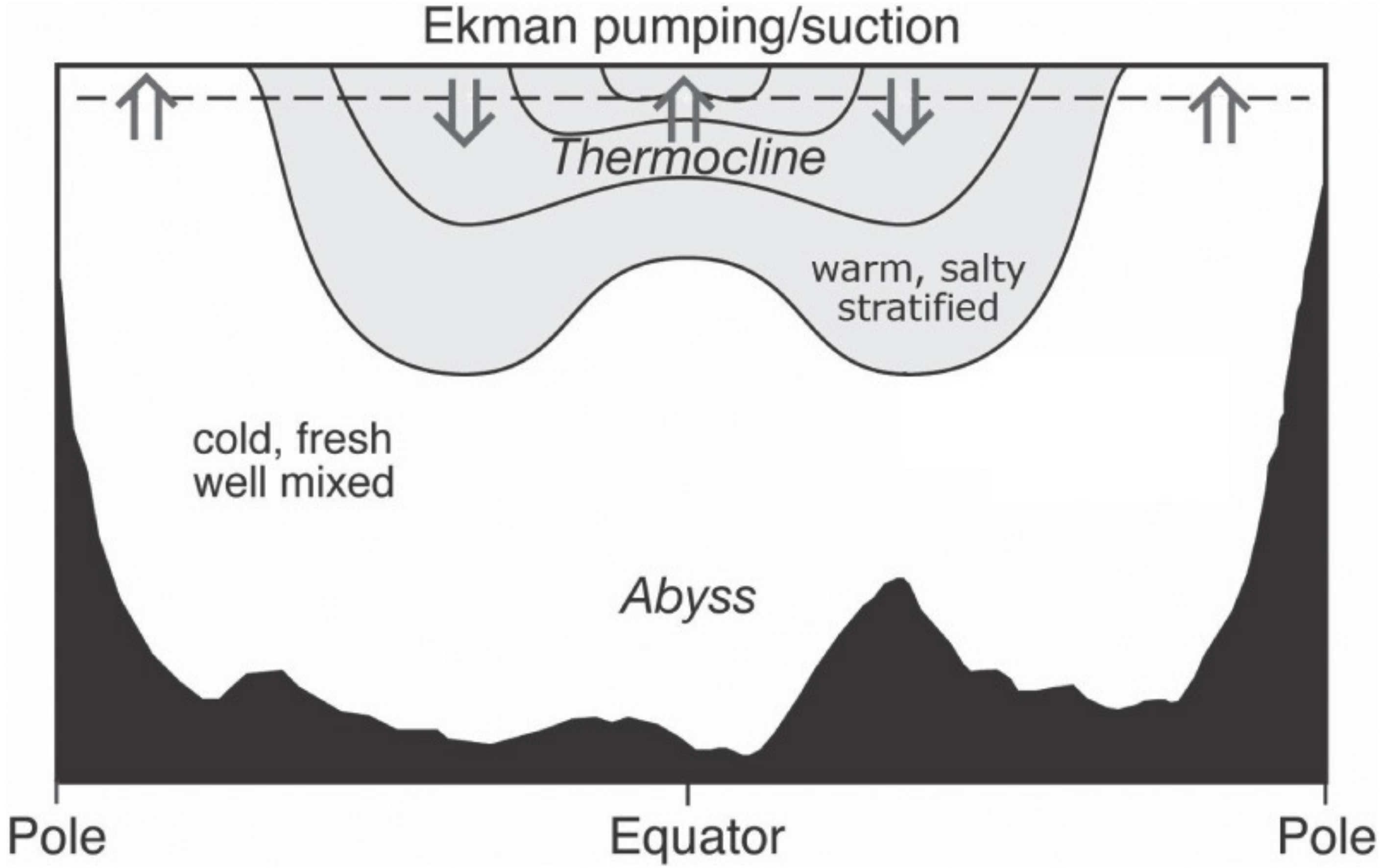
Ocean layers

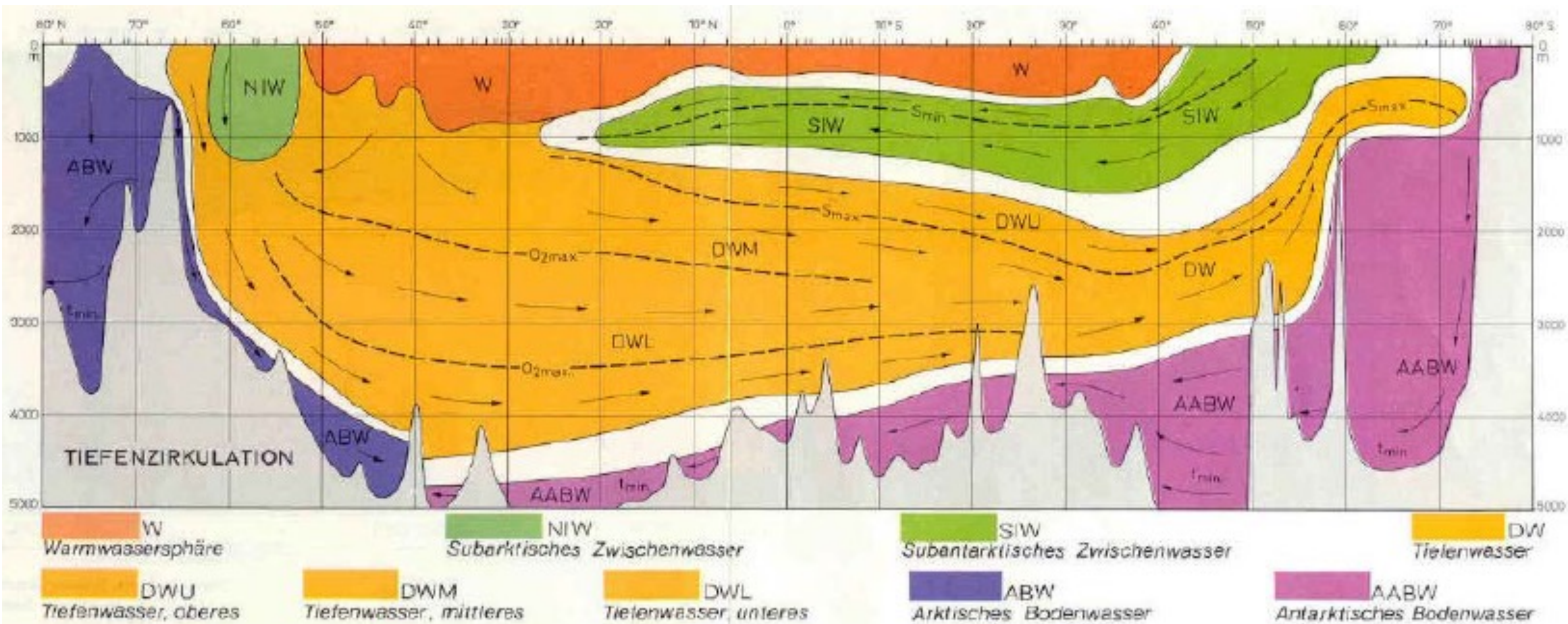


- Steady winds blowing on the sea surface produce a thin, horizontal boundary layer, the *Ekman layer.*
- *thin: a few-hundred meters thick, compared with the depth of the water in the deep ocean.*
- *A similar boundary layer exists at the bottom of the ocean and at the bottom of the atmosphere just above the sea surface, the planetary boundary layer or frictional layer .*

Atlantic Ocean

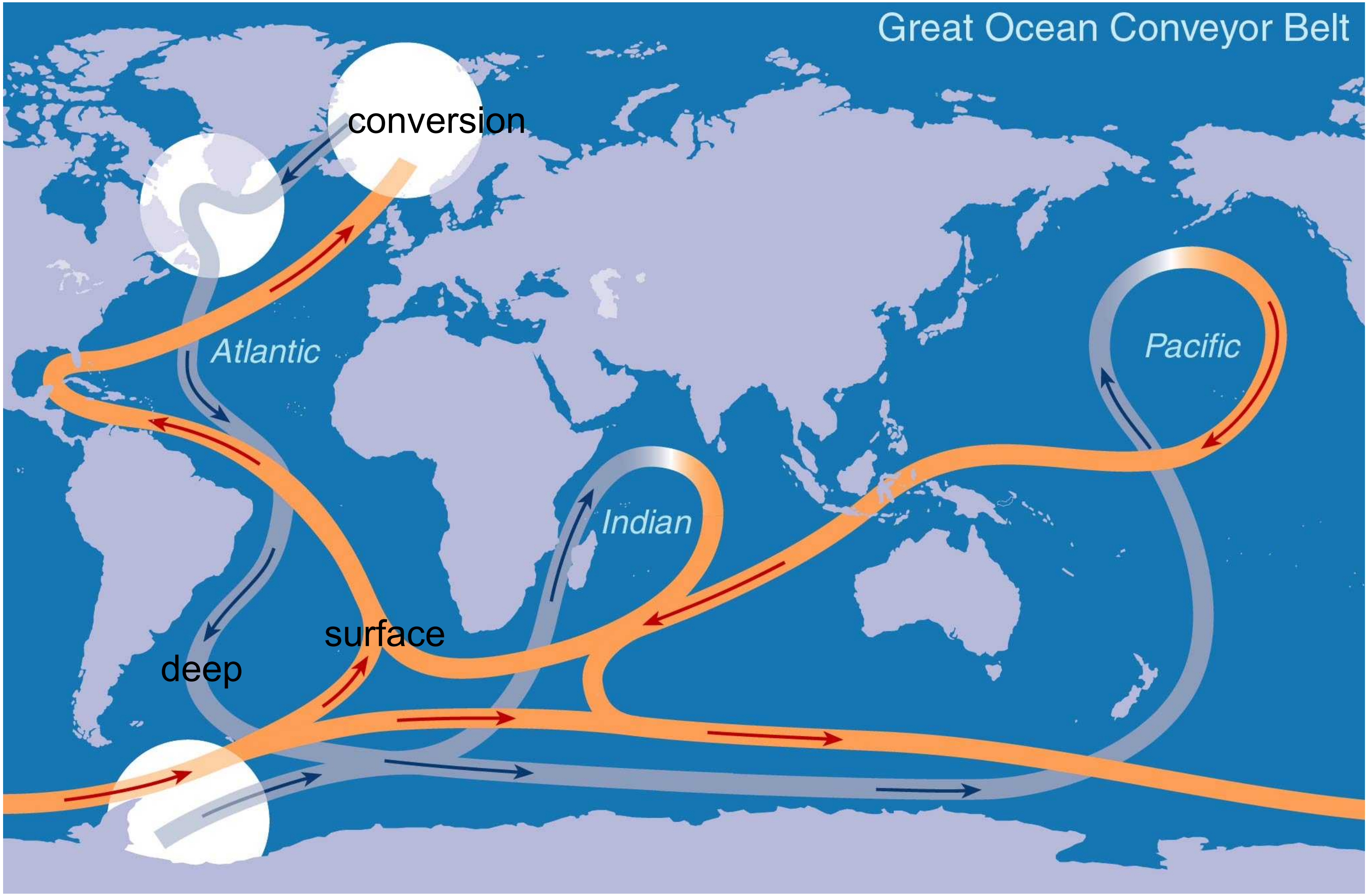
Water masses (schematic)

N S

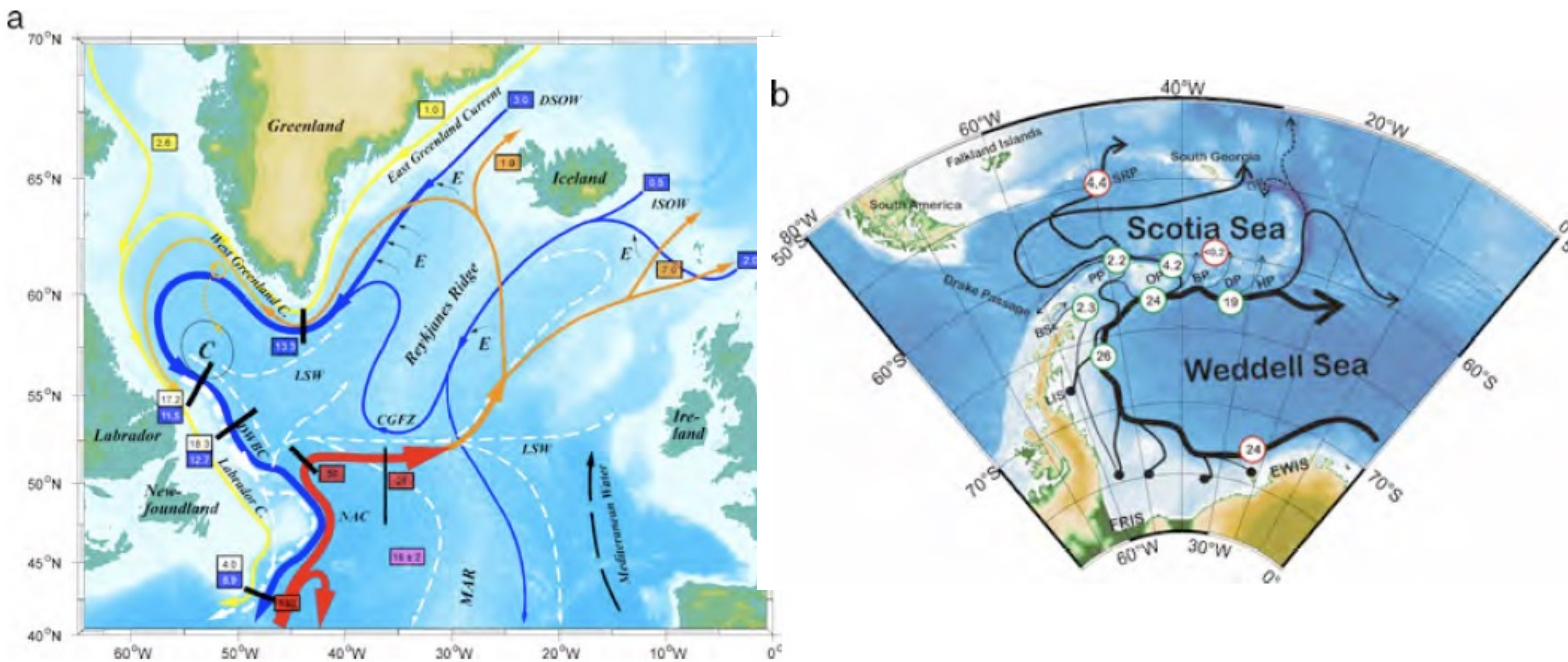


Source: Dietrich, Kalle, Kraus, Siedler

Ocean Circulation Methaphor



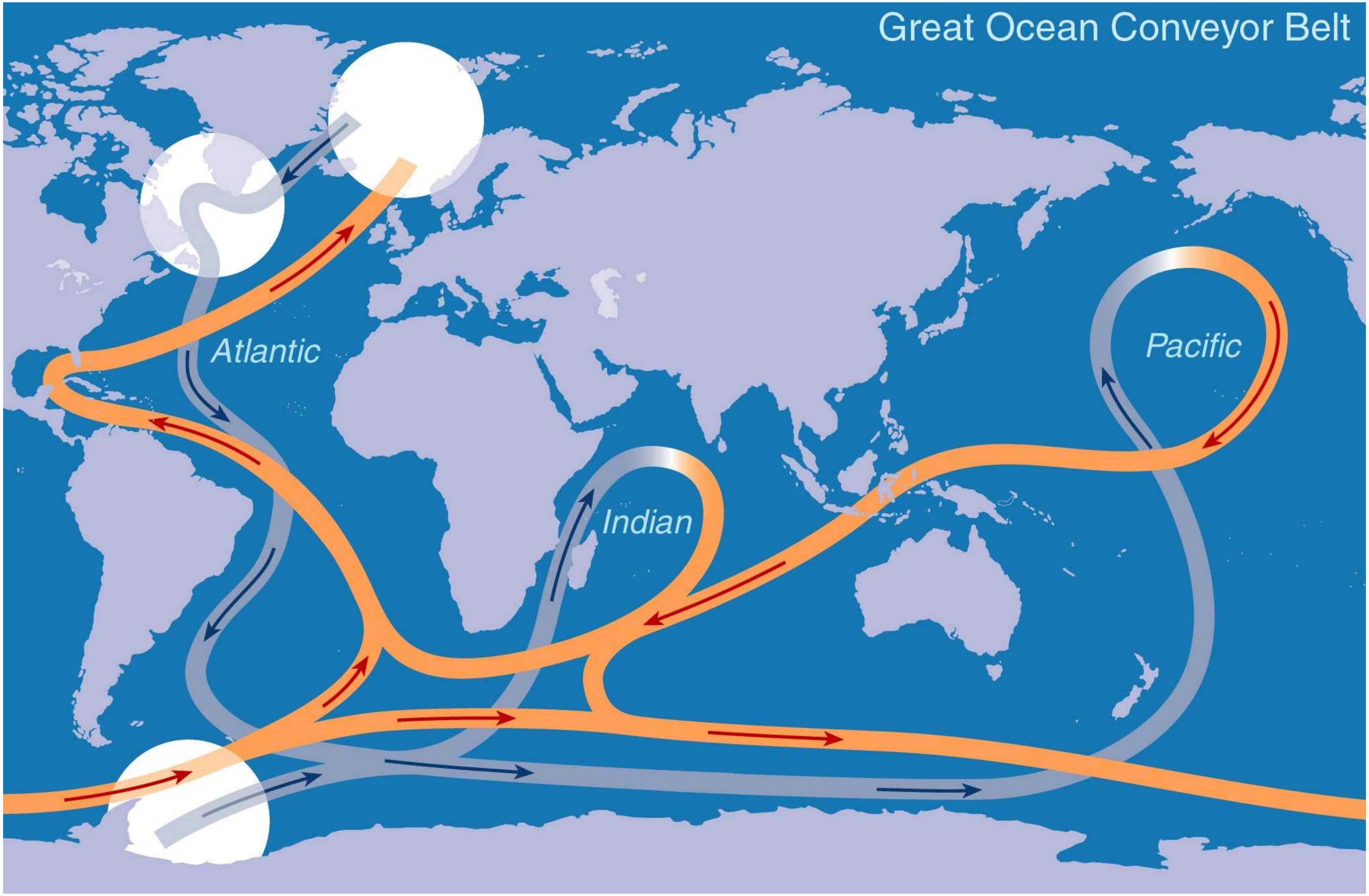
High latitudes: Schematic of the flow of important water masses



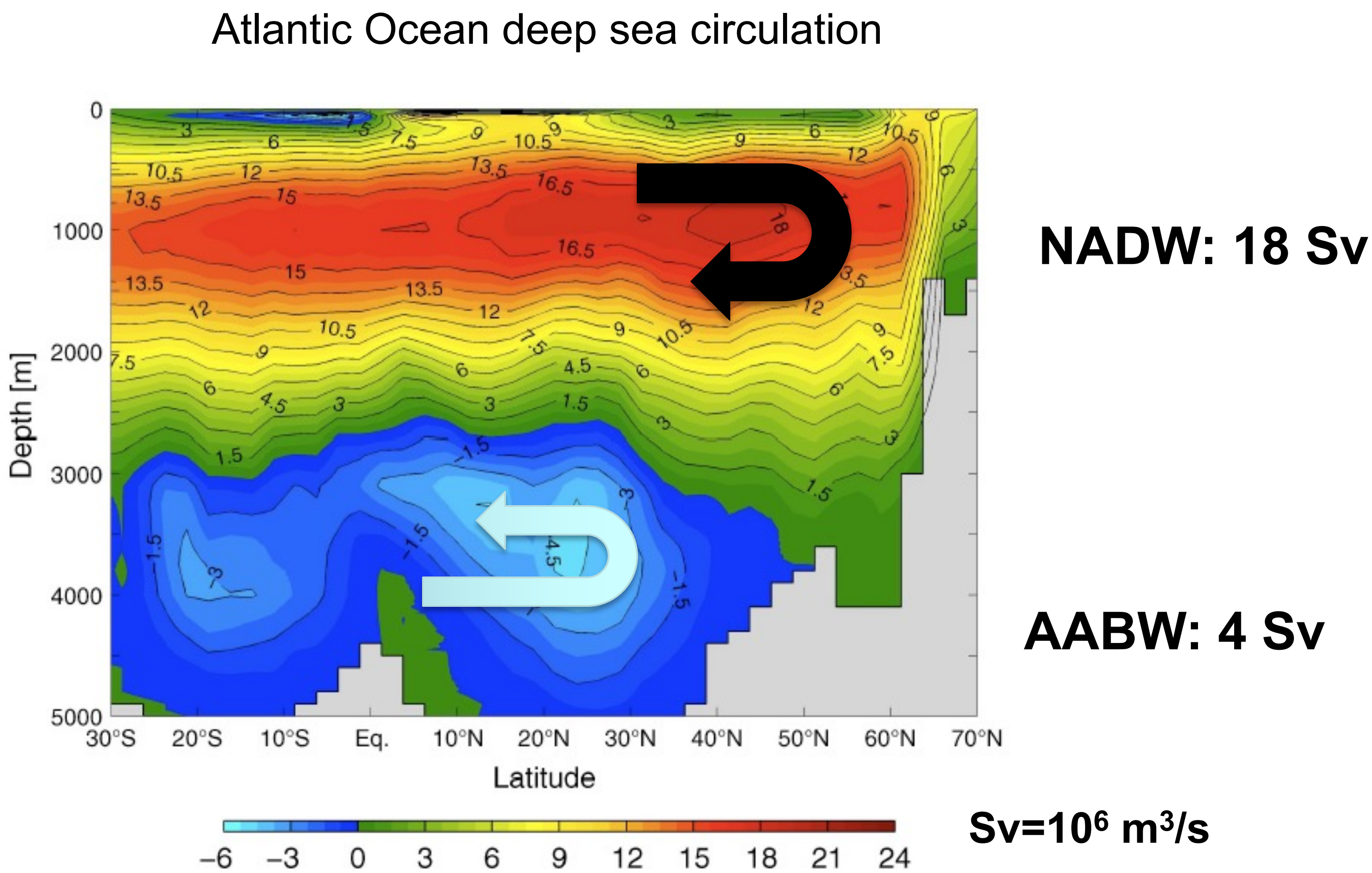
NADW

- Although the overflows themselves have only a strength of 1-2 Sv, turbulent entrainment during the downslope flow increases the volume transport of the overflows considerably.
- The overflow water from the Denmark Strait and the Faroer Bank Channel then forms the deep part of the western boundary current of the Labrador Sea and joins with a deep water outflowout of the central Labrador Sea where deep open ocean convection takes place.
- Overflow and Labrador convection contribute to about equal to the sum of around 10-15 Sv

- **we are concerned here with the dynamics of the large-scale circulation *after* the water-mass formation process**


Great Ocean Conveyor Belt
Atlantic
Pacific
Indian

Meridional overturning circulation



Symmetric solution

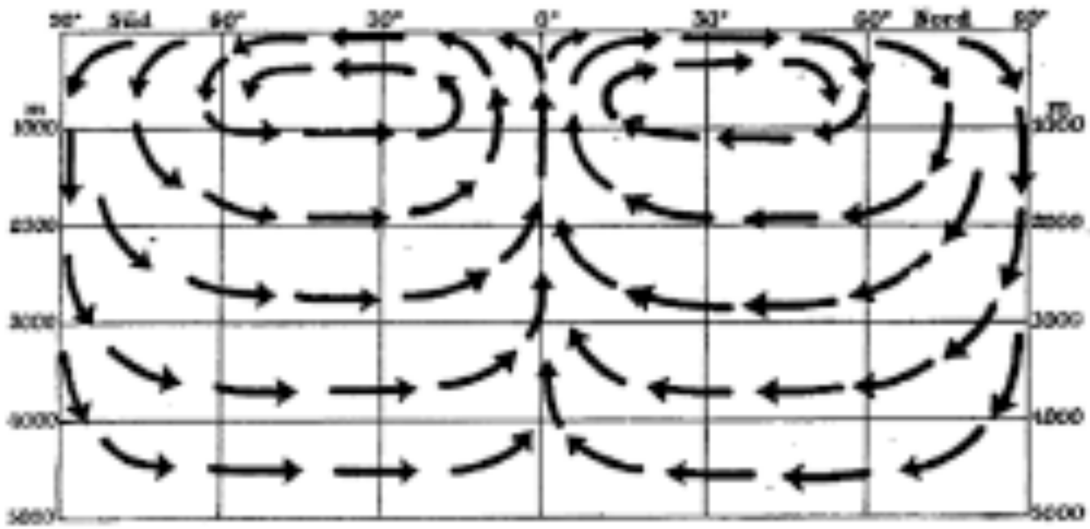


Figure 2.15: Atlantic circulation model according to (von Lenz, 1847a, b), figure after (Merz and Wüst, 1922)

Simple Model of MOC

It is instructive to derive a simple concept of the meridional overturning based on vorticity dynamics in the (y,z)-plane. The dynamical model in two dimensions read

$$\frac{d}{dt}v = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} - fu - \kappa v \tag{2.92}$$
$$\frac{d}{dt}w = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g}{\rho_0}(\rho - \rho_0) - \kappa w \tag{2.93}$$

with κ as parameter for Rayleigh friction. Using the continuity equation

$$0 = \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{2.94}$$

one can introduce a streamfunction $\Phi(y,z)$ with $v = \partial_z \Phi$ and $w = -\partial_y \Phi$. The associated vorticity equation in the (y,z)-plane is therefore

$$\frac{d}{dt}\nabla^2\Phi = -f\frac{\partial u}{\partial z} + \frac{g}{\rho_0}\frac{\partial \rho}{\partial y} - \kappa\nabla^2\Phi \tag{2.95}$$

We can choose the ansatz† satisfying that the normal velocity at the boundary vanishes, $\Phi = 0$:

$$\Phi = \Phi_{max}\,\sin(\pi y/L) \times \sin(\pi z/D) \tag{2.97}$$

MOC continued

The parameters L and D dentote the meridional and depth extend. With the assumption that the Coriolis term is absorbed into the viscous terms, we derive:

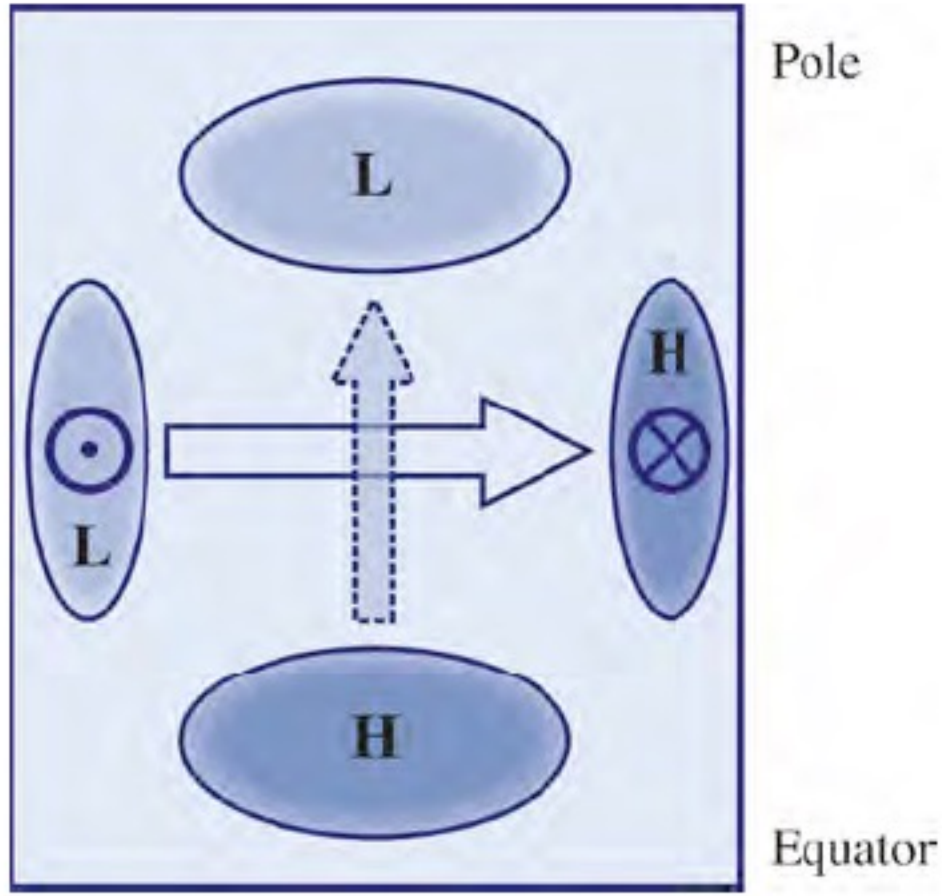
$$\frac{d}{dt}\Phi_{max} \quad = \quad \frac{a}{\rho_0}(\rho_{north} - \rho_{south}) \; - \; \kappa\Phi_{max} \tag{2.98}$$

with $a = gLD^2/4(L^2 + D^2)$.[‡]

This shows that the overturning circulation depends on the density differences on the right and left boxes. In the literature, (2.98) is simplified to a diagnostic relation

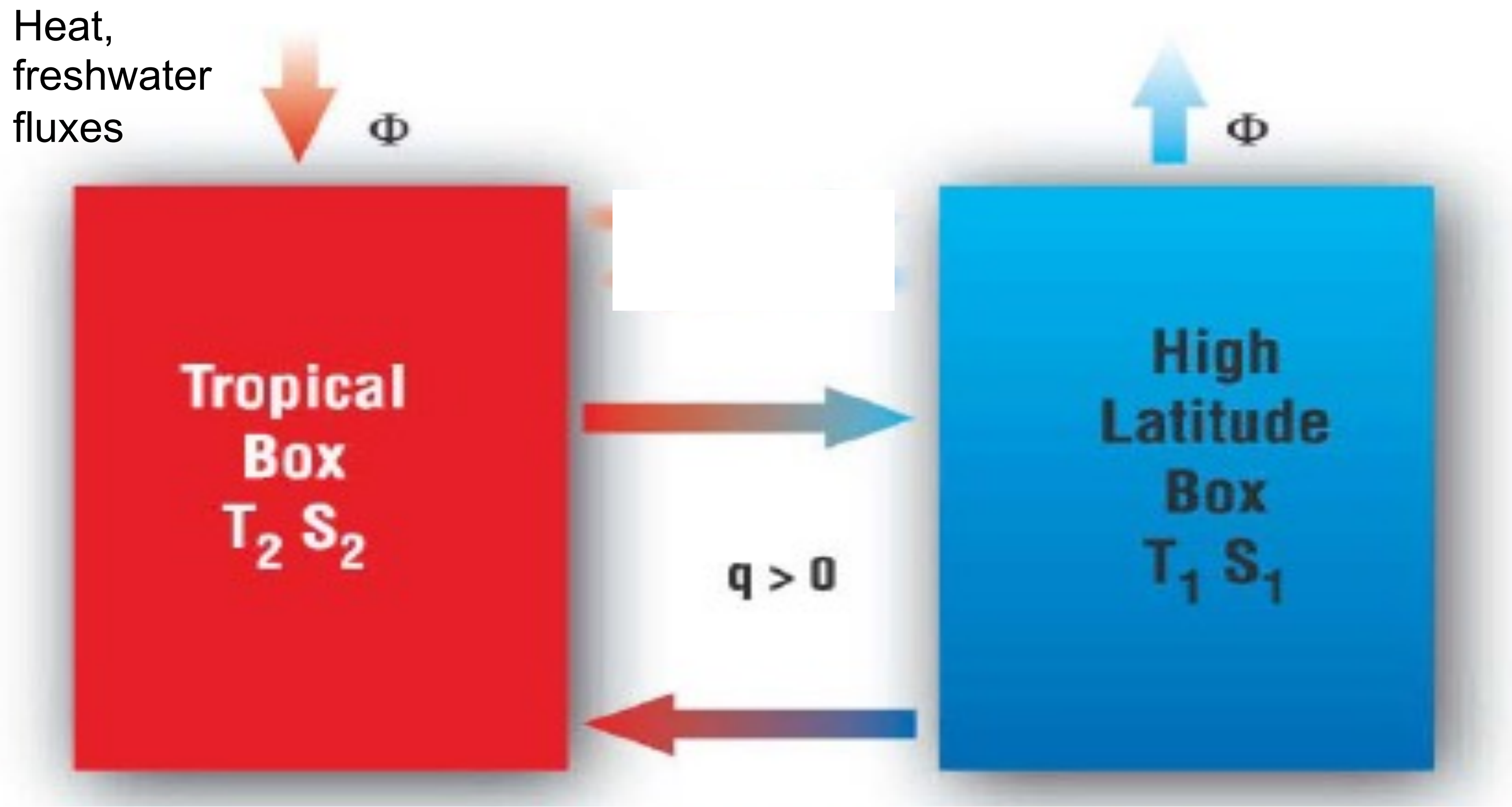
$$\Phi_{max} = \frac{a}{\rho_0\, \kappa}\; (\rho_{north} - \rho_{south}) \tag{2.99}$$

because the adjustement of Φ_{max} is quasi-instantaneous due to Kelvin waves (section 2.3).

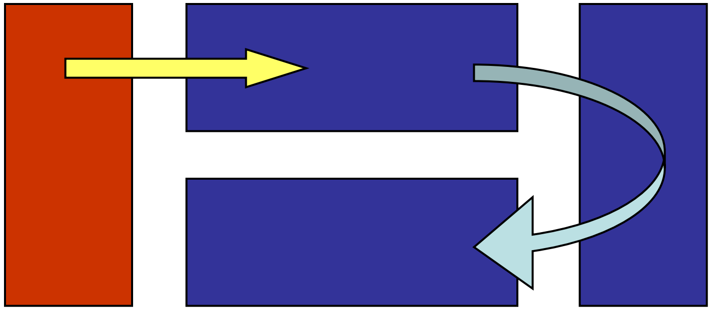


Schematic of the surface flow driven by a north-south density gradient in an ocean basin. The primary north-south gradient – as a result of the surface forcing – is in balance with an eastward geostrophic current which generates a secondary high and low pressure system. This, in turn, drives a northward geostrophic current, the upper branch of the

Stommel (1961) Box Model



Conceptual Model of MOC



South- Equator North- Atlantic

T and S

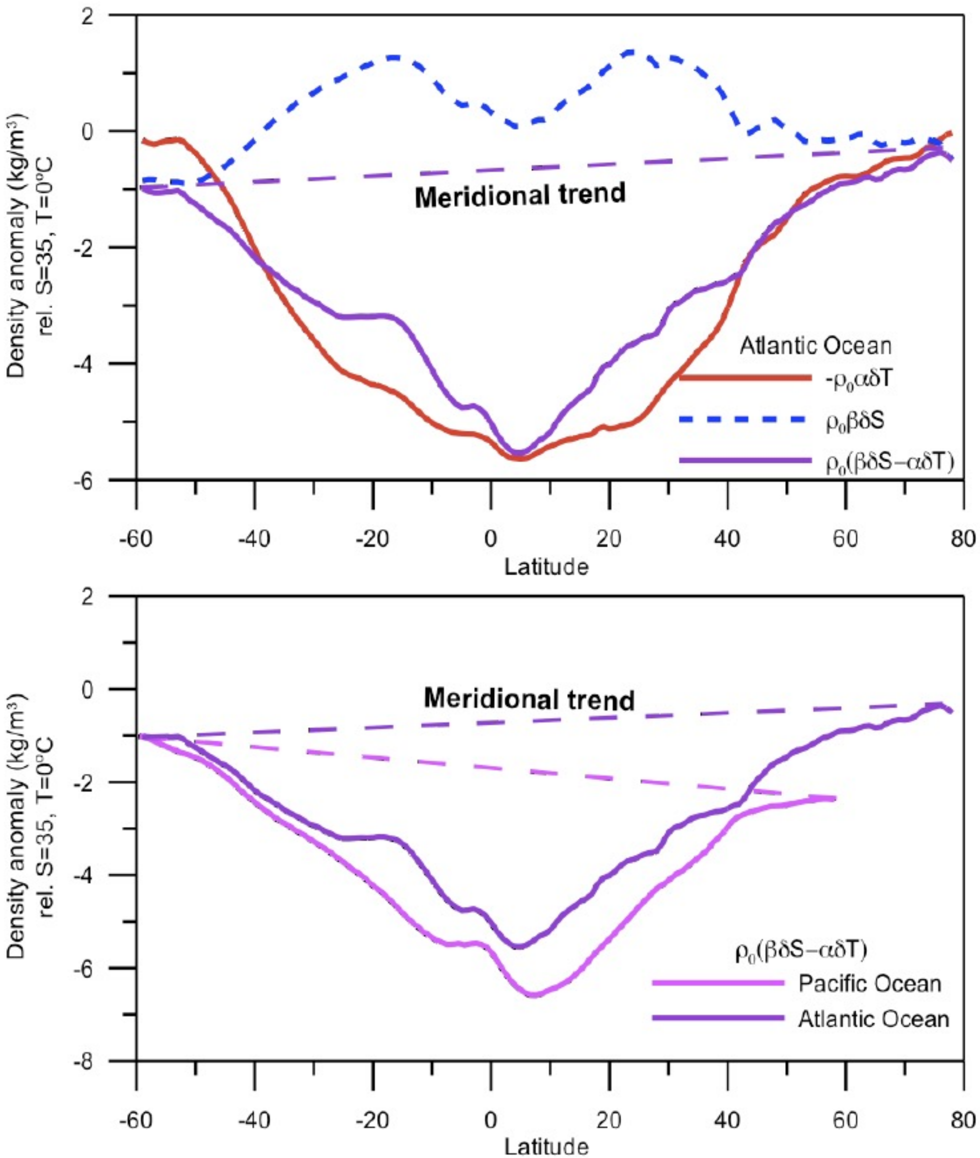
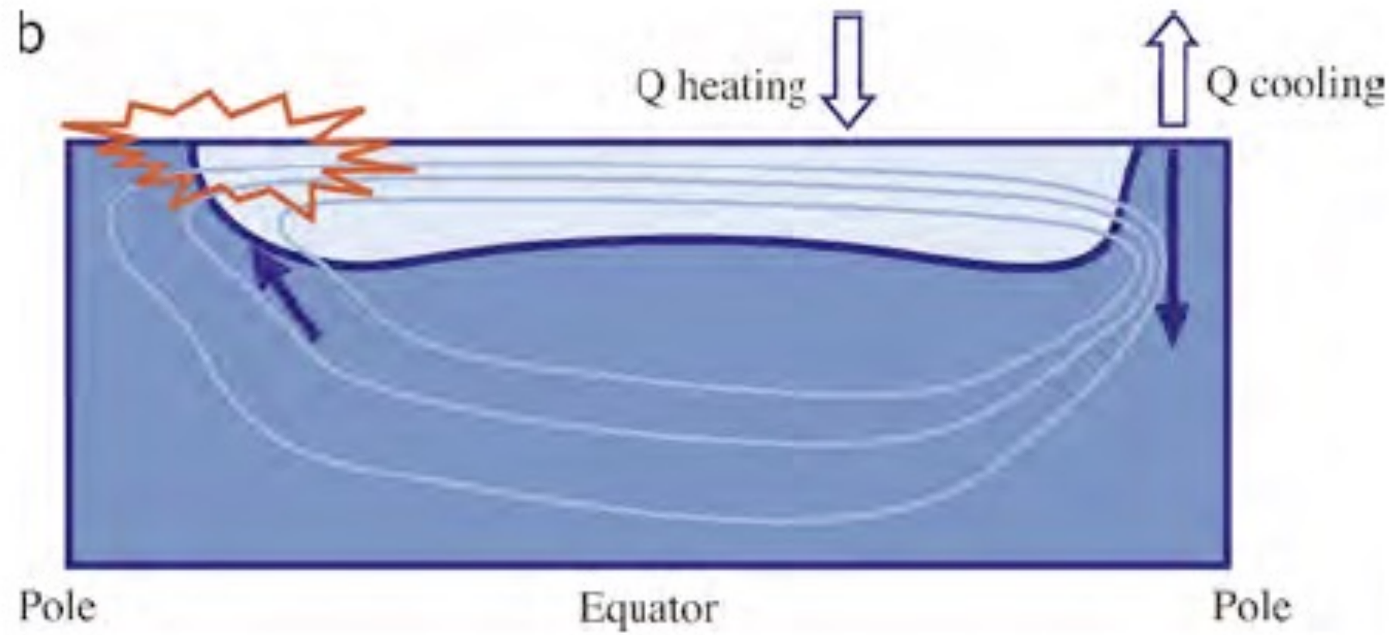


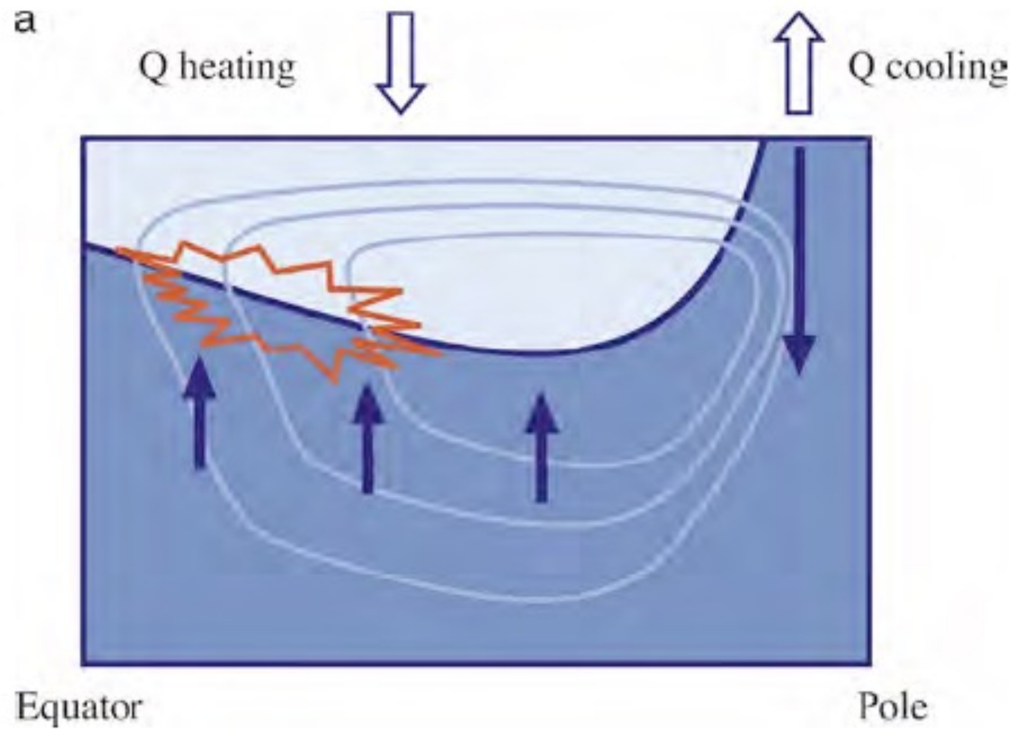
Figure 2.16: a) The Atlantic surface density is mainly related to temperature differences. b) But the pole-to-pole differences are caused by salinity differences.

MOC and mixing



the upwelling can be wind-induced (Ekman pumping), isopycnals must outcrop at the surface as in the Southern Ocean.

What drives the AMOC?



Mixing in the Interior, the interior mixing drives the MOC

the windstress and breaking surface waves drive the MOC

Box Models

- Stommel's model almost completey ignored (25 years)
- Rooth, 1982: Two hemisphere counterpart,
- Unaware of Stommel (1961) model
- Rooth suggested to F. Bryan: test with a GCM

Bryan 1986: Single basin, 2 hemispheres, surface density (almost) prescribed, symmetric about equator

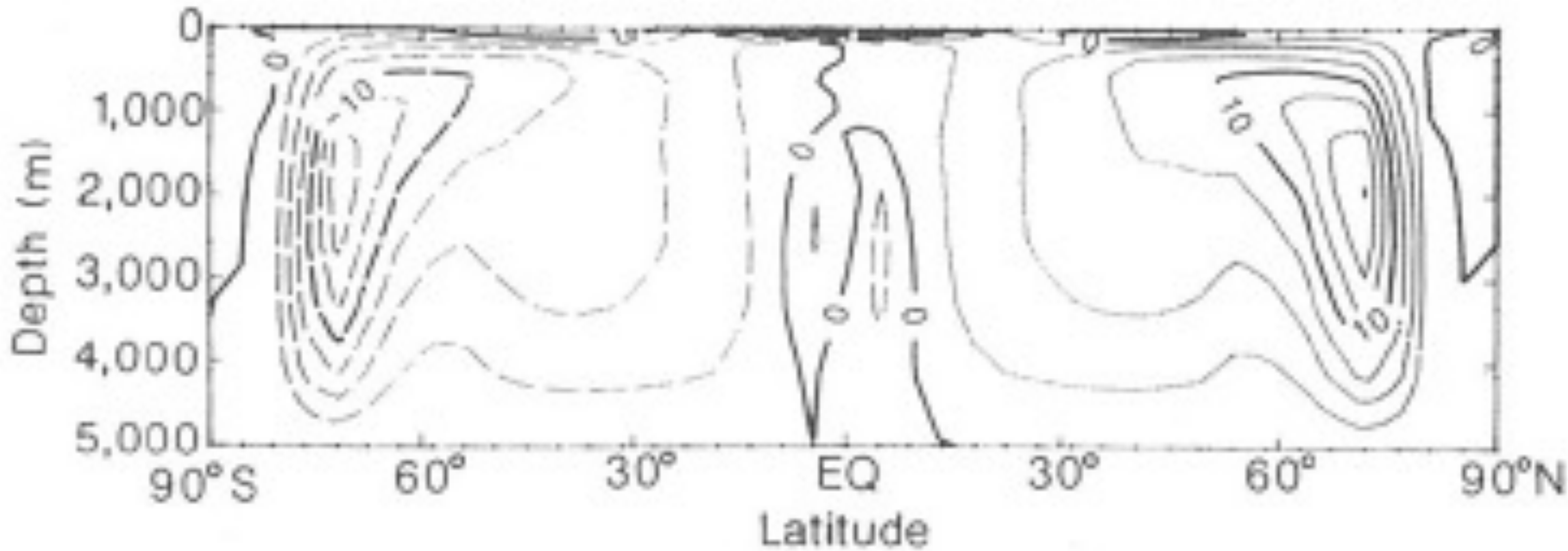
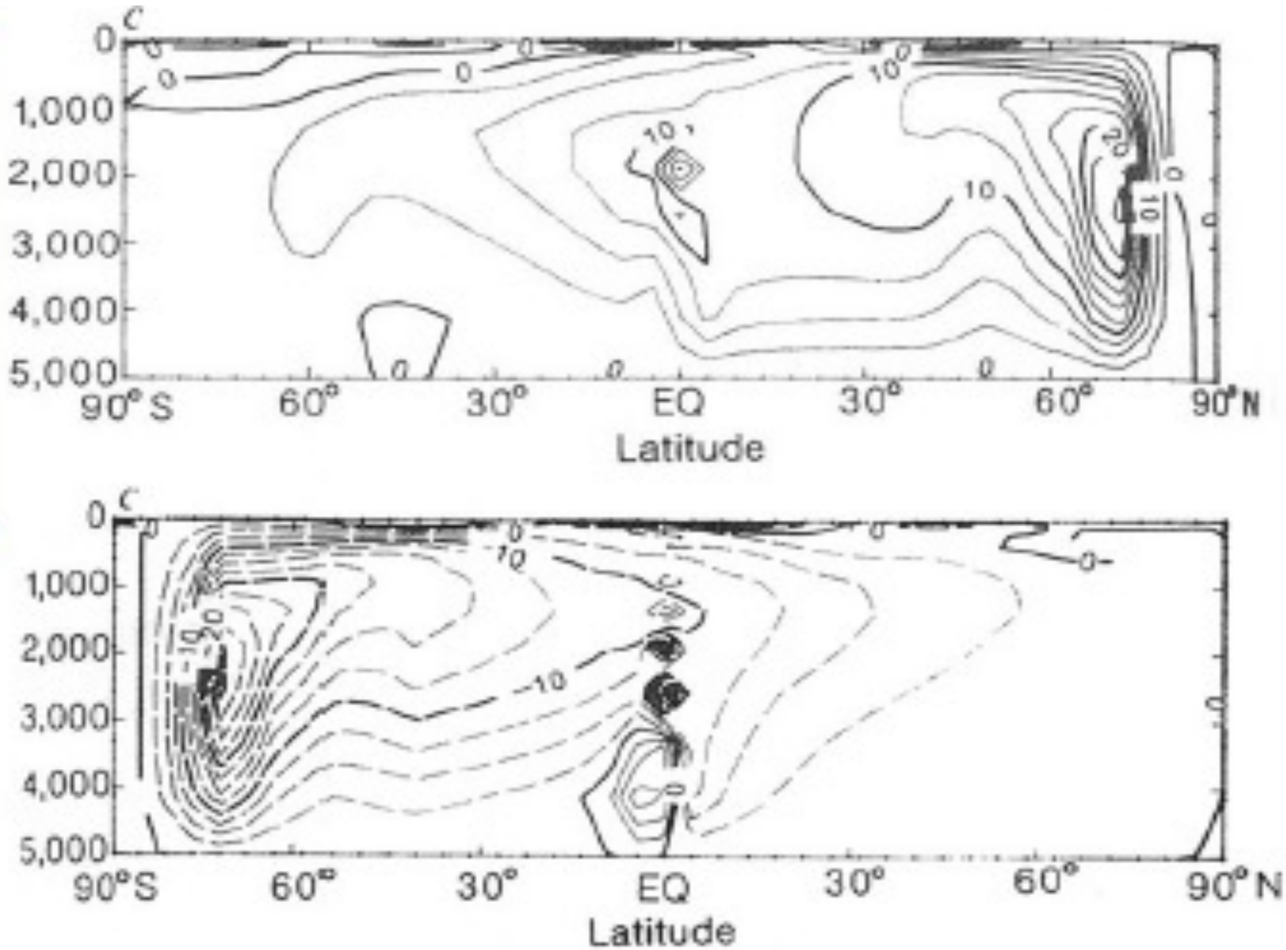


Fig. 2 Stream-function for the zonally integrated meridional overturning circulation for experiment 1 at the end of the integration. Positive values indicate a clockwise circulation; contour interval, $2.5 \times 10^6\ m^3\ s^{-1}$.

Bryan 1986: Single basin, 2 hemispheres, symmetric forcing (SST and E-P prescribed): Two stable equilibria



Global circulation: 4 states

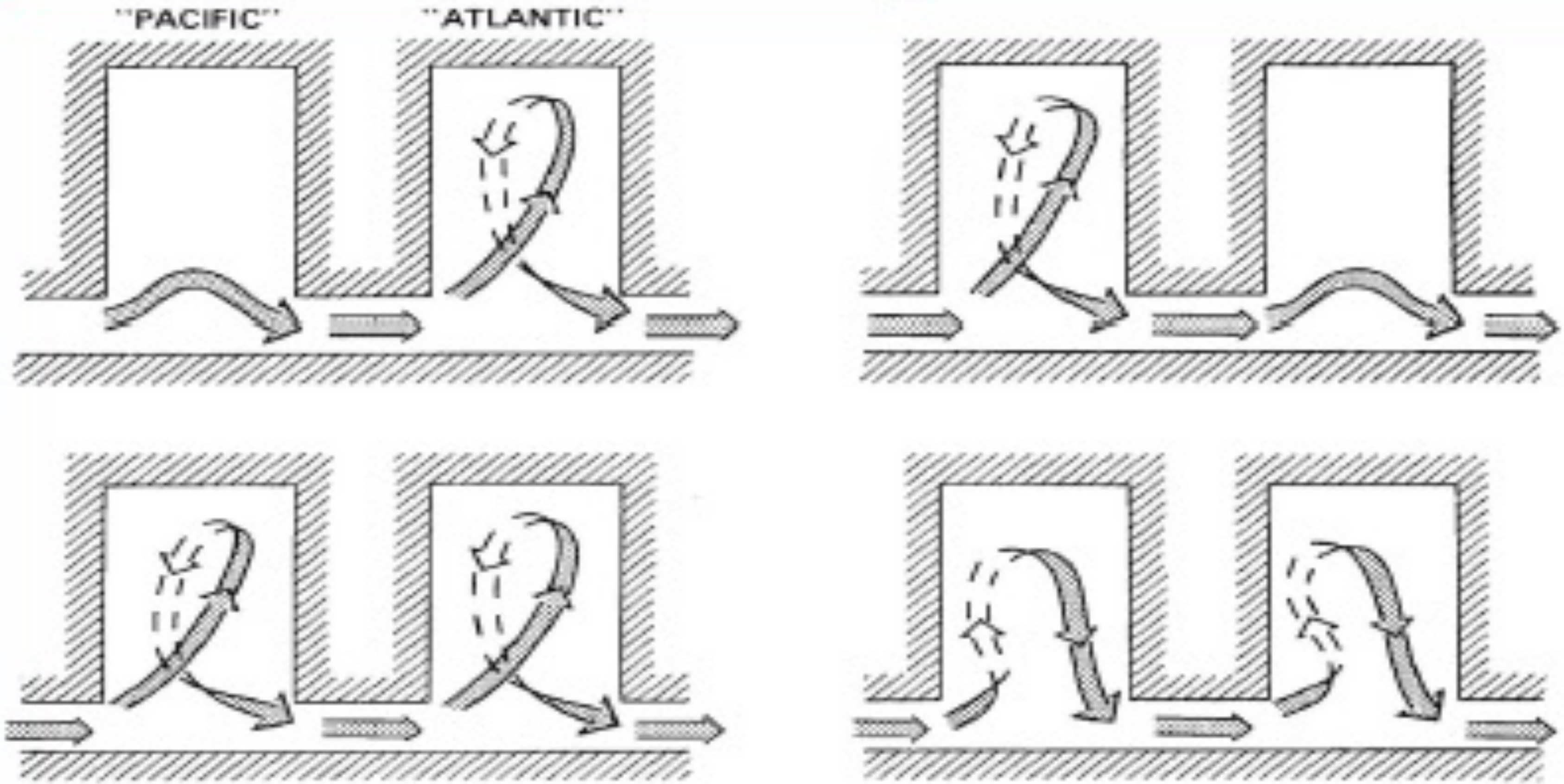
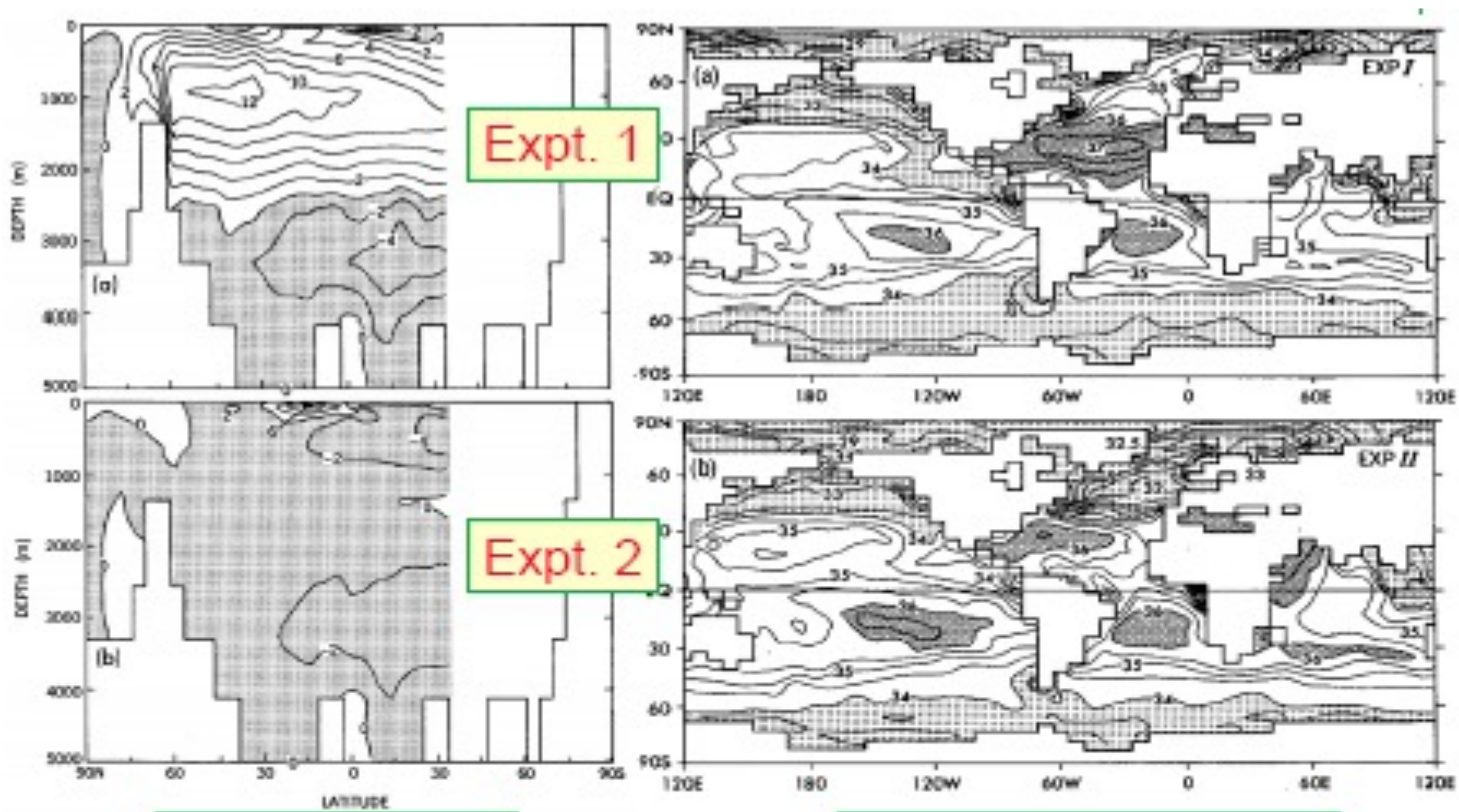


FIG. 2. A schematic of the four distinct climatic states obtained by Marotzke and Willebrand (1991) in a numerical ocean model.

Coupled model. Manabe & Stouffer (1988)

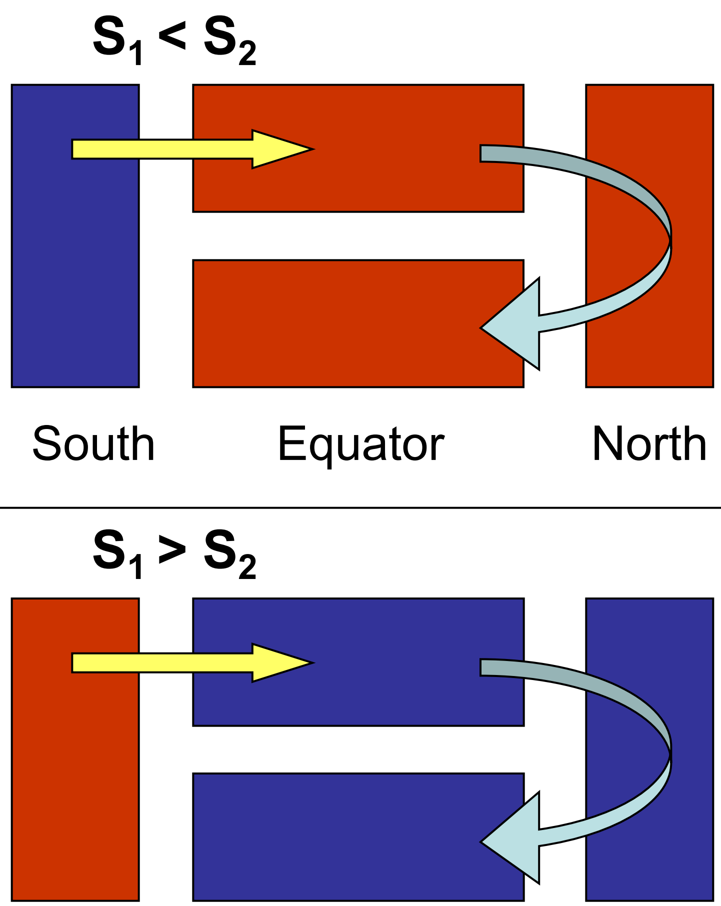
Overtunrning **Sea surface salinity**



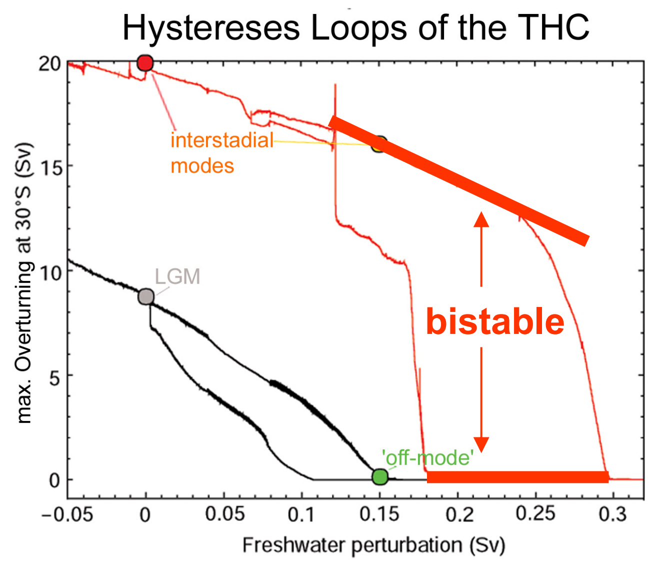
Conceptual model of the Atlantic THC

Glacial

mono-stable



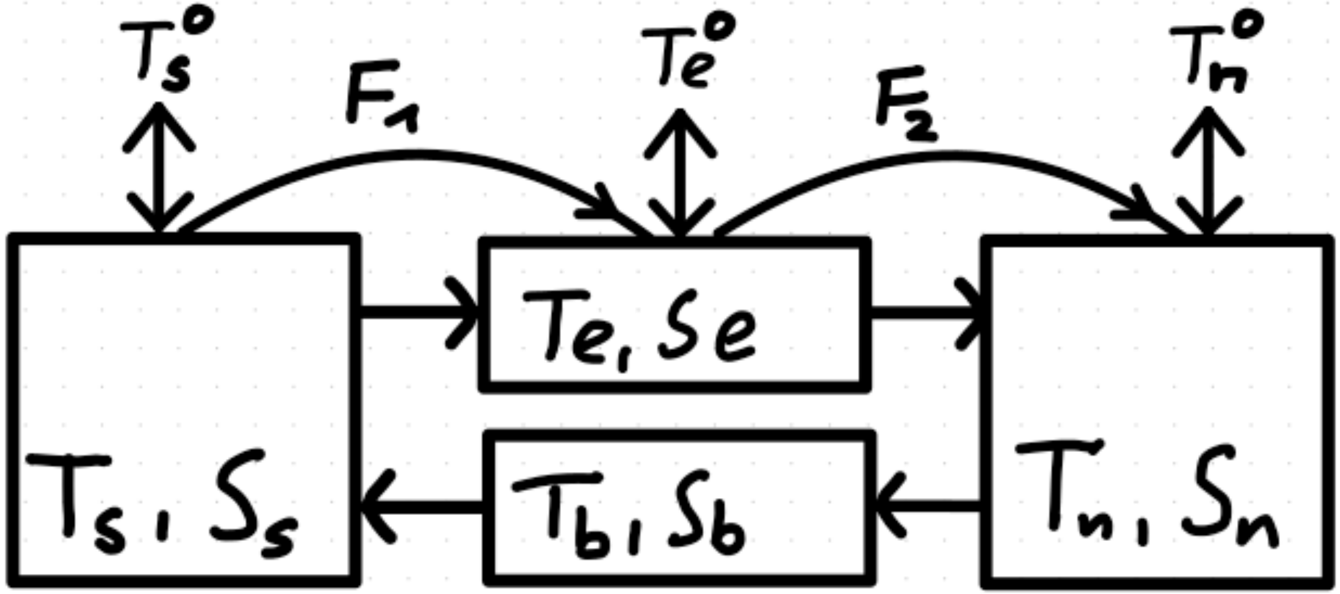
bistable



Peeters et al. 2004

→ Importance of the "warm water route"

→ Bistability: Potential for the YD event



T_s^o
F_1
T_e^o
F_2
T_n^o
T_e, S_e
T_s, S_s
T_b, S_b
T_n, S_n

- https://www.awi.de/fileadmin/user_upload/AWI/Forschung/Klimawissenschaft/Dynamik_des_Palaeoklimas/EnergyBalanceModels/index.html

EBM with sea ice

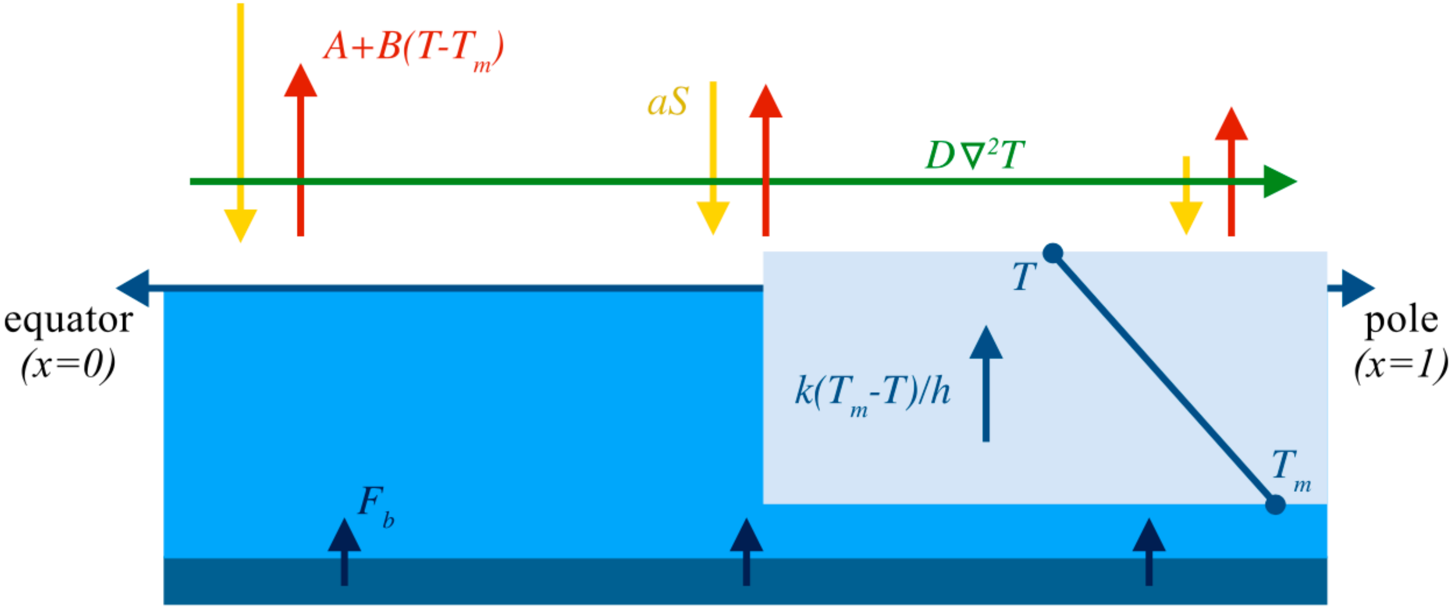


FIG. 1. Schematic of the global model of climate and sea ice described in section 2, showing the fluxes included in the model: insolation (yellow), OLR (red), horizontal heat transport (green), ocean heating (dark blue), and vertical heat flux through the ice (blue). The temperature of the ice is given by T at the surface and T_m at the base.

How Climate Model Complexity Influences Sea Ice Stability
TILL J. W. WAGNER AND IAN EISENMAN
JOURNAL OF CLIMATE
VOLUME 28

The equations

$$\frac{\partial E}{\partial t} = \underbrace{aS}_{\text{solar}} - \underbrace{L}_{\text{OLR}} + \underbrace{D\nabla^2 T}_{\text{transport}} + \underbrace{F_b}_{\substack{\text{ocean}\\ \text{heating}}} + \underbrace{F}_{\text{forcing}}$$

$$E \equiv \begin{cases} -L_f h, & E < 0 \quad \text{(sea ice)}, \\ c_w(T - T_m), & E \geq 0 \quad \text{(open water)}, \end{cases}$$

$$S(t,x) = S_0 - S_1 x \cos\omega t - S_2 x^2$$

$$a(x,E) = \begin{cases} a_0 - a_2 x^2, & E > 0 \quad \text{(open water)}, \\ a_i, & E < 0 \quad \text{(ice)}, \end{cases}$$

$$D\nabla^2 T = D \frac{\partial}{\partial x}\left[(1 - x^2)\frac{\partial T}{\partial x}\right]$$

$$L = A + B(T - T_m)$$

EBM with sea ice

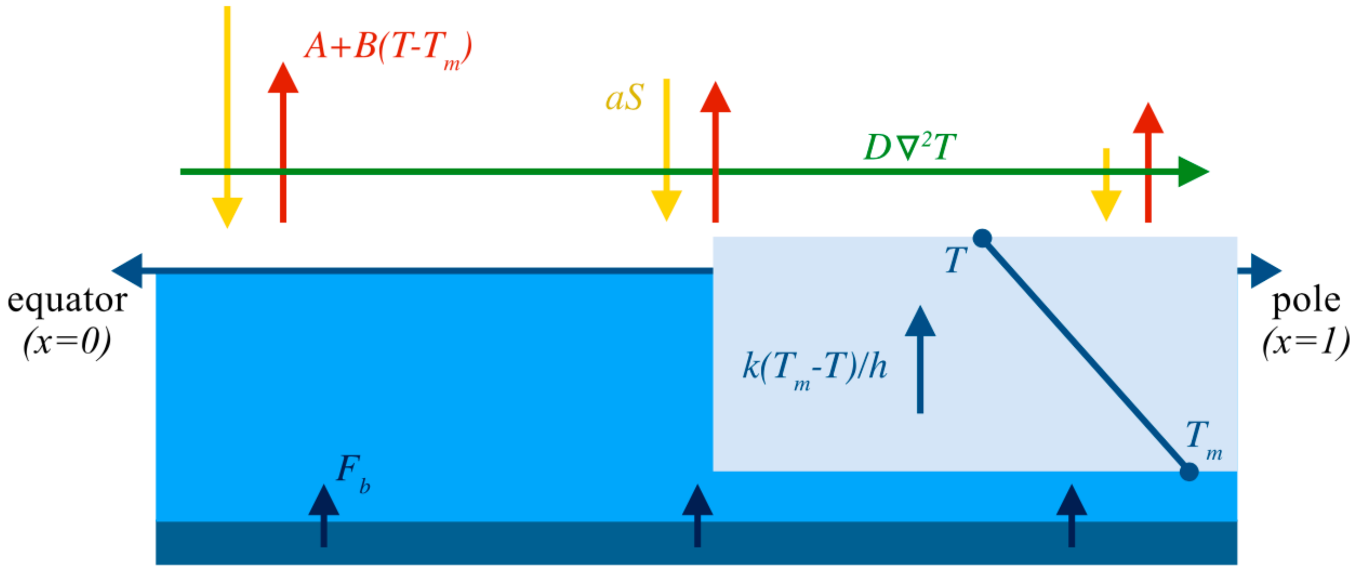


FIG. 1. Schematic of the global model of climate and sea ice described in section 2, showing the fluxes included in the model: insolation (yellow), OLR (red), horizontal heat transport (green), ocean heating (dark blue), and vertical heat flux through the ice (blue). The temperature of the ice is given by T at the surface and T_m at the base.

$$k(T_m - T_0)/h = -aS + A + B(T_0 - T_m) - D\nabla^2 T - F$$

$$T = \begin{cases} T_m + E/c_w, & E > 0 \qquad\qquad\qquad \text{(open water)}, \\ T_m, & E < 0, \quad T_0 > T_m \quad \text{(melting ice)}, \\ T_0, & E < 0, \quad T_0 < T_m \quad \text{(freezing ice)}. \end{cases}$$

output

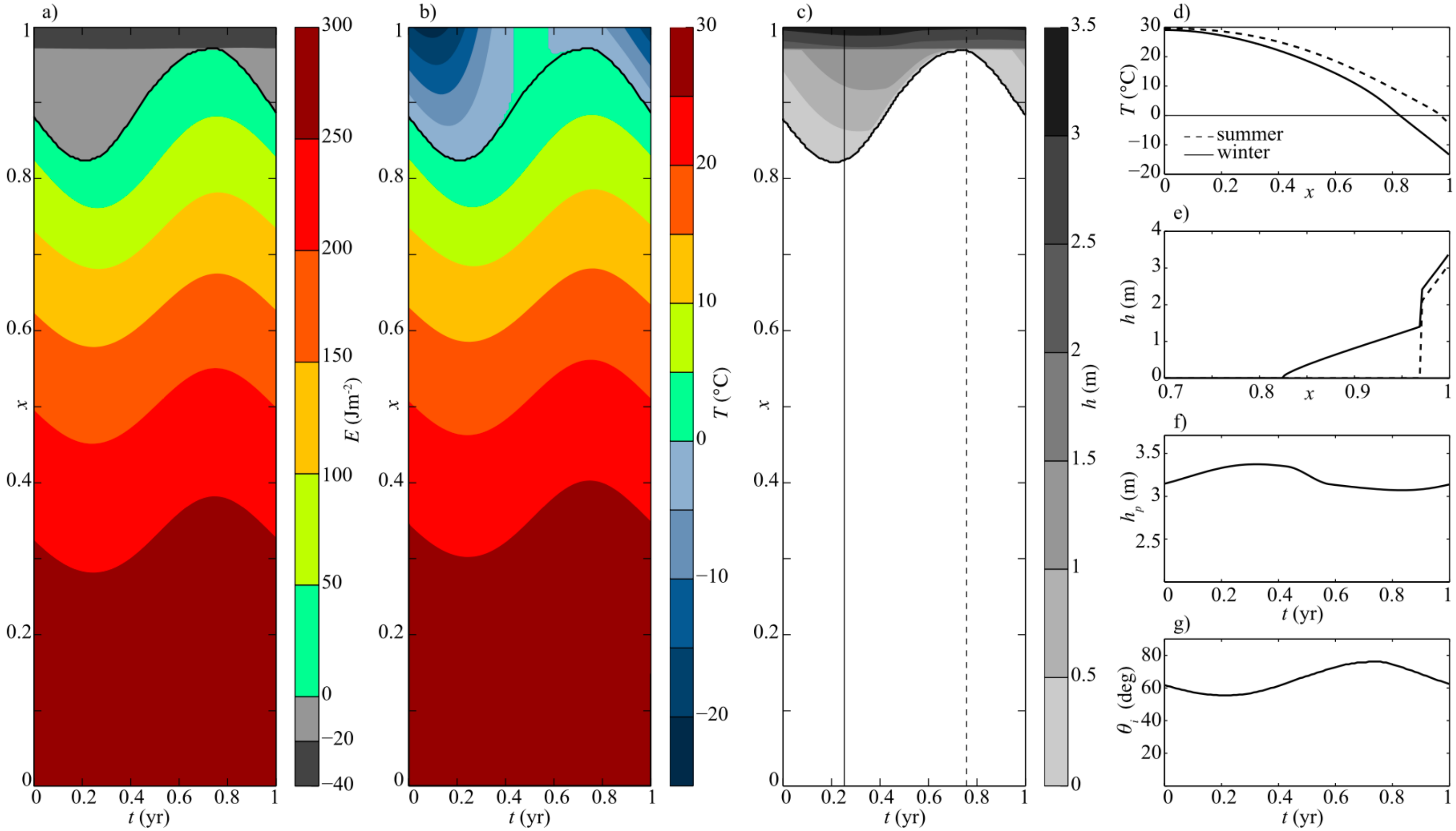


FIG. 2. Simulated climate in the default parameter regime. Contour plot of the seasonal cycle of (a) surface enthalpy $E(x, t)$, (b) surface temperature $T(x, t)$, and (c) sea ice thickness $h(x, t)$. The black curve in (a)–(c) indicates the ice edge. (d) Surface temperature T in summer and winter, corresponding to dashed and solid vertical lines in (c). (e) Ice thickness h in summer and winter where $x > 0.7$. (f) Seasonal cycle of ice thickness at the pole h_p. (g) Seasonal cycle of the latitude of the sea ice edge θ_i.

Exercise 1: EBM

D∗=Wm−2K−1 Diffusivity for heat transport: 0.6

A=Wm−2 OLR:193

B=Wm−2K−1 OLR temperature dependence: 2.1

cw=Wyrm−2K−1 Ocean mixed layer heat capacity: 9.8

S0=Wm−2 Insolation at equator: 420

S2=Wm−2 Insolation spatial dependence: 240

A0 Ice-free coalbedo at equator: 0.7

A2 Ice-free coalbedo spatial dependence: 0.1

Ai Coalbedo when there is sea ice: 0.4

Wm−2 Radiative forcing: 0

γ Gamma: 1

1) What will happen if the CO_2 content in the atmosphere is doubled? Radiative forcing= 4 W/m^2
2) What will happen if the factor γ is 1% higher/lower in the long-wave radiation?
3) Describe the effect if the diffusivity is enhanced by a factor of 2!
4) The coalbedo of sea ice can vary between 0.3 and 0.4. Describe the effect when varying the value!

https://www.awi.de/fileadmin/user_upload/AWI/Forschung/Klimawissenschaft/Dynamik_des_Palaeoklimas/EnergyBalanceModels/index.html

https://paleodyn.uni-bremen.de/study/MES/ebm/

Exercise 2: **Aquaplanet EBM with seasonal cycle**

a) What will happen if the CO_2 content in the atmosphere is doubled? Radiative forcing= 4 W/m^2= lowering of A

b) Describe the effect if the diffusivity is enhanced by a factor of 2!

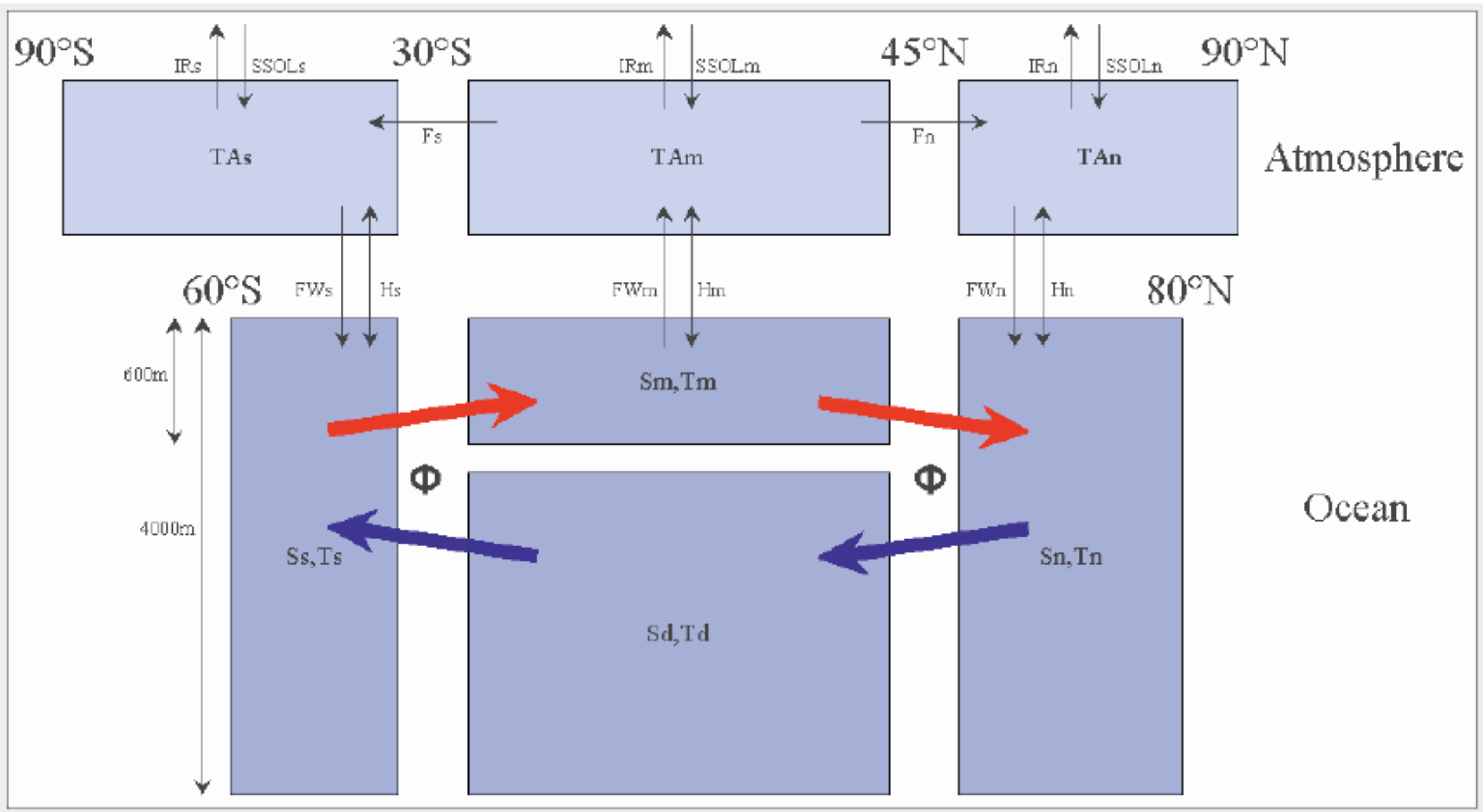
c) The coalbedo of sea ice can vary. Describe the effect when enhancing the value by 0.1!

Not:Reduce and enhance the sea ice thermal conductivity by 20% mimiking more /less snow on top of sea ice !

Discuss the sea ice evolution during the year !

https://paleodyn.uni-bremen.de/study/MES/ebm/

Application: Climate-Box-Model



$$\frac{d}{dt}T_N = -\left(T_N - T_M\right)\frac{\Phi}{V_N} + \frac{H_N}{\rho_0 c_p dz_2}, \tag{4.1}$$

$$\frac{d}{dt}T_M = -\left(T_M - T_S\right)\frac{\Phi}{V_M} + \frac{H_M}{\rho_0 c_p dz_1}, \tag{4.2}$$

$$\frac{d}{dt}T_S = -\left(T_S - T_D\right)\frac{\Phi}{V_S} + \frac{H_S}{\rho_0 c_p dz_2} \quad \text{and} \tag{4.3}$$

$$\frac{d}{dt}T_D = -\left(T_D - T_N\right)\frac{\Phi}{V_D} \tag{4.4}$$

where ρ_0 denotes a reference density for saltwater and c_p the specific heat capacity of water. The factors dz_i and V_i indicate the depths and volumes of the discrete ocean boxes, respectively.

$$\frac{d}{dt}S_N = -\left(S_N - S_M\right)\frac{\Phi}{V_N} - S_{ref}\frac{(P-E)_N}{dz_N}, \tag{4.7}$$

$$\frac{d}{dt}S_M = -\left(S_M - S_S\right)\frac{\Phi}{V_M} + S_{ref}\frac{(P-E)_M}{dz_M}, \tag{4.8}$$

$$\frac{d}{dt}S_S = -\left(S_S - S_D\right)\frac{\Phi}{V_S} - S_{ref}\frac{(P-E)_S}{dz_S}, \tag{4.9}$$

$$\frac{d}{dt}S_D = -\left(S_D - S_N\right)\frac{\Phi}{V_D}. \tag{4.10}$$

The reference salinity S_{ref} is a characteristic average value for the entire Atlantic Ocean, and the freshwater fluxes are denoted as precipitation minus evaporation (P-E). These freshwater fluxes are calculated by the divergence of the latent heat transport in the atmosphere and are assumed to be proportional to the meridional moisture gradient explained below.

T_{A_i} follow the prognostic equations

$$c_2 \frac{d}{dt} T_{A_N} = \frac{\partial \left(F_{s_N} + F_{l_N} \right)}{\partial y} + R_N - H_N, \tag{4.11}$$
$$c_2 \frac{d}{dt} T_{A_M} = \frac{\partial \left(F_{s_S} + F_{l_S} \right)}{\partial y} + R_M - H_M, \tag{4.12}$$
$$c_2 \frac{d}{dt} T_{A_S} = \frac{\partial \left(F_{s_S} + F_{l_S} \right)}{\partial y} + R_S - H_S. \tag{4.13}$$

c_2 is related to the specific heat of air. The sensible F_{s_i} and latent F_{l_i} heat transport are described in dependence of the meridional gradient of the surface temperature and moisture q

$$F_s = K_s \frac{\partial T_A}{\partial y} \tag{4.14}$$
$$F_l = K_l \left(\frac{\partial q}{\partial y} \right). \tag{4.15}$$

The surface heat fluxes follow the equations from Haney (1971):

sensible and latent heat transports. The radiation terms $\boldsymbol{R_i}$ in (4.11) to (4.13) consist of an incoming solar shortwave $\boldsymbol{S_i}$ and an outgoing infrared longwave $\boldsymbol{I_i}$ part. The extraterrestrial solar radiation is not absorbed entirely, and a latitude-dependent average albedo $\boldsymbol{\alpha_i}$ is introduced to account for the reflectance. The outgoing infrared radiation $\boldsymbol{I_i}$ is calculated through a linear formula of Budyko (1969). Thus, the equation for the net radiation balance is

$$\boldsymbol{R_i = S_i - I_i = S_{sol,i}\,(1 - \alpha_i) - (A + BT_{A_i})\,.} \tag{4.16}$$

The model calculates the freshwater fluxes from the divergence of the latent heat transport ($\boldsymbol{P - E \sim \partial F_l / \partial y}$). The integration of the system is implemented with an Euler-forward scheme.

Exercise 5 – Investigations with the box-model

1. In the regions of deep water formation in the North Atlantic, relatively small amounts of fresh water added to the surface can stabilize the water column to the extent that convection can be prevented from occurring. Such interruption decreases the poleward mass transport Φ in the ocean. Furthermore, perturbations of the meridional transport in the ocean can be amplified by positive feedbacks: a weaker northward salt transport brings less dense water to high latitudes, which further reduces the meridional transport. Discuss the case where the initial conditions in salinity at latitudes is changed.

2. Which feedbacks are acting for global warming? Please evaluate the hydrological cycle and atmospheric heat transports!

3. The coupled model shall be used to investigate the sensitivity of the system with respect

to radiative forcing and stochastic weather perturbations. Additional radiative forcing may come from increased tracer gas concentrations in the atmosphere, whereas the atmospheric weather fluctuations may reflect unresolved effects of the atmospheric transports modeled as white noise.

4. The initial values of the model represent averages for present-day climate conditions. Determine which parameters in the model affect the climate conditions most.

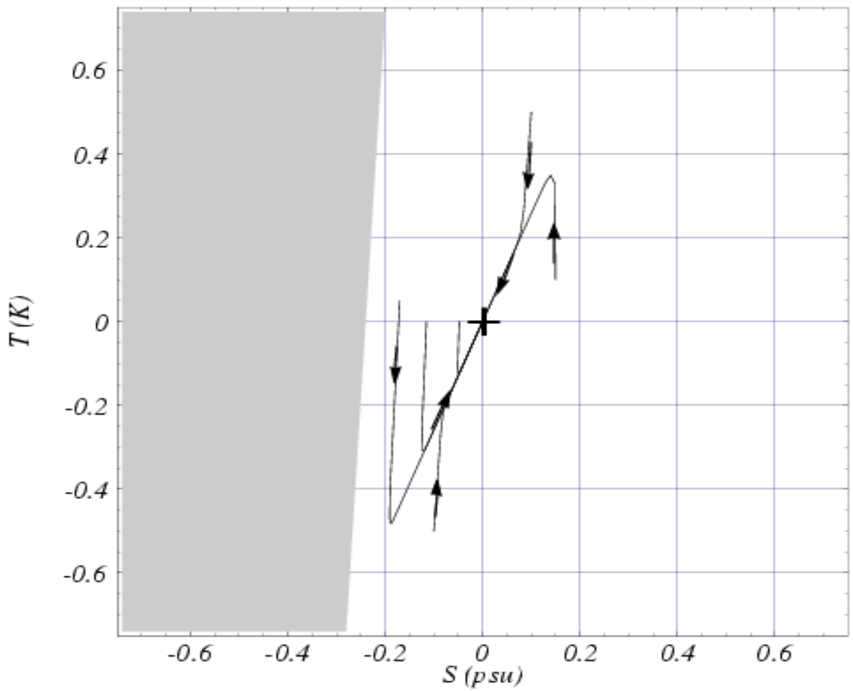
Matlab Version

Fortran 90 Version

```
gfortran-mp-4.8 box_model_oop.f90
a.out

gnuplot
plot './box_model_oop.out' using 1:2 w l
set pointsize 0.1
```

```
% Euler forward for ocean temperature

    Tnln = Tnl + dts * ((Hfnl)/(rcz2)-(Tnl-Tml)*phi/Vnl);

    Tmln = Tml + dts * ((Hfml)/(rcz1)-(Tml-Tsl)*phi/Vml);

    Tsln = Tsl + dts * ((Hfsl)/(rcz2)-(Tsl-Td)*phi/Vsl);

    Tdn = Td + dts * (-(Td-Tnl)*(phi/Vd));
```

Phase space dynamics



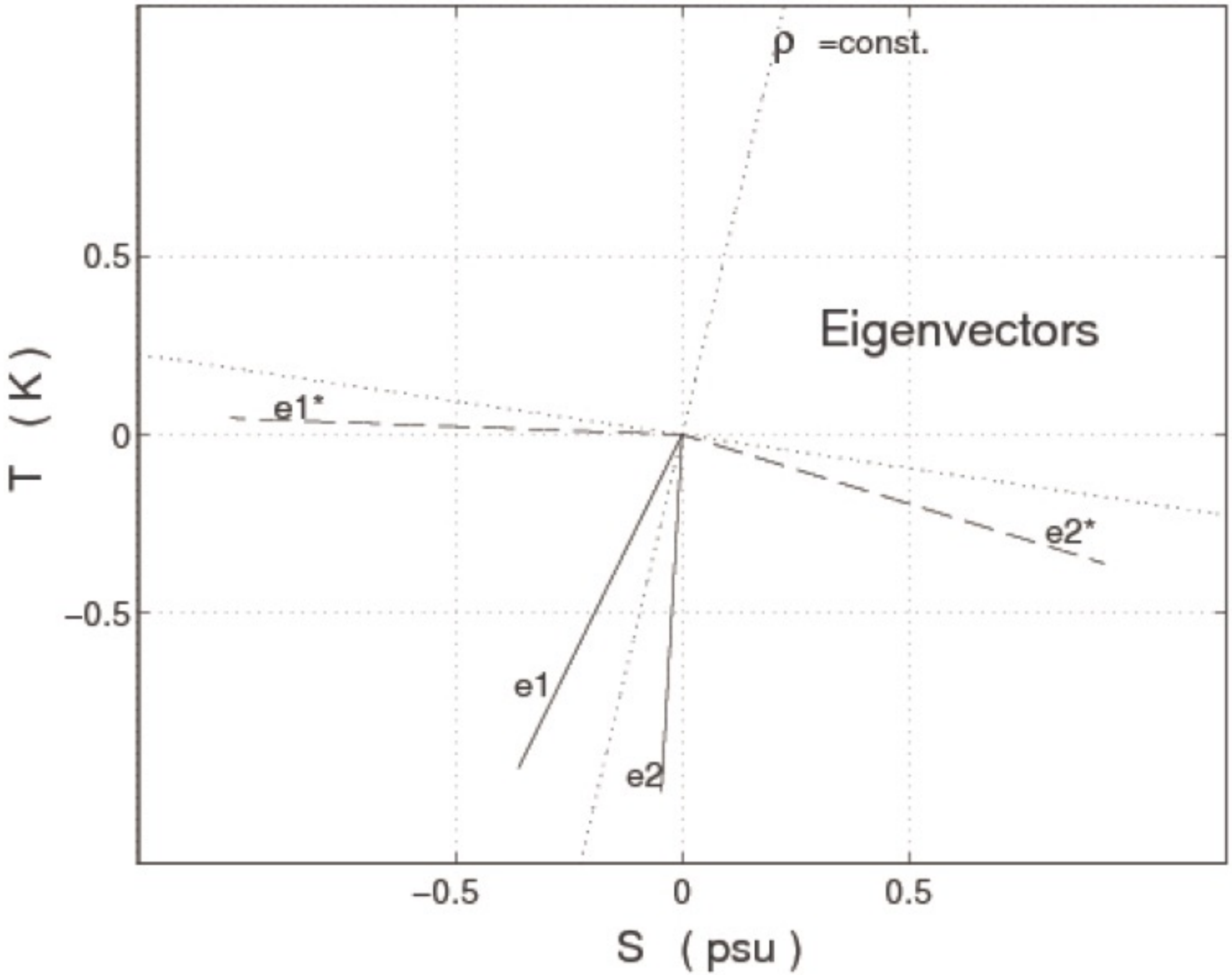


Figure 10: Eigenvectors e_1, e_2, and adjoint eigenvectors e_1^*, e_2^* of the tangent linear operator $A^\dagger$. The dotted lines show the line of constant density and the perpendicular.

T and S

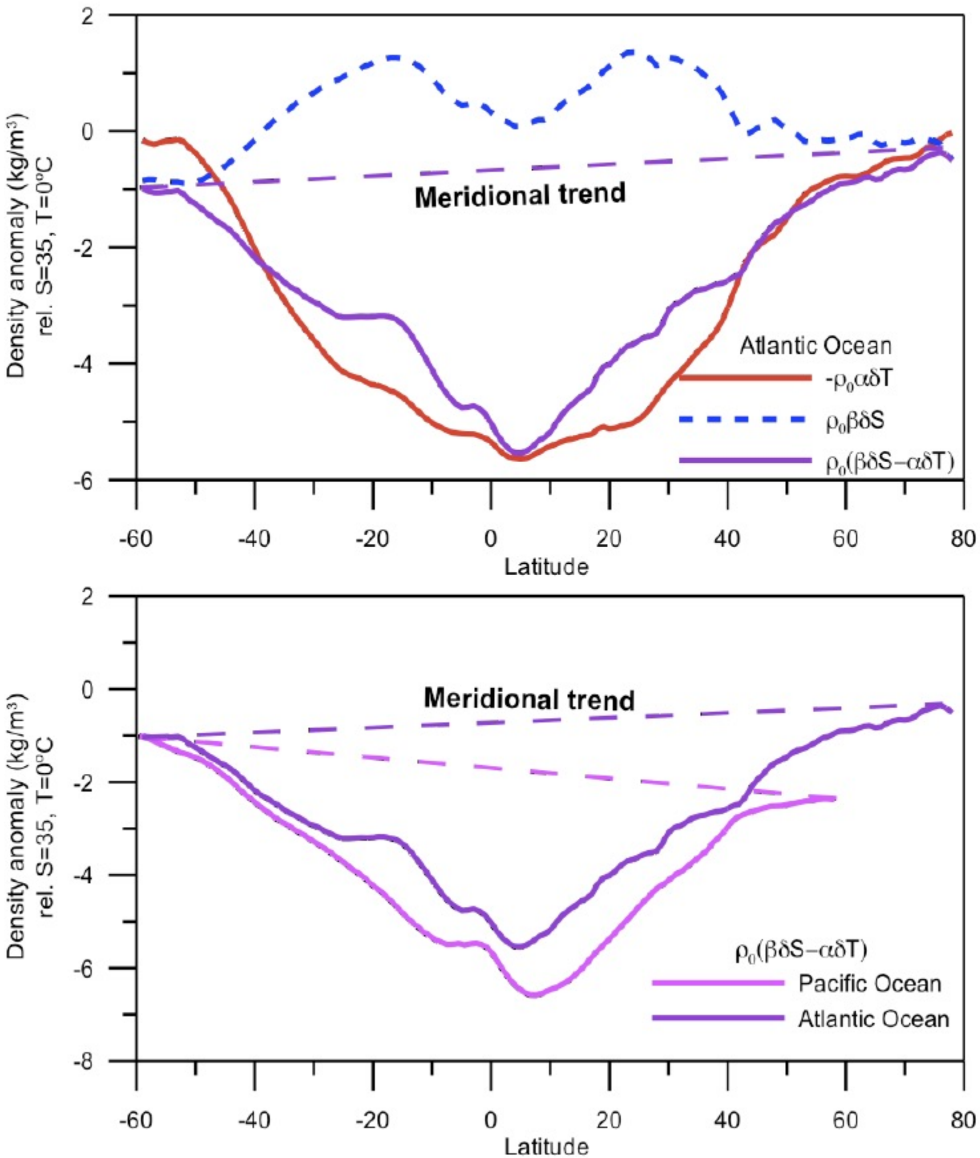
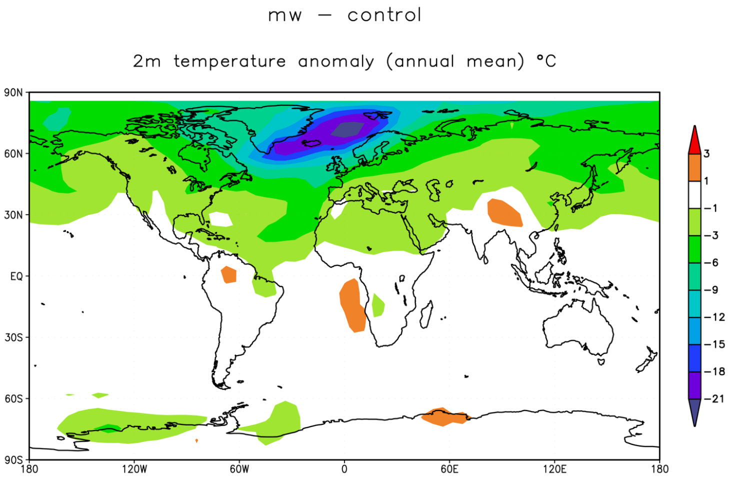


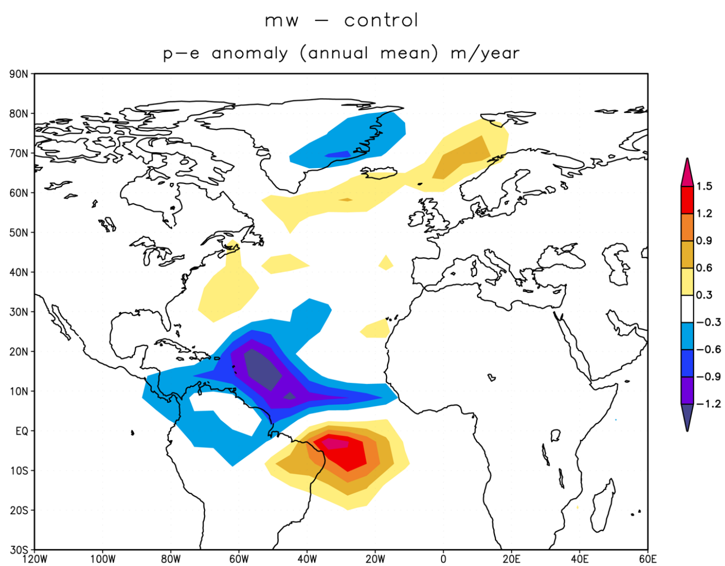
Figure 2.16: a) The Atlantic surface density is mainly related to temperature differences. b) But the pole-to-pole differences are caused by salinity differences.

Model Results



2 meter temperature

interhemispheric Seesaw



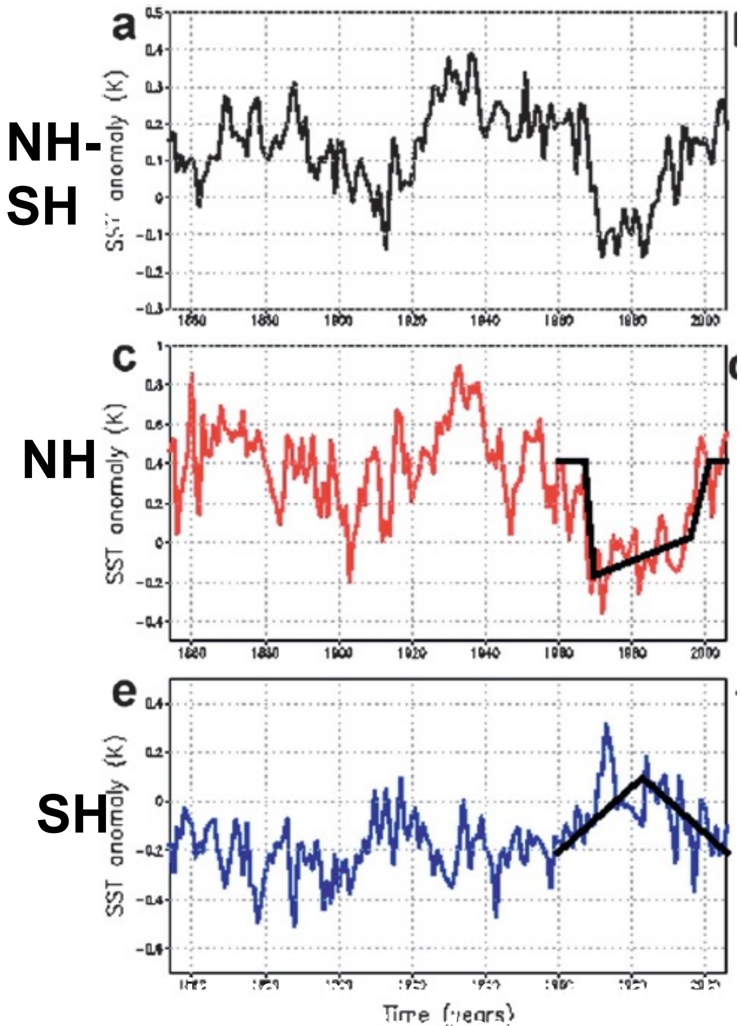
Southward Shift of the ITCZ

→ **high Noise level**

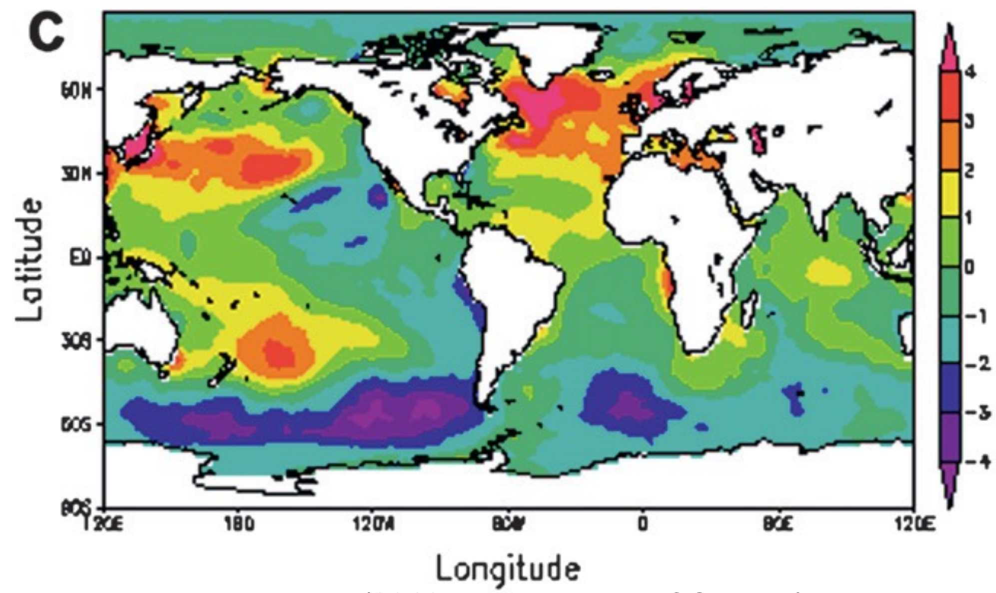
→ **Projection onto several Phenomena**

Tellus 2003

Present dynamics: Atlantic Multidecadal Mode & seesaw



Linear SST trends (°C/ century) for 1980–2006

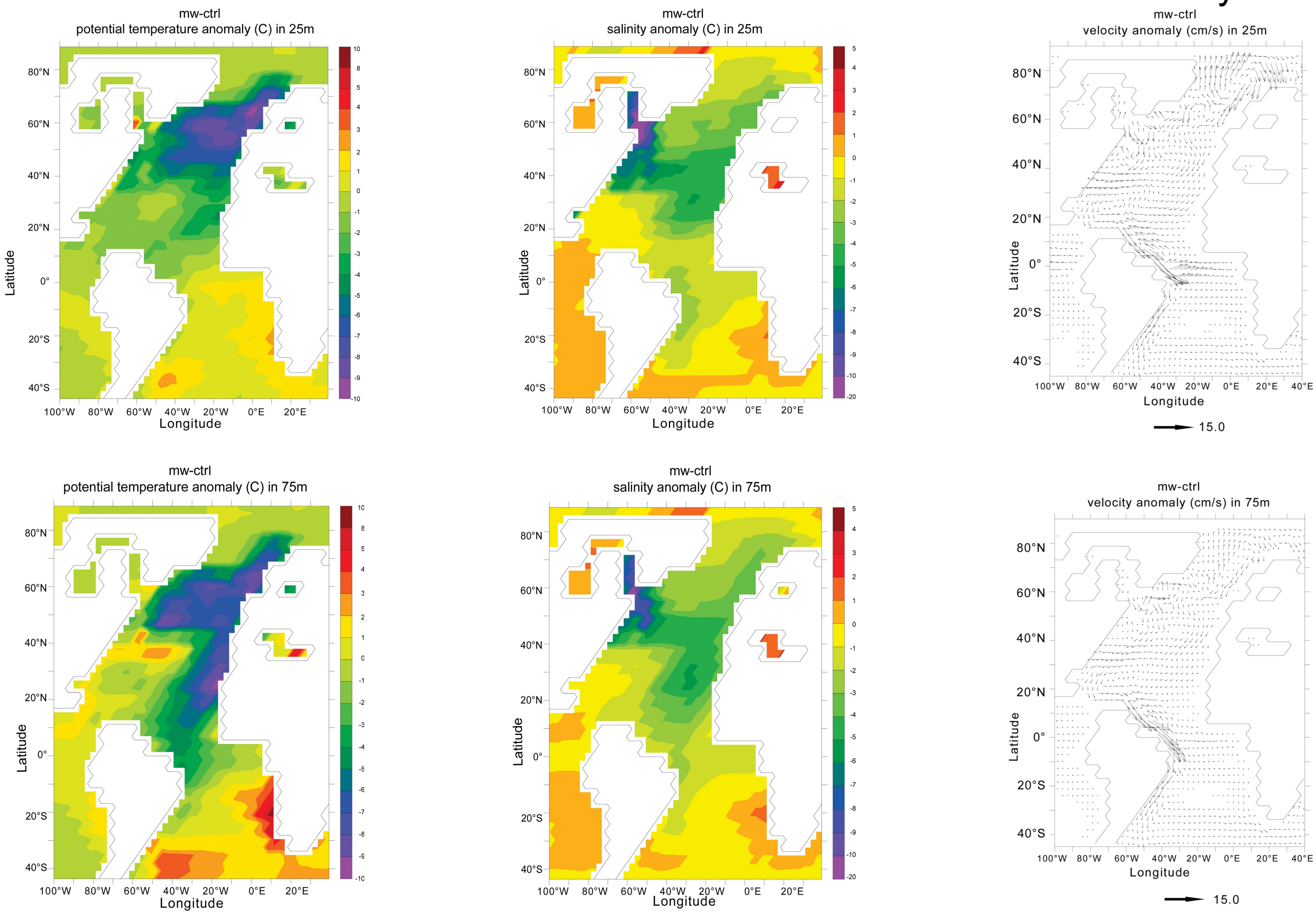


Dima and Lohmann (2010, Jan., Journal of Climate).

Also: „An abrupt drop in Northern Hemisphere sea surface temperature around 1970“ Thompson et al. 2010

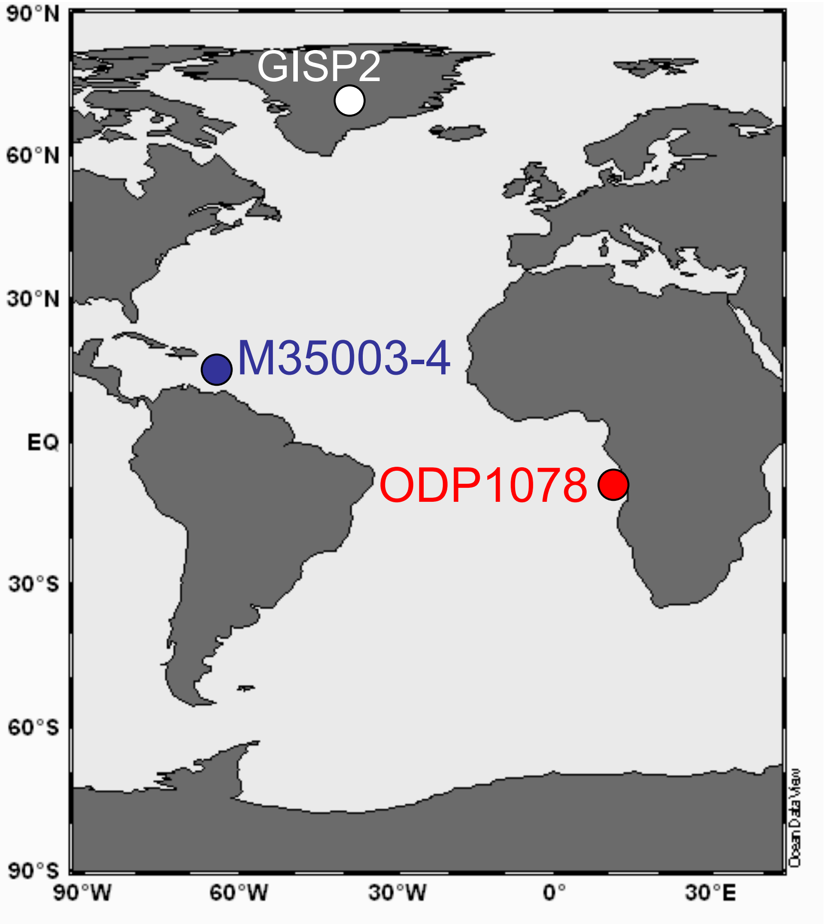
Spatial signature of a simulated meltwater event

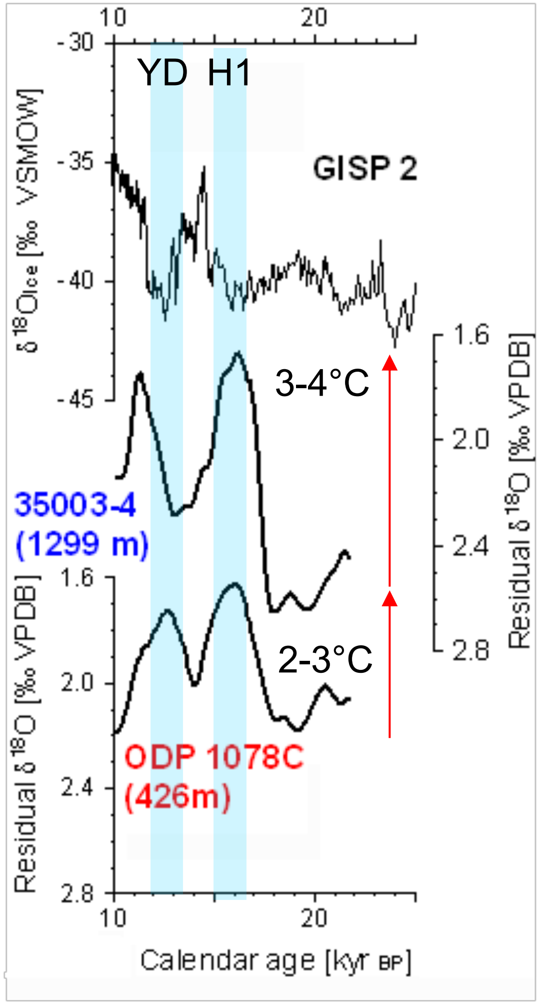
Temperature | Salinity | Velocity



Lohmann, 2003

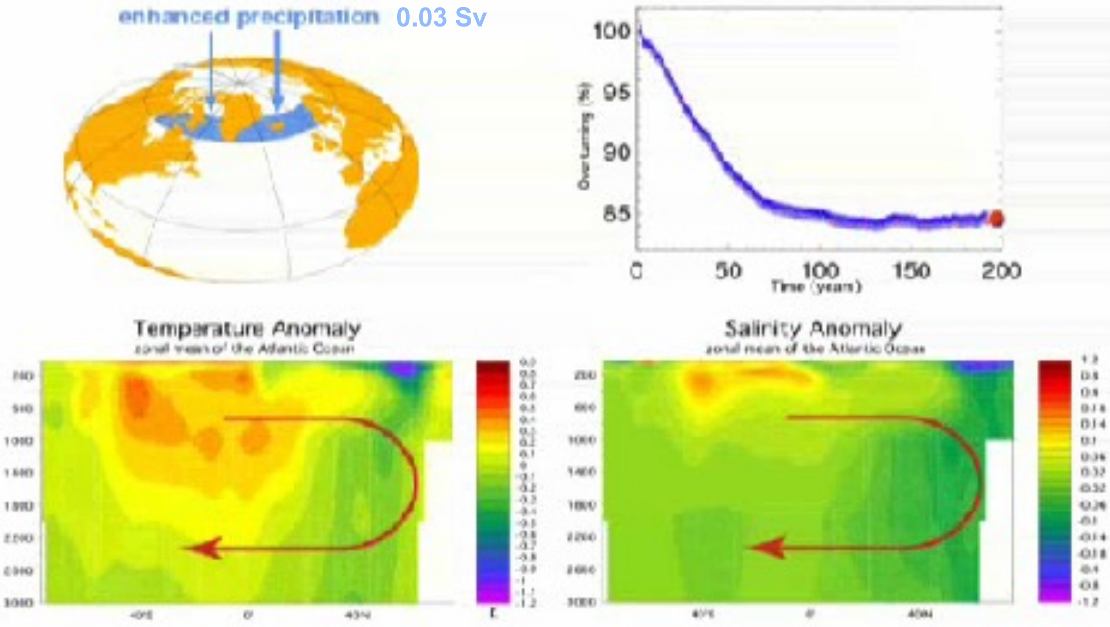
Lessons from Paleoclimate Data

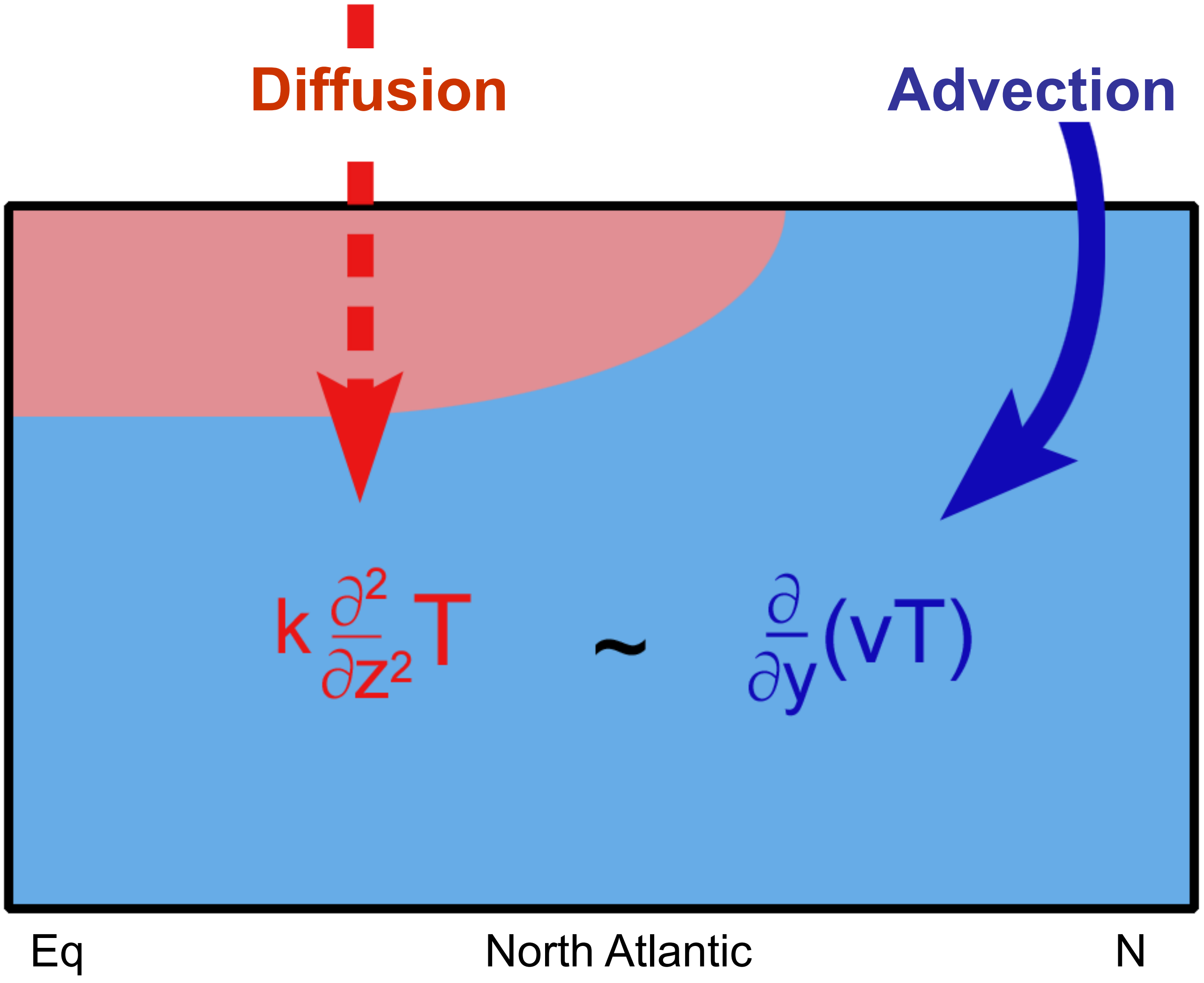




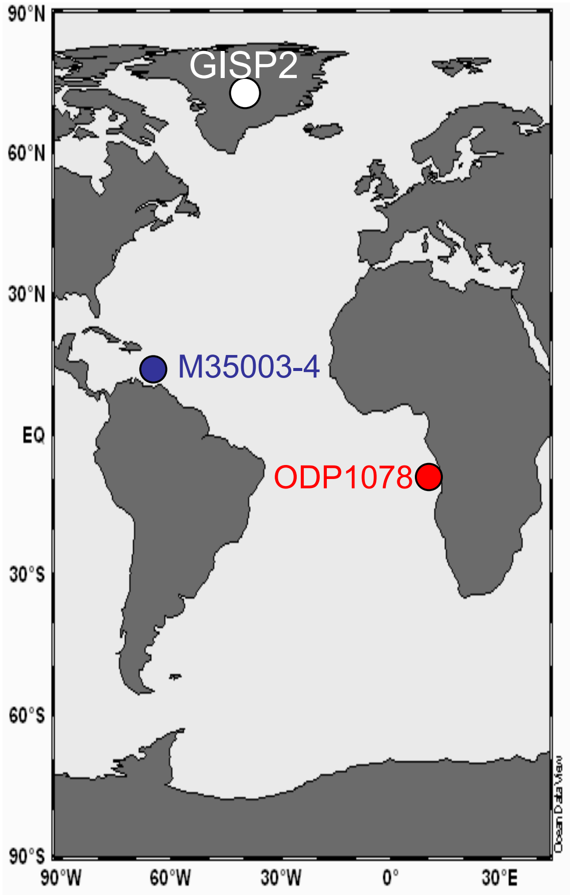
GISP2: Grootes et al. 2000
M35003-4: Hüls and Zahn 2000
Science (under rev.)

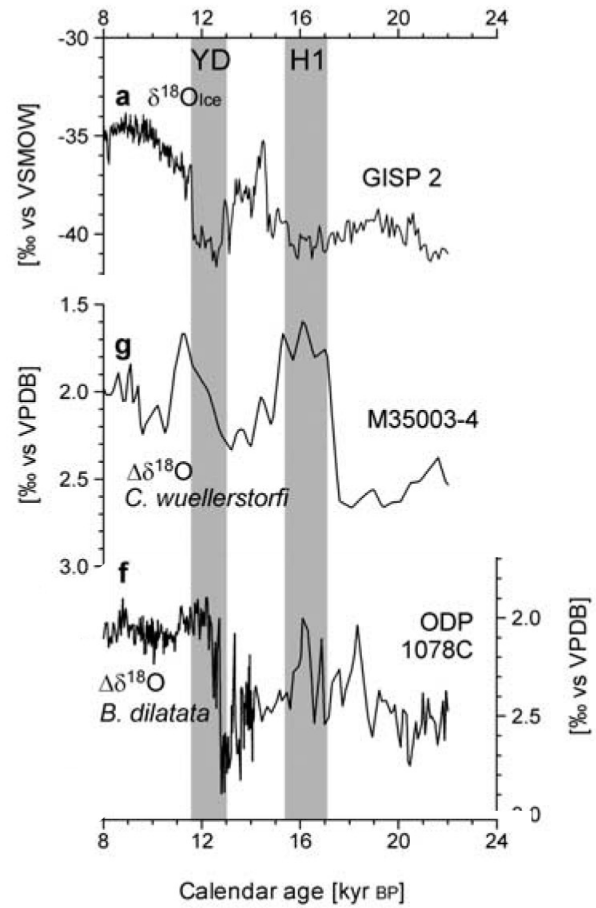
Hydrographic changes due to reduced overturning




Diffusion
Advection
$k \frac{\partial^2}{\partial z^2} T \sim \frac{\partial}{\partial y}(vT)$
Eq
North Atlantic
N

Ein Blick auf Paläoklimadaten





GISP2

M35003-4 (1299 m)

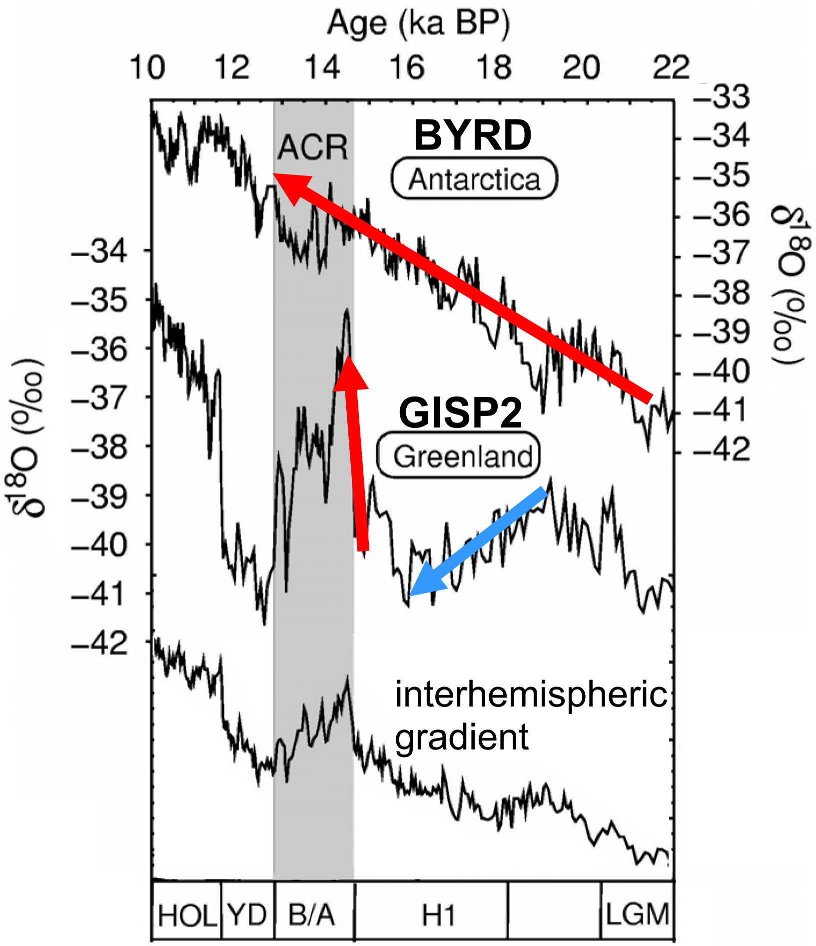
ODP 1078C (426 m)

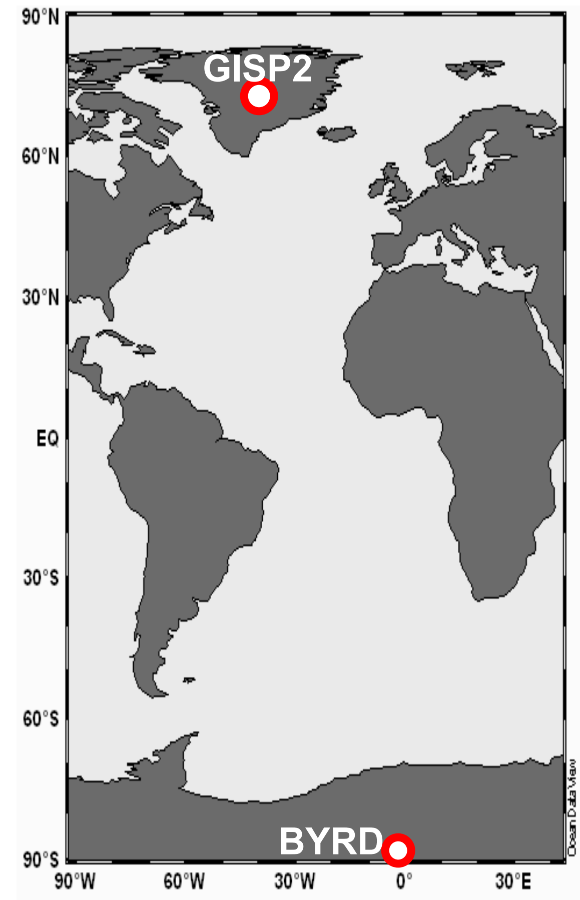
← time

Rühlemann et al. 2004

Motivation: Geological Data

Transition Glacial-Interglacial





← Time

Blunier und Brook 2001

Ocean Dynamics

- **Dynamics for the atmosphere-ocean system**
- **Theory, numerical models**
- **Concepts of flow, energetics, vorticity, wave motion**
- **Oceanic wind driven and thermohaline circulation**

Schedule:

Theory; Programing in R (please bring your laptops), Stability

Schedule: 2nd part

Theory; Programing in Matlab or Fortran (please bring your laptops),

3rd part

1.3 Non-dimensional parameters: The Reynolds number

For the case of an incompressible flow, assuming the temperature effects are negligible, they consist of conservation of mass

$$\nabla \cdot \mathrm{u} = 0 \tag{1.13}$$

and conservation of momentum

$$\partial_t \mathrm{u} + (\mathrm{u} \cdot \nabla) \cdot \mathrm{u} = -\nabla p + \frac{1}{Re} \nabla^2 \mathrm{u} \tag{1.14}$$

where u is the velocity vector and p is the pressure. The equations have been made dimensionless by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. The dimensionless parameter $Re = UL/\nu$ is the Reynolds number, ν denotes the kinematic viscosity. For large Reynolds numbers, the flow is turbulent. In most practical flows Re is rather large $(10^4 - 10^8)$, large enough for the flow to be turbulent. A large Reynolds number allows

Dynamics of the Ocean System

Momentum equation

$$\overbrace{\rho\left(\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{\substack{\text{Unsteady}\\\text{acceleration}}} + \underbrace{\mathbf{v}\cdot\nabla\mathbf{v}}_{\substack{\text{Advective}\\\text{acceleration}}}\right)}^{\text{Inertia (per volume)}} = \overbrace{\underbrace{-\nabla p}_{\substack{\text{Pressure}\\\text{gradient}}} + \underbrace{\mu\nabla^2\mathbf{v}}_{\text{Viscosity}}}^{\text{Divergence of stress}} + \underbrace{\mathbf{f}}_{\substack{\text{Other}\\\text{body}\\\text{forces}}} .$$

$$\left(\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{10^{-4}} + \underbrace{\mathbf{v}\cdot\nabla\mathbf{v}}_{10^{-4}}\right) = \underbrace{-\frac{1}{\rho}\nabla p}_{10^{-3}} + \underbrace{\nu\nabla^2\mathbf{v}}_{10^{-12}} + \underbrace{2\boldsymbol{\Omega}\times\mathbf{v}}_{10^{-3}} .$$

Geostrophic balance: Pressure gradient Coriolis force

Scaling: Ocean circulation

The Coriolis effect is one of the dominating forces for the large-scale dynamics of the oceans and the atmosphere. In meteorology and ocean science, it is convenient to use a rotating frame of reference where the Earth is stationary. Starting from (3.7), we can estimate the relative contributions in the horizontal momentum equations:

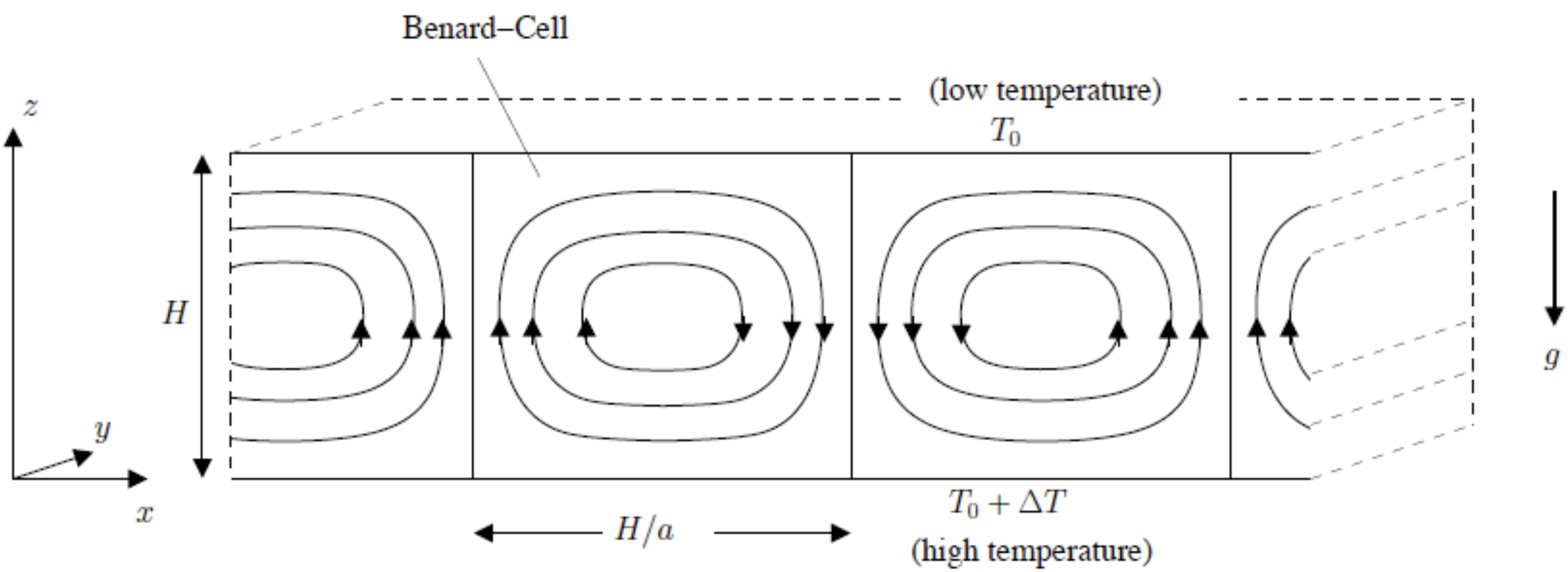
$$\Big(\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{} + \underbrace{\mathbf{v}\cdot\nabla\mathbf{v}}_{10^{-4}}\Big) = \underbrace{-\frac{1}{\rho}\nabla p}_{} + \underbrace{\nu\nabla^2\mathbf{v}}_{10^{-12}} + \underbrace{2\Omega\times\mathbf{v}}_{10^{-3}}. \tag{3.7}$$

Rossby number Ro is defined as:

$$Ro = \frac{(U^2/L)}{(fU)} = \frac{U}{fL} \tag{3.8}$$

where U and L are, respectively, characteristic velocity and length scales of the phenomenon and $f = 2\Omega\sin\varphi$ is a typical Coriolis frequency. The procedure is analogous to the dimensionless Reynolds number (section 1.3).

Applicaton: Rayleigh-Bénard convection



$$T(x, y, z = H) = T_0$$

$$T(x, y, z = 0) = T_0 + \Delta T$$

Zero Solution

Such a system possesses a steady-state solution in which there is no motion, and the temperature varies linearly with depth:

$$\begin{aligned} \boldsymbol{v} &= 0 \\ T &= T_0 + \left(1 - \frac{z}{H}\right)\Delta T \end{aligned} \tag{2.3}$$

When this solution becomes unstable, convection should develop.

In the case where all motions are parallel to the $x-z$-plane, and no variations in the direction of the y-axis occur, the governing equations may be written (see Saltzman (1962)) as:

$$
\begin{aligned}
\partial_t u + u\partial_x u + w\partial_z u &= -\frac{1}{\rho_0}\partial_x p + \nu\nabla^2 u && (2.4)\\
\partial_t w + u\partial_x w + w\partial_z w &= -\frac{1}{\rho_0}\partial_z p + \nu\nabla^2 w + g(1-\alpha(T-T_0)) && (2.5)\\
\partial_t T + u\partial_x T + w\partial_z T &= \kappa\nabla^2 T && (2.6)\\
\partial_x u + \partial_z w &= 0 && (2.7)
\end{aligned}
$$

where w and u are the vertical and horizontal components of the velocity, respectively. Furthermore, $\nu = \eta/\rho_0$, $\kappa = \lambda/(\rho_0 C_v)$ the momentum diffusivity (kinematic viscosity) and thermal diffusivity, respectively.

It is useful to define the stream function Ψ for the two-dimensional motion, i.e.

$$
\begin{aligned}
\frac{\partial\Psi}{\partial x} &= w && (2.8)\\
\frac{\partial\Psi}{\partial z} &= -u \quad . && (2.9)
\end{aligned}
$$

Then, the dynamics can be formulated for Ψ and Θ, which is the departure of temperature from that occurring in the state of no convection[1] (2.3):

$$\partial_t \nabla^2 \Psi = -\frac{\partial(\Psi, \nabla^2 \Psi)}{\partial(x,z)} + \nu \nabla^4 \Psi - g\alpha \frac{\partial \Theta}{\partial x} \tag{2.10}$$

$$\partial_t \Theta = -\frac{\partial(\Psi, \Theta)}{\partial(x,z)} + \frac{\Delta T}{H}\frac{\partial \Psi}{\partial x} + \kappa \nabla^2 \Theta \quad . \tag{2.11}$$

The notation

$$\frac{\partial(\Psi, \nabla^2 \Psi)}{\partial(x,z)} \quad ,$$

known as the determinant of the Jacobian matrix (or simply “the Jacobian”), stands for

$$\frac{\partial \Psi}{\partial x}\frac{\partial \nabla^2 \Psi}{\partial z} - \frac{\partial \Psi}{\partial z}\frac{\partial \nabla^2 \Psi}{\partial x} \quad .$$

The constants $g,\ \alpha,\ \nu$, and κ denote, respectively, the acceleration of gravity, the coefficient of thermal expansion, the kinematic viscosity, and the thermal conductivity of the fluid. The problem is most tractable (analytically) when both the upper- and the lower-boundaries are taken to be free, in which case Ψ and $\nabla^2 \Psi$ vanish at both boundaries.

2 Approaches

$$\frac{a}{1+a^2}\kappa \quad \Psi = X\sqrt{2}\sin\left(\frac{\pi a}{H}x\right)\sin\left(\frac{\pi}{H}z\right) \tag{2.16}$$

$$\pi\frac{R_a}{R_c}\frac{1}{\Delta T} \quad \Theta = Y\sqrt{2}\cos\left(\frac{\pi a}{H}x\right)\sin\left(\frac{\pi}{H}z\right) - Z\sin\left(2\frac{\pi}{H}z\right) \tag{2.17}$$

$$R_a = \frac{g\alpha H^3 \Delta T}{\nu\kappa} \quad , \tag{2.14}$$

exceeds a critical value

$$R_c = \pi^4 a^{-2}(1+a^2)^3 \quad . \tag{2.15}$$

$$\dot{X} = -\sigma X + \sigma Y \tag{2.18}$$

$$\dot{Y} = rX - Y - XZ \tag{2.19}$$

$$\dot{Z} = -bZ + XY \tag{2.20}$$

Numerical: LBM

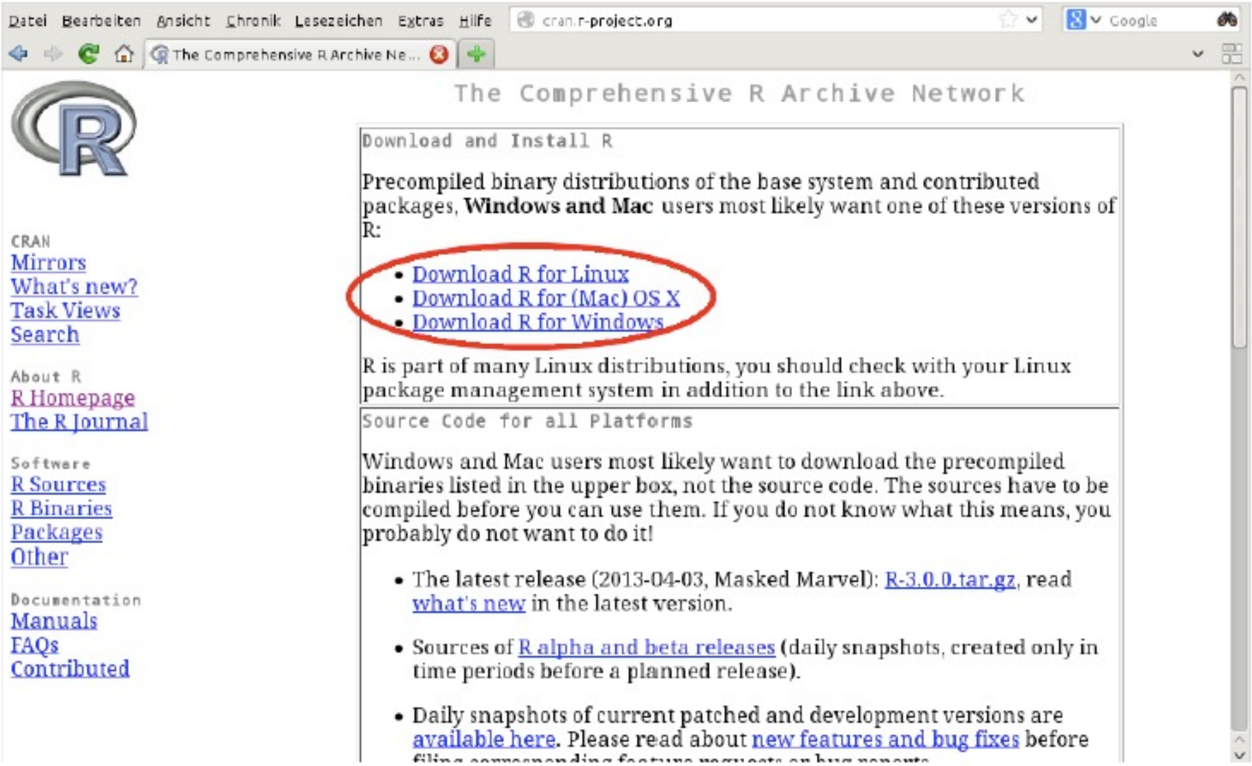


Figure 5: R is available for download from the CRAN webpage: `http://cran.r-project.org`

```
setwd("/Users/glohmann/Desktop/DynII_2013/Rayleigh-Benard_R")
source("rayleigh-benard.r")
```

From Statistical Mechanics to Fluid Dynamics

One of the most significant theoretical breakthroughs in statistical physics was due to Ludwig Boltzmann (1864) (see Boltzmann (1995) for a recent reprint of his famous lectures on kinetic theory), who pioneered non-equilibrium statistical mechanics. Boltzmann postulated that a gas was composed of a set of interacting particles, whose dynamics could be (at least in principle) modelled by classical dynamics. Due to the very large number of particles in such a system, a statistical approach was adopted, based on simplified physics composed of particle streaming in space and billiard-like inter-particle collisions (which are assumed elastic). Instrumental to the theory is the single-particle distribution function (hereafter SPDF), $f(\vec{x}, \vec{p}, t)$ which represents the probability density of having a particle at the point $(\vec{x}, \vec{p})$ in the phase space. Hence, the quantity

$$f(\vec{x}, \vec{p}, t)\, d\vec{x}\, d\vec{p} \tag{5.1}$$

represents the probability of finding a particle inside an infinitesimal space cubelet centered around $\vec{x}$, and inside an infinitesimal momentum-space cubelet around $\vec{p}$ at any given time t. In

$$f(\vec{x}+d\vec{x},\vec{p}+d\vec{p},t+dt)d\vec{x}d\vec{p} = f(\vec{x},\vec{p},t)d\vec{x}d\vec{p} + (\Gamma_+ - \Gamma_-)\, d\vec{x}d\vec{p}dt \tag{5.3}$$

Inserting Eq. (5.4) into Eq. (5.3) and cancelling terms, we easily obtain Boltzmann's Equation:

$$\boxed{\frac{\partial f}{\partial t} + \vec{u}\nabla_{\vec{x}} f + \vec{F}\nabla_{\vec{p}} f = \Gamma_+ - \Gamma_-} \tag{5.5}$$

Eq. (5.5). The collision operator, which is in itself a complex integro-differential expression, reads

$$\Gamma_+ - \Gamma_- \equiv \int d^3\vec{u}_1 \int d\Omega\, \sigma(\Omega)\, |\vec{u}-\vec{u}_1|\, \left[f(\vec{u}')f(\vec{u}_1') - f(\vec{v})f(\vec{v}_1)\right] \tag{5.6}$$

where σ is the differential cross-section in the case of the 2-particle collisions (which is a function of the solid angle Ω), unprimed velocities are incoming (before collision) and primed velocities are outgoing (after collision).

A fundamental property of the collision operator Cercignani (1987) is that it conserves mass, momentum and kinetic energy (hence also a linear combination thereof). Also, it can be shown that the local Maxwell-Boltzmann distribution pertains to a certain class of positive SPDFs for which the collision integral vanishes.[2] It can be shown that this equilibrium distribution is given by

$$f_0(\vec{x},\vec{v}) = \rho(\vec{x}) \left[\frac{m}{2\pi kT(\vec{x})}\right]^{3/2} \exp\{-m\left[\vec{v}-\vec{v}_0(\vec{x})\right]^2/2kT(\vec{x})\} \qquad (5.7)$$

This implies that, if this distribution is attained, we also have a state where incoming SPDFs exactly balance the outgoing ones, maintaining a local dynamic equilibrium. This observation is of paramount importance for our method, which uses the (discretized) Maxwell-Boltzmann distribution as the equilibrium distribution functions. The other important feature of this equation is that the integral

$$H = \int\int d\vec{x}d\vec{p}\; f(\vec{x},\vec{p},t)\ln f(\vec{x},\vec{p},t) \qquad (5.8)$$

can only decrease[3]. For a system of N statistically independent particles, H is related to the thermodynamic entropy S through:

$$S \stackrel{\text{def}}{=} -NkH \qquad (5.9)$$

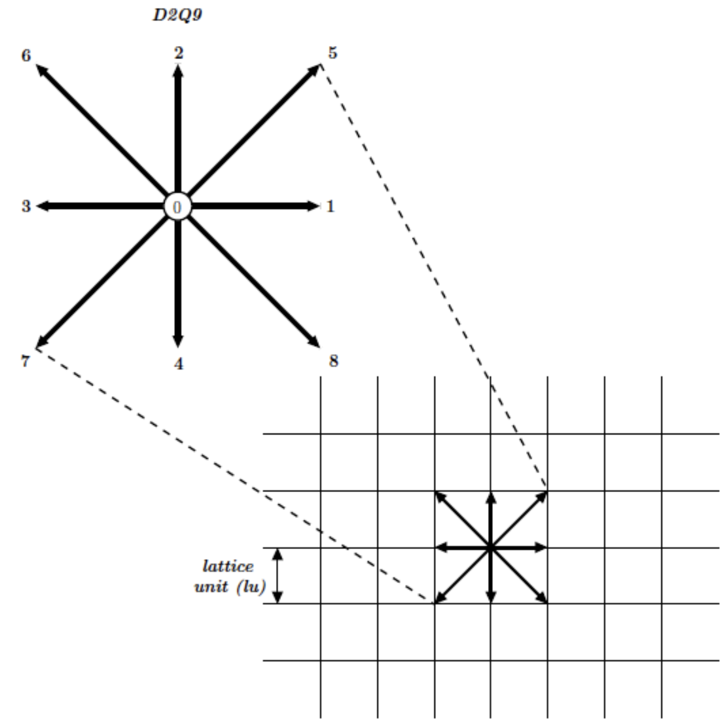


Figure 5.1: Discrete lattice velocities for the $D2Q9$ model

The macroscopic variables are defined as functions of the particle distribution functions (hereafter DFs) according to:

$$\rho = \sum_{a=0}^{\beta-1} f_a \qquad \text{(macroscopic fluid density)} \tag{5.10}$$

and

$$\vec{u} = \frac{1}{\rho}\sum_{a=0}^{\beta-1} f_a \vec{e}_a \qquad \text{(macroscopic velocity).} \tag{5.11}$$

3.1 Provided files

Your copy of the Rayleigh-Bénard.r application should come with the following files:

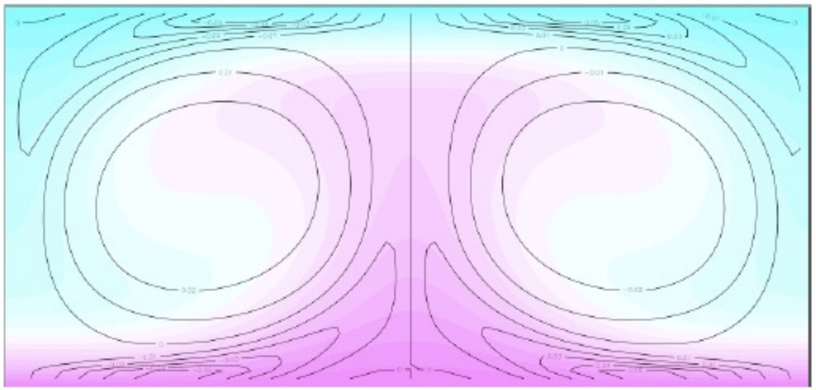
- rayleigh-benard.r | The R source code
- rb_functions.r | Some extra R functions needed by the model
- rb_plot_functions.r | Some R functions for plotting the results

3.2 Parameters

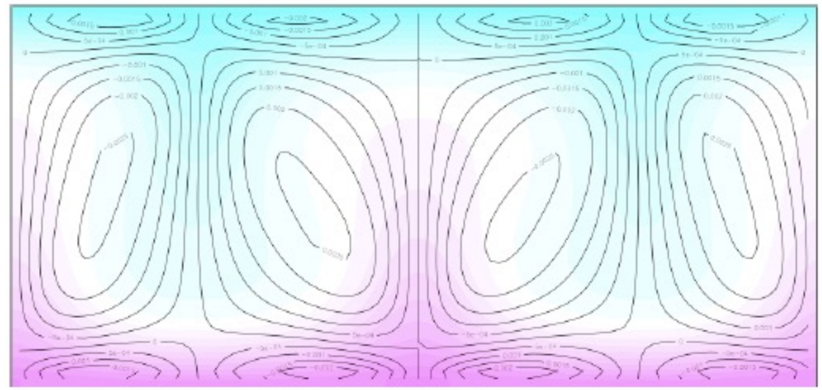
This is a short overview of the parameters which can be set by editing the 'Parameter Definitions' section in the 'rayleigh-benard.r'-file. There are two different types of parameters that can be edited: the 'model parameters' (which define the 'physical' values needed for the simulation), and the 'output parameters' (which define the frequency and kind of output).

Below we provide a list with the explained parameter list:

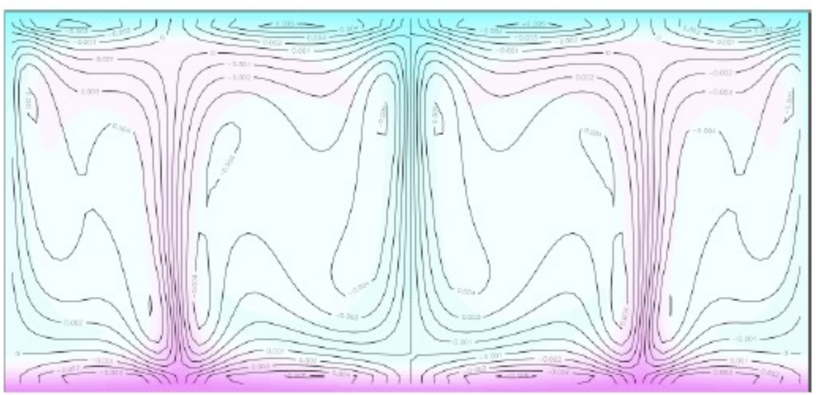
- `lx` : number of gridpoints in X-direction;
- `ly` : number of gridpoints in Y-direction;
- `N_t0` : total simulation time measured in time unit scales $\boldsymbol{t_{0p}}$ ($\boldsymbol{T_{max} = N_{t0} t_{0p}}$);
- `Pr` : Prandtl number;
- `Ra` : Rayleigh number;
- `beta` : parameter that describes the coupling between $\boldsymbol{\delta_x}$ and $\boldsymbol{\delta_t}$;
- `N_out`: number of outputs;
- `out_dir` : parameter that defines the path where the output should be saved;



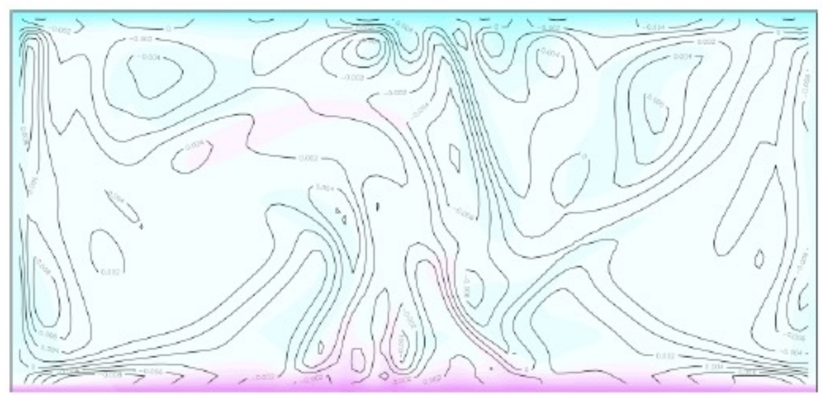
(a) $Ra = 2 \cdot 10^4, Pr = 0.1$



(b) $Ra = 2 \cdot 10^4, Pr = 10$



(c) $Ra = 5 \cdot 10^5, Pr = 10$



(d) $Ra = 1 \cdot 10^7, Pr = 10$

Dynamics of the Ocean System

Momentum equation

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Dynamics of the Ocean System

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Geostrophic balance: Pressure gradient Coriolis force

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Rossby number Ro is defined as:

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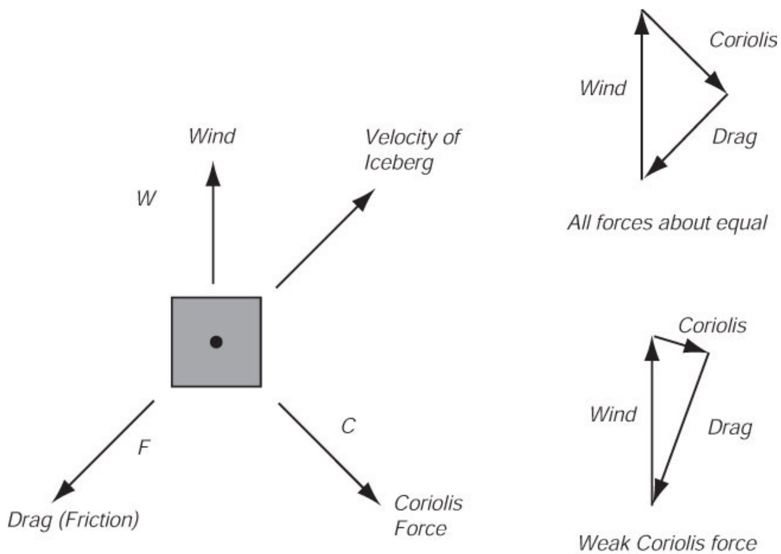
where U and L are, respectively, characteristic velocity and length scales of the phenomenon and $f = 2\Omega \sin \varphi$ is a typical Coriolis frequency. The procedure is analogous to the dimensionless Reynolds number (section 1.3).

Nansen's Qualitative Arguments



Fridtjof Nansen noticed that wind tended to blow icebergs 20° –40° to the right of the wind in the Arctic.

Nansen argued that three forces must be important:

1) Wind Stress W

2) Friction F (otherwise the iceberg would move as fast as the wind)
Drag must be opposite the direction of the ice's velocity

3) Coriolis Force C.
Coriolis force must be perpendicular to the velocity

The forces must balance for steady flow: **W + F + C =** 0