Tellus A*, 46(5):671–680.

Manabe, S. and Stouffer, R. (1993). Century-scale effects of increased atmospheric CO_2 on the ocean atmosphere system. *Nature*, 364:215–218.

Mandelbrot, B. B. (1967). How long is the coast of britain: Statistical self-similarity and fractal dimension. *Science*, 155:636–638.

Mandelbrot, B. B. (1983). *The fractal geometry of nature*. Macmillan.

Matsuno, T. (1966). Quasi-geostrophic motions in the equatorial area. *J. Meteor. Soc. Japan*, 44(1):25–43.

Mori, H. (1965). Transport, collective motion, and brownian motion. *Progress of Theoretical Physics*, 33(3):423–455.

Mori, H., Fujisaka, H., and Shigematsu, H. (1974). A new expansion of the master equation. *Progress of Theoretical Physics*, 51(1):109–122.

Müller and Maier-Reimer (2000). Trapped rossby waves. *Phys. Rev. E*, 61:1468 – 1485.

Müller, D., Kelly, B., and O'brien, J. (1994). Spheroidal eigenfunctions of the tidal equation. *Physical review letters*, 73(11):1557.

Müller, D. and O'Brien, J. (1995). Shallow water waves on the rotating sphere. *Physical Review E*, 51(5):4418.

Olbers, D. (2001). A gallery of simple models from climate physics. *In: Stochastic Climate Models, Progress in Probability (Eds.: P. Imkeller and J. von Storch)*, 49:3–63.

Peitgen, H.-O. and Richter, P. (1986). *The Beauty of Fractals*. Heidelberg: Springer-Verlag.

Proudman, J. (1916). On the motion of solids in a liquid possessing vorticity. *Proc. R. Soc. Lond. A*, 92:408–424.

Rahmstorf, S. (1996). On the freshwater forcing and transport of the Atlantic thermohaline circulation. *Climate Dynamics*, 12:799–811.

Rayleigh, L. (1916). On convection currents in a horizontal layber of fluid, when the higher temperature is on the under side. *Phil. Mag.*, 6:529–546.

Rooth, C. (1982). Hydrology and ocean circulation. *Progress in Oceanography*, 11:131–149.

Rossby, C.-G. (1939). "relation between variations in the intensity of the zonal circulation of the atmosphere and the displacements of the semi-permanent centers of action". *Journal of Marine Research*, 2 (1):38–55.

Saltzman, B. (1962). Finite amplitude free convection as an initial value problem – i. *Journal of the Atmoshpheric Sciences*, 19:329–341.

Shannon, C. E. (1948). A Mathematical Theory of Communication. *Bell System Technical Journal*, 27 (3):379–423.

Stommel, H. (1961). Thermohaline convection with two stable regimes of flow. *Tellus*, 13:224–230.

Strogatz, S. (2000). *Non-linear Dynamics and Chaos: With applications to Physics, Biology, Chemistry and Engineering*. Perseus Books.

Taylor, G. (1917). Motion of solids in fluids when the flow is not irrotational. *Proc. R. Soc. Lond. A*, 93:92–113.

Townsend, S., Lenosky, T., Muller, D., Nichols, C., and Elser, V. (1992). Negatively curved graphitic sheet model of amorphous carbon. *Physical Review Letters*, 69(6):921–924.

Tritton, D. J. (1988). *Physical Fluid Dynamics*. Oxford University Press, Science Publication. ISBN 978-0-19-854493-7.

Uhlenbeck, G. E. and Ornstein, L. S. (1930). On the theory of the brownian motion. *Physical review*, 36(5):823.

van Kampen, N. G. (1981). *Stochastic processes in physics and chemistry*. North Holland. IISBN 978-0-444-52965-7.

Wüst, G. (1935). Schichtung und Zirkulation des Atlantischen Ozeans. Das Bodenwasser und die Stratosphäre. *Wiss. Ergebn. Dtsch. Atlant. Exped. 'Meteor' 1925-1927*, 6(1):1–288.

Zwanzig, R. (1960). Ensemble method in the theory of irreversibility. *The Journal of Chemical Physics*, 33:1338.

Zwanzig, R. (1980). Problems in nonlinear transport theory. In *Systems far from equilibrium*, pages 198–225. Springer.

Zwanzig, R. (2001). *Nonequilibrium statistical mechanics*. Oxford University Press, USA.