1. **Concept of dynamic similarity** *(4 points)*

For the case of an incompressible flow, assuming the temperature effects are negligible and external forces are neglected, the Navier-Stokes equations consist of conservation of mass

$$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

and conservation of momentum

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu \nabla^2 \mathbf{u} \tag{2}$$

where $\mathbf{u}$ is the velocity vector and p is the pressure, ν denotes the kinematic viscosity.

a) Show: The equations (1,2) can be made dimensionless by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. For example: $u = U \cdot u_d$.

Note: the units of $[\rho_0] = kg/m^3$, $[p] = kg/(ms^2)$, and $[p]/[\rho_0] = m^2/s^2$. Therefore the pressure gradient term in (2) has the scaling U^2/L.

b) Show: The scalings vanish completely in front of the terms except for the $\nabla^2\mathbf{u_d}$-term! The dimensionless parameter is the Reynolds number and the only parameter left!

Remark: For large Reynolds numbers, the flow is turbulent. In most practical flows Re is rather large $(10^4 - 10^8)$, *large enough for the flow to be turbulent.*

2. **Elimination of the pressure term** *(5 points)*
Assume a 2D flow without non-linear terms and friction, where the equations reduce to:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{3}$$
$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} \quad . \tag{4}$$

a) Show: Subtract $\partial/\partial y$ of (3) from $\partial/\partial x$ of (4) results in the elimination of pressure.

b) Show: Defining the stream function ψ through

$$u = -\frac{\partial \psi}{\partial y} \quad ; \quad v = \frac{\partial \psi}{\partial x} \tag{5}$$

(mass continuity being unconditionally satisfied), the incompressible Newtonian 2D momentum and mass conservation degrade into one equation:

$$\partial_t \left(\nabla^2 \psi \right) = 0 \tag{6}$$

c) We now consider the rotating framework and add the Coriolis terms $-\rho f v$ and $\rho f u$ to the left hand side of (3,4). Subtract $\partial/\partial y$ of (3) from $\partial/\partial x$ of (4) to eliminate the pressure terms to derive the vorticity equation! Show that (6) changed into

$$\partial_t \left(\nabla^2 \psi \right) + \beta v = 0 \tag{7}$$

3. **Scaling of the dynamical equations** *(2 points)*

We work in the rotating frame of reference of the Earth. The equation can be scaled by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. We can estimate the relative contributions in units of m/s^2 in the horizontal momentum equations:

$$\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{U/T \sim 10^{-8}} + \underbrace{\mathbf{v} \cdot \nabla \mathbf{v}}_{U^2/L \sim 10^{-8}} = \underbrace{-\frac{1}{\rho} \nabla p}_{\delta \mathbf{P}/(\rho \mathbf{L}) \sim \mathbf{10^{-5}}} + \underbrace{2\mathbf{\Omega} \times \mathbf{v}}_{\mathbf{f_0 U} \sim \mathbf{10^{-5}}} + \underbrace{fric}_{\nu U/H^2 \sim 10^{-13}} \tag{8}$$

where fric denotes the contributions of friction due to eddy stress divergence (usually $\sim \nu \nabla^2 \mathbf{v}$). Typical values are given in Table 1. The values have been taken for the ocean.

a) Please repeat the estimate for the atmosphere using Table 1.

b) The Rossby number Ro is the ratio of inertial (the left hand side in (8)) to Coriolis (second term on the right hand side in (8)) terms

$$Ro = \frac{(U^2/L)}{(fU)} = \frac{U}{fL} \quad . \tag{9}$$

Ro is small when the flow is in a so-called geostrophic balance. Please calculate Ro for the atmosphere and ocean using Table 1.

	Quantity	Atmosphere	Ocean
horizontal velocity	U	$10 \quad ms^{-1}$	$10^{-1}\, ms^{-1}$
horizontal length	L	$10^6\, m$	$10^6\, m$
horizonal Pressure changes	δP (horizontal)	$10^3\, Pa$	$10^4\, Pa$
time scale	T	$10^5\, s$	$10^7\, s$
Coriolis parameter at 45°N	$f_0 = 2\Omega \sin \varphi_0$	$10^{-4}\, s^{-1}$	$10^{-4}\, s^{-1}$
density	ρ	$1 \quad kgm^{-3}$	$10^3\, kgm^{-3}$
viscosity (turbulent)	ν	$10^{-5}\, kgm^{-3}$	$10^{-6}\, kgm^{-3}$

Table 1: Table shows the typical scales in the atmosphere and ocean system.

Notes on submission form of the exercises: *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00) to Hanna Knahl (hanna.knahl@awi.de), Alexander Thorneloe (alexander.thorn@awi.de).*