1. **Elimination of the pressure term** *(4 points)*
   Assume a 2D flow without non-linear terms and friction, where the equations reduce to:

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{1}$$
$$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} \quad . \tag{2}$$

   a) Show: Subtract $\partial/\partial y$ of (1) from $\partial/\partial x$ of (2) results in the elimination of pressure.

   b) Show: Defining the stream function ψ through

$$u = -\frac{\partial \psi}{\partial y} \quad ; \quad v = \frac{\partial \psi}{\partial x} \tag{3}$$

   (mass continuity being unconditionally satisfied), the incompressible Newtonian 2D momentum and mass conservation degrade into one equation:

$$\partial_t \left(\nabla^2 \psi \right) = 0 \tag{4}$$

   c) We now consider the rotating framework and add the Coriolis terms $-\rho f v$ and $\rho f u$ to the left hand side of (1,2). Subtract $\partial/\partial y$ of (1) from $\partial/\partial x$ of (2) to eliminate the pressure terms to derive the vorticity equation! Show that (4) changed into

$$\partial_t \left(\nabla^2 \psi \right) + \beta v = 0 \tag{5}$$

2. **Conservation of potential vorticity:** *(3 points)*
   An air column at 53°N with $\zeta = 0$ initially streches from the surface to a fixed tropopause at 10 km height. If the air column moves until it is over a mountain barrier of 2 km height at 30°N, what is its absolute vorticity and relative vorticity as it passes the mountain top?

   Assume: $\sin 53° = 0.8; \sin 30° = 0.5$
   The angular velocity of the Earth $\Omega = 2\pi/(1 \text{ day})$.
   Potential vorticity: $(\zeta + f)/h$

3. **Graphical method for bifurcations** *(3 points)*

We introduce a graphical method to obtain stability or instability. Consider the "'saddle-node bifurcation"', one of the equilibrium points is unstable (the saddle), while the other is stable (the node). In Fig. 1, we can plot $\frac{dx}{dt} = f(x)$ dependent on x (left panel) for

$$\frac{dx}{dt} = b + x^2 \tag{6}$$

with $b < 0$ in this particular case (For $b > 0$ we would have no equilibrium, and we have no point x_e with $f(x_e) = 0$.). We just consider the slope $f'(x_e)$ and see that the filled circles with positive slope are unstable, the open circles with negative slopes are stable (right panel in Fig. 1).

(a) Draw the bifurcations as in Fig. 1 for the pitchfork bifurcation.

$$\frac{dx}{dt} = r \cdot x + x^3 \tag{7}$$

(b) Draw the bifurcations as in Fig. 1 for the transcritical bifurcation.

$$\frac{dx}{dt} = r \cdot x - x^2 \tag{8}$$

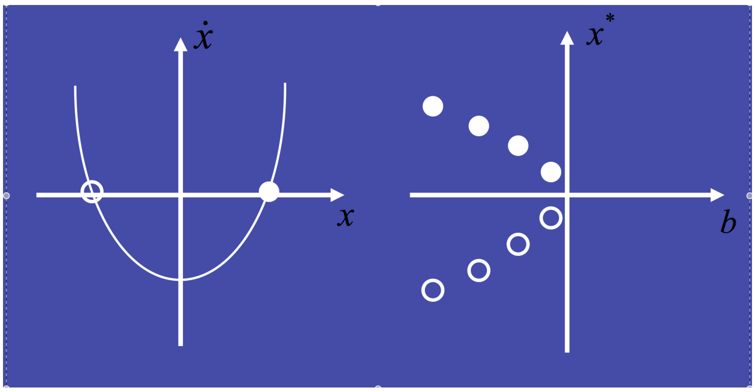

Figure 1: Saddle-node bifurcation diagram using the graphical method.

4. **Numerical Solution** *(3 points)*

   Solve (6, 7, 8) numerically using the Euler forward scheme:

$$\frac{dx}{dt} \rightarrow \frac{(x_{n+1} - x_n)}{\Delta t}$$

   and the right hand side is $f(x_n)$.

   (a) Write down the iteration for x_{n+1} for (6, 7, 8).

   (b) Plot the solutions for (6, 7, 8). Initial conditions x_0 shall be very close, but not identical to an unstable equilibrium point.

   Here is the solution for $\frac{dx}{dt} = A \cdot x$:

```
# ODE1.R
#demonstration of Euler forward method in 1st order ODE: dx/dt= A x

#constants
A<-  -0.5  #growth / decay rate
T<-  20    #integration time in time units
dt<-  .1     #step size in time units
x0<- 100   #inital value


n<-T/dt            #number of time steps (time / timestep)
t<-(0:(n-1))*dt   #create a vector of discrete timesteps
x<-vector()       #define an empty vector for the state variable y(t)
x[1]<-x0          #assign initial value


for (i in 1:(n-1))
{
        x[i+1]<-x[i]+dt*A*x[i]
}

plot(t,y,type="l") #plot the result against time

#additionaly plot the analytical solution in red
lines(t,Y0*exp(A*t),col="red")
```

<u>Notes on submission form of the exercises:</u> *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00) to Justus Contzen (Justus.Contzen@awi.de), Lars Ackermann (Lars.Ackermann@awi.de).*