Dynamics II, Summer semester 2025 | Exercise 2
Lecturer: Prof. Dr. G. Lohmann, Dr. M. Ionita | 14.4.2024
Tutors: Alessandro Gagliardi, Georg Hüttner | Due date: 25.4.2024

1. **Concept of dynamic similarity** *(3 points)*

   For the case of an incompressible flow, assuming the temperature effects are negligible and external forces are neglected, the Navier-Stokes equations consist of conservation of mass

   $$\nabla \cdot \mathbf{u} = 0 \tag{1}$$

   and conservation of momentum

   $$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho_0}\nabla p + \nu \nabla^2 \mathbf{u} \tag{2}$$

   where $\mathbf{u}$ is the velocity vector and p is the pressure, ν denotes the kinematic viscosity.

   a) Show: The equations (1,2) can be made dimensionless by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. For example: $u = U \cdot u_d$.

   Note: the units of $[\rho_0] = kg/m^3$, $[p] = kg/(ms^2)$, and $[p]/[\rho_0] = m^2/s^2$. Therefore the pressure gradient term in (2) has the scaling U^2/L.

   b) Show: The scalings vanish completely in front of the terms except for the $\nabla^2 \mathbf{u_d}$-term! The dimensionless parameter is the Reynolds number and the only parameter left!

   Remark: For large Reynolds numbers, the flow is turbulent. In most practical flows Re is rather large $(10^4 - 10^8)$, *large enough for the flow to be turbulent.*

2. **Elimination of the pressure term** *(3 points)*
   Assume a 2D flow without non-linear terms and friction, where the equations reduce to:

   $$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{3}$$
   $$\rho \frac{\partial v}{\partial t} = -\frac{\partial p}{\partial y} \quad . \tag{4}$$

   a) Show: Subtract $\partial/\partial y$ of (3) from $\partial/\partial x$ of (4) results in the elimination of pressure.

b) Show: Defining the stream function ψ through

$$u = -\frac{\partial \psi}{\partial y} \quad ; \quad v = \frac{\partial \psi}{\partial x} \tag{5}$$

(mass continuity being unconditionally satisfied), the incompressible Newtonian 2D momentum and mass conservation degrade into one equation:

$$\partial_t \left(\nabla^2 \psi \right) \ = \ 0 \tag{6}$$

c) We now consider the rotating framework and add the Coriolis terms $-\rho f v$ and $\rho f u$ to the left hand side of (3,4). Subtract $\partial/\partial y$ of (3) from $\partial/\partial x$ of (4) to eliminate the pressure terms to derive the vorticity equation! Show that (6) changed into

$$\partial_t \left(\nabla^2 \psi \right) \ + \ \beta v \ = \ 0 \tag{7}$$

3. **Question about Rayleigh-Bénard instability (*2 points*)**

Describe in words the Rayleigh-Bénard instability. The basic state possesses a steady-state solution in which there is no motion, and the temperature varies linearly with depth:

$$u \ = \ w = 0 \tag{8}$$

$$T_{eq} \ = \ T_0 + \left(1 - \frac{z}{H}\right) \Delta T \tag{9}$$

When this solution becomes unstable, ... (please continue)

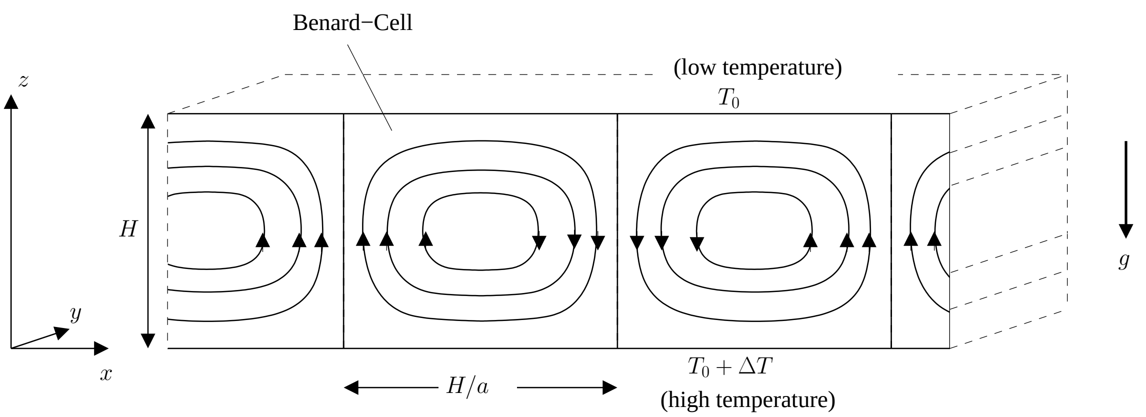

Figure 1: Geometry of the Rayleigh-Bénard system.

4. **Elimination of the pressure term in the Rayleigh-Bénard system** *(3 points)*

As above, derive the vorticity equation for

$$D_t u = -\frac{1}{\rho_0}\partial_x p + \nu\nabla^2 u \quad (10)$$

$$D_t w = -\frac{1}{\rho_0}\partial_z p + \nu\nabla^2 w + g(1-\alpha(T-T_0)) \quad (11)$$

using

$$\partial_x u + \partial_z w = 0$$

$$D_t T = \kappa\nabla^2 T$$

$$T = T_{eq} + \Theta \quad \text{where } \Theta \text{ is the anomaly to the equilibrium solution (9)}$$

For the calculation ignore the non-linear terms.

Show in analogy to (7)

$$D_t\left(\nabla^2\Psi\right) = \nu\nabla^4\Psi - g\alpha\frac{\partial\Theta}{\partial x}$$

5. **Graphical method for bifurcations** *(3 points)*

   We introduce a graphical method to obtain stability or instability. Consider the "'saddle-node bifurcation"', one of the equilibrium points is unstable (the saddle), while the other is stable (the node). In Fig. 2, we can plot $\frac{dx}{dt} = f(x)$ dependent on x (left panel) for

$$\frac{dx}{dt} = b + x^2 \tag{12}$$

   with $b < 0$ in this particular case (For $b > 0$ we would have no equilibrium, and we have no point x_e with $f(x_e) = 0$.). We just consider the slope $f'(x_e)$ and see that the filled circles with positive slope are unstable, the open circles with negative slopes are stable (right panel in Fig. 2).

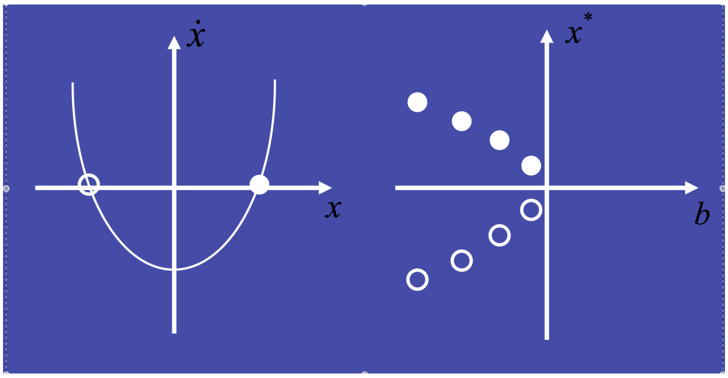

Figure 2: Saddle-node bifurcation diagram using the graphical method.

   (a) Draw the bifurcations as in Fig. 2 for the pitchfork bifurcation.

$$\frac{dx}{dt} = r \cdot x + x^3 \tag{13}$$

   (b) Draw the bifurcations as in Fig. 2 for the transcritical bifurcation.

$$\frac{dx}{dt} = r \cdot x - x^2 \tag{14}$$

<u>Notes on submission form of the exercises:</u> *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00 pm) to Alessandro Gagliardi (Alessandro.Gagliardi@awi.de), Georg Huettner (Georg.Huettner@awi.de).*