Dynamics II, Summer semester 2024 — Exercise 2
Lecturer: Prof. Dr. G. Lohmann, Dr. M. Ionita — 15.4.2023
Tutors: Hanna Knahl, Alexander Thorneloe — Due date: 22.4.2024

1. **Questions about Rayleigh-Bénard instability (*2 points*)**

   Describe in words the Rayleigh-Bénard instability. The basic state possesses a steady-state solution in which there is no motion, and the temperature varies linearly with depth:

$$u = w = 0 \tag{1}$$

$$T_{eq} = T_0 + \left(1 - \frac{z}{H}\right)\Delta T \tag{2}$$

   When this solution becomes unstable, ... (please continue)

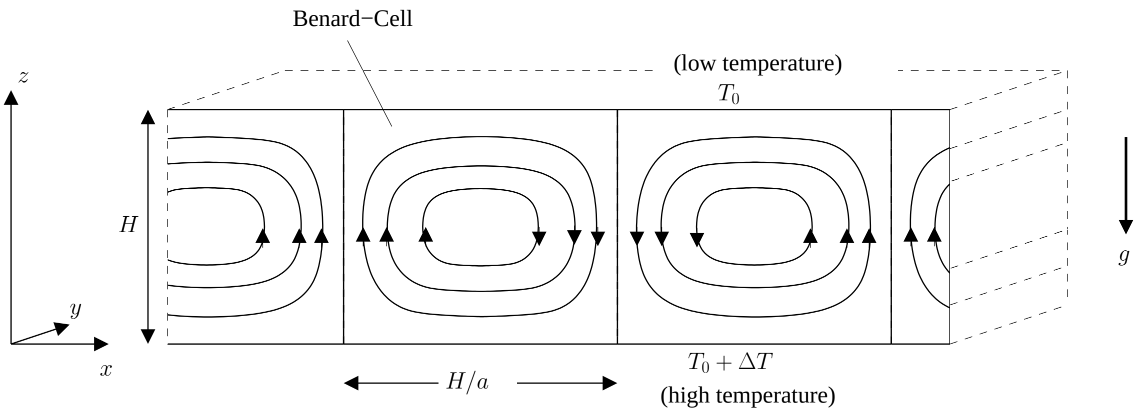

Figure 1: Geometry of the Rayleigh-Bénard system.

2. **Lorenz equations:** *(5 points)*

   Consider the Lorenz equations which were derived from the Rayleigh-Bénard system (Fig. 1) in the lecture:

$$\dot{x} = \sigma(y - x) \tag{3}$$

$$\dot{y} = rx - xz - y \tag{4}$$

$$\dot{z} = xy - bz \tag{5}$$

   with $\sigma, r, b > 0$. σ is the Prandtl number. Furthermore, Rayleigh number $R_a \sim \Delta T$, critical Rayleigh number R_c, and $r = R_a/R_c$.

   (a) Evaluate the equilibrium points.
   (b) Determine the stability of the $(0,0,0)-$equilibrium through linearization! Control parameter is r.
   (c) Show the symmetry: The Lorenz equation has the following symmetry $(x, y, z) \to (-x, -y, z)$ independent on the parameters σ, r, b.

3. **Graphical method for bifurcations** *(3 points)*

We introduce a graphical method to obtain stability or instability. Consider the "'saddle-node bifurcation"', one of the equilibrium points is unstable (the saddle), while the other is stable (the node). In Fig. 2, we can plot $\frac{dx}{dt} = f(x)$ dependent on x (left panel) for

$$\frac{dx}{dt} = b + x^2 \tag{6}$$

with $b < 0$ in this particular case (For $b > 0$ we would have no equilibrium, and we have no point x_e with $f(x_e) = 0$.). We just consider the slope $f'(x_e)$ and see that the filled circles with positive slope are unstable, the open circles with negative slopes are stable (right panel in Fig. 2).

(a) Draw the bifurcations as in Fig. 2 for the pitchfork bifurcation.

$$\frac{dx}{dt} = r \cdot x + x^3 \tag{7}$$

(b) Draw the bifurcations as in Fig. 2 for the transcritical bifurcation.

$$\frac{dx}{dt} = r \cdot x - x^2 \tag{8}$$

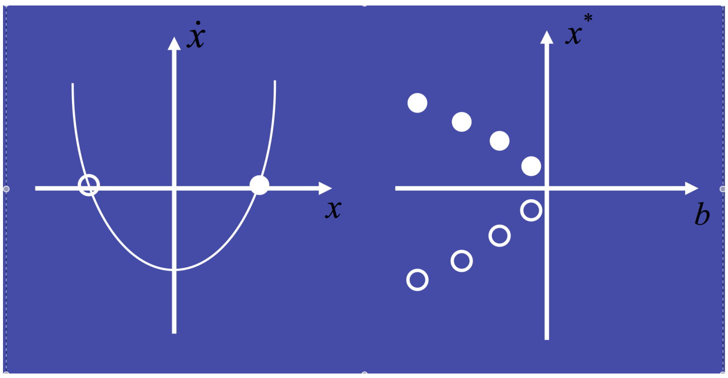

Figure 2: Saddle-node bifurcation diagram using the graphical method.

Notes on submission form of the exercises: *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00) to Hanna Knahl (hanna.knahl@awi.de), Alexander Thorneloe (alexander.thorn@awi.de).*