1. **Ocean thermohaline circulation** *(3 points)*

   Consider a geostrophic flow (u, v)

   $$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{1}$$

   $$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \quad . \tag{2}$$

   The meridional overturning stream function $\Phi(y, z)$ is defined via

   $$\Phi(y, z) = \int_0^z \frac{\partial \Phi}{\partial \tilde{z}}\, d\tilde{z} \tag{3}$$

   $$\frac{\partial \Phi}{\partial \tilde{z}} = \int_{x_e}^{x_w} v(x, y, \tilde{z})\, dx \quad \text{(zonally integrated transport)}, \tag{4}$$

   where x_e and x_w are the eastward and westward boundaries in the ocean basin (think e.g. of the Atlantic Ocean). Units of Φ are $m^3 s^{-1}$. At the surface $\Phi(y, 0) = 0$.

   Calculate $\Phi(y, z)$ as a function of density ρ at the basin boundaries!

   Use

   $$v(x, y, z) = \frac{1}{\rho_0 f}\frac{\partial p}{\partial x}$$

   and the hydrostatic approximation

   $$\frac{\partial p}{\partial z} = -g\rho$$

2. **Laplace transform** *(2 points)*

   $$\text{is given by} \quad \mathcal{L}\{x(t)\} = L(s) = \int_0^\infty e^{-st} x(t)\, dt \tag{5}$$

   a) Show that

   $$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sL(s) - x(0) \tag{6}$$

   through integration by parts.
   b) Show furthermore that

   $$\mathcal{L}\{\exp(-at)\} = \frac{1}{s + a} \tag{7}$$

3. **Laplace transformation of mixed layer model** *(5 points)*
   Imagine that the temperature of the ocean mixed layer is governed by

   $$\frac{dT}{dt} = -\lambda T + Q(t),\tag{8}$$

   where λ is the typical damping rate of a temperature anomaly and $Q(t)$ a forcing.

   (a) Use the Laplace transformation to show

   $$L(s) = \frac{Q(s) + T(0)}{s + \lambda}.\tag{9}$$

   where $Q(s) = \mathcal{L}\{Q(t)\}$

   (b) Consider the special case $Q(t) = \exp(i\omega_0 t)$, then $Q(s) = \frac{1}{s - i\omega_0}$. The forcing and
   the temperature is of course a real number, but by representing it as a complex
   number we can simultaneously keep track of both phase components.
   Show that

   $$L(s) = \frac{T(0) + Q(s)}{s + \lambda} = \frac{T(0)}{s + \lambda} + \frac{1}{(s + \lambda)}\frac{1}{(s - i\omega_0)}\tag{10}$$

   and via the Laplace back-transformation and (7) of the exercise above as well as
   (15) that

   $$T(t) = \exp(-\lambda t)T(0) + \frac{[\exp(i\omega_0 t) - \exp(-\lambda t)]}{\lambda + i\omega_0}.\tag{11}$$

   (c) Calculate the real and imaginary part of (11).

   (d) Take the real part. Show: At low frequencies, the output $T(t)$ is similar to the
   forcing $Q(t)$. At high frequencies it rolls off as $1/\omega$ (it is a low-pass filter) and is
   out of phase by 90°.

   (e) Instead of b), consider now the special case $Q(t) = c \cdot u(t)$ with $u(t)$ as unit step or
   the so-called Heaviside step function (step forcing). Show via Laplace transform
   that

   $$<T(t)> = \mathcal{L}^{-1}\{L(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{<T(0)>}{s + \lambda} + \frac{c}{s} \cdot \frac{1}{s + \lambda}\right\}\tag{12}$$

   $$= T(0) \cdot \exp(-\lambda t) + \frac{c}{\lambda}(1 - \exp(-\lambda t))\tag{13}$$

   the equilibrium response

   $$\Delta T = \lim_{t \to \infty} <T(t)> = \frac{c}{\lambda}.\tag{14}$$

Dynamics 2   Exercise 5, Summer Semester 2021
Lecturer: Prof. Dr. G. Lohmann   Tutors: Justus Contzen, Lars Ackermann
Due date: 31.5.2021   17.5.2021

Hint:

$$\mathcal{L}\left\{-\exp(-at)+\exp(-bt)\right\} \;\; = \;\; \frac{-1}{s+a}+\frac{1}{s+b}=\frac{a-b}{(s+a)}\frac{1}{(s+b)} \tag{15}$$

4. **Explain in words** *(3 points)*

... how the the winds and sea level pressure in Figure 1 are related !

Use the following keywords: Geostrophy, friction, High and Low pressure system, Northern Hemisphere, Coriolis force, pressure gradient.

Notes on submission form of the exercises: *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00) to Justus Contzen (Justus.Contzen@awi.de), Lars Ackermann (Lars.Ackermann@awi.de).*
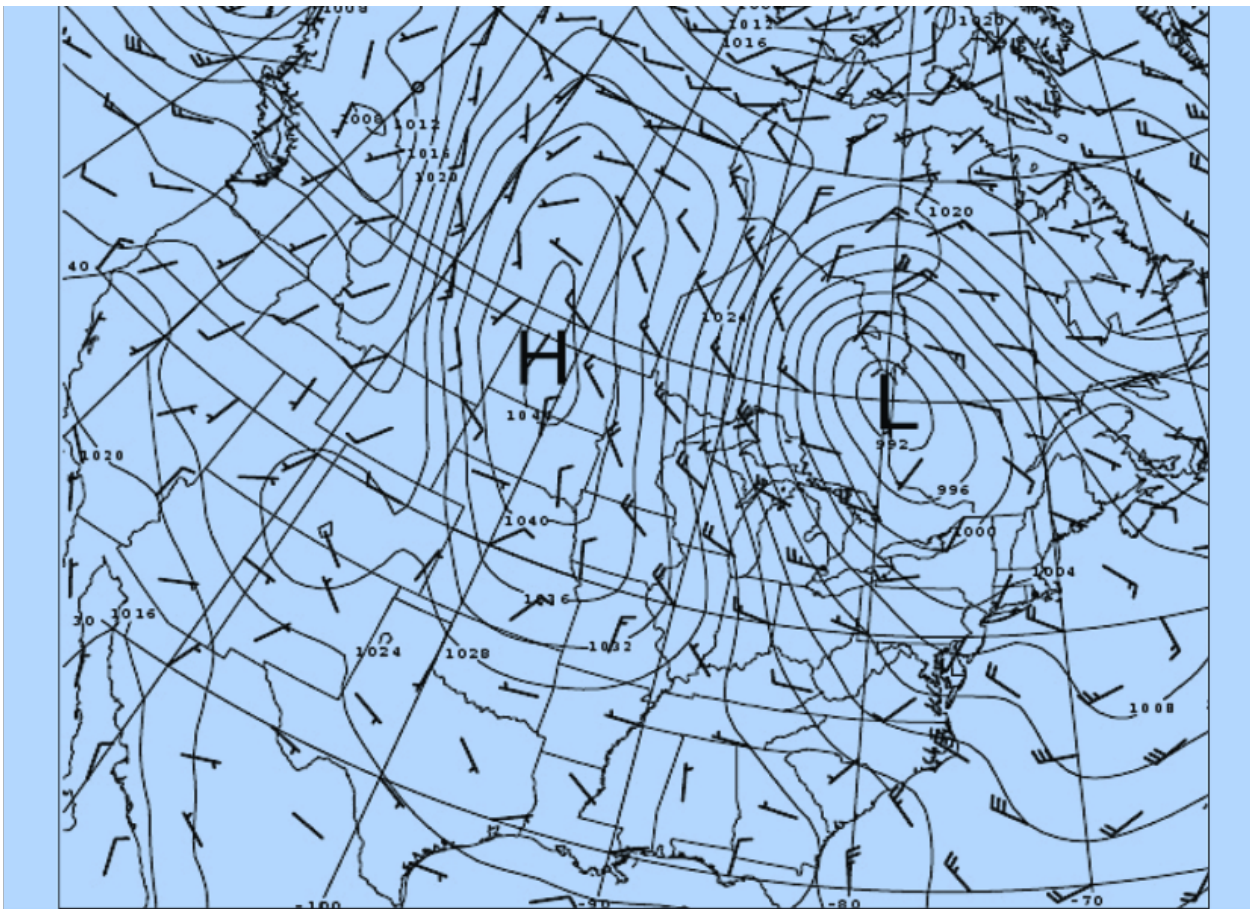
Figure 1: Surface pressure and surface winds, February 10, 2006. One full quiver represents 5 m/s blowing in the direction of the arrow shaft, away from the quiver.