Mixed layer model and linear system theory

Imagine that the temperature anomaly of the ocean mixed layer is governed by

$$\frac{dT}{dt} = -\lambda T + Q(t)\,, \tag{1}$$

where λ is the typical damping rate of a temperature anomaly and $Q(t)$ a forcing.

1. **Laplace transformation of mixed layer model** *(6 points) (2,2,2)*
The Laplace transform $L(s)$ is used for the solution of differential equations and the analysis of filters. From the previous exercise sheet, we know that the Laplace transform is given by

$$\mathcal{L}\{x(t)\} = L(s) = \int_0^\infty e^{-st}x(t)dt \tag{2}$$

$$\text{and} \quad \mathcal{L}\{\exp(-at)\} = \frac{1}{s+a} \tag{3}$$

$$\mathcal{L}\{-\exp(-at)+\exp(-bt)\} = \frac{-1}{s+a} + \frac{1}{s+b} = \frac{a-b}{(s+a)}\frac{1}{(s+b)} \quad . \tag{4}$$

For the mixed layer model (1)

$$L(s) = \frac{Q(s)+T(0)}{s+\lambda} \quad . \tag{5}$$

where $Q(s) = \mathcal{L}\{Q(t)\}$.

(a) Consider the special case $Q(t) = \exp(i\omega_0 t)$, then $Q(s) = \frac{1}{s-i\omega_0}$. The forcing and the temperature is of course a real number, but by representing it as a complex number we can simultaneously keep track of both phase components. Show

$$L(s) = \frac{T(0)+Q(s)}{s+\lambda} = \frac{T(0)}{s+\lambda} + \frac{1}{(s+\lambda)}\frac{1}{(s-i\omega_0)} \tag{6}$$

and via the Laplace back-transformation $x(t) = \mathcal{L}^{-1}F(s)$ using (3,4) that

$$T(t) = \exp(-\lambda t)T(0) + \frac{[\exp(i\omega_0 t) - \exp(-\lambda t)]}{\lambda + i\omega_0} \quad . \tag{7}$$

Calculate the real and imaginary part of (7).

(b) Take the real part. Show: At low frequencies, the output $T(t)$ is similar to the forcing $Q(t)$. At high frequencies it rolls off as $1/\omega$ (it is a low-pass filter) and is out of phase by 90°.

(c) Instead of b), consider now the special case $Q(t) = c \cdot u(t)$ with $u(t)$ as unit step or the so-called Heaviside step function (step forcing). Show via Laplace transform that

$$T(t) = \mathcal{L}^{-1}\{L(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{T(0)}{s+\lambda} + \frac{c}{s} \cdot \frac{1}{s+\lambda}\right\} \tag{8}$$

$$= T(0) \cdot \exp(-\lambda t) + \frac{c}{\lambda}(1 - \exp(-\lambda t)) \tag{9}$$

Note that the equilibrium response is

$$\lim_{t \to \infty} T(t) = \frac{c}{\lambda} . \tag{10}$$

which is related to the climate sensitivity.

2. **Stochastic climate model** *(4 points)(1,1,2)*

For the mixed layer model (1) assume now that Q is the air-sea fluxes due to weather systems are represented by a white-noise process with zero average $< Q >= 0$ and δ-correlated in time

$$Cov_Q(\tau) =< Q(t)\, Q(t+\tau) >= a \cdot \delta(\tau) \quad . \tag{11}$$

The Fourier transform of the auto-correlation function $Cov_Q(\tau)$ is called spectrum

$$S_Q(\omega) = \int_{\mathsf{R}} Cov_Q(\tau) e^{i\omega\tau}\, d\tau \quad = \int_{\mathsf{R}} a \cdot \delta(\tau) e^{i\omega\tau}\, d\tau \quad = \quad a \tag{12}$$

a) Solve Eq. (1) for the temperature response $T = \hat{T}(\omega) e^{-i\omega t}$ and hence show that:

$$\hat{T}(\omega) = \frac{\hat{Q}(\omega)}{(\lambda - i\omega)} \tag{13}$$

b) Show that it has a spectral density $\hat{T}(\omega)\, \hat{T}^*(\omega)$ is given by:

$$\hat{T}\, \hat{T}^* = \frac{\hat{Q}\, \hat{Q}^*}{(\lambda^2 + \omega^2)} \tag{14}$$

where $\hat{Q}^*$ is the complex conjugate.

c) Show that the spectrum of the T is

$$S_T(\omega) =< \hat{T}\, \hat{T}^* >= \frac{< \hat{Q}\, \hat{Q}^* >}{(\lambda^2 + \omega^2)} = \frac{a}{(\lambda^2 + \omega^2)}. \tag{15}$$

The brackets $< \ldots >$ denote the ensemble mean. Make a sketch of the spectrum S_T using a log-log plot.

<u>Notes on submission form of the exercises:</u> *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00 pm) to Alessandro Gagliardi (Alessandro.Gagliardi@awi.de), Georg Huettner (Georg.Huettner@awi.de).*