Dynamics II, Summer semester 2023 | Exercise 6
Lecturer: Prof. Dr. G. Lohmann, Dr. M. Ionita | 5.6.2023
Tutors: Fernanda Matos, Ahmadreza Masoum | Due date: 12.6.2023

1. **Graphical method for bifurcations** *(3 points)*

We introduce a graphical method to obtain stability or instability. Consider the "'saddle-node bifurcation"', one of the equilibrium points is unstable (the saddle), while the other is stable (the node). In Fig. 1, we can plot $\frac{dx}{dt} = f(x)$ dependent on x (left panel) for

$$\frac{dx}{dt} = b + x^2 \tag{1}$$

with $b < 0$ in this particular case (For $b > 0$ we would have no equilibrium, and we have no point x_e with $f(x_e) = 0$.). We just consider the slope $f'(x_e)$ and see that the filled circles with positive slope are unstable, the open circles with negative slopes are stable (right panel in Fig. 1).

(a) Draw the bifurcations as in Fig. 1 for the pitchfork bifurcation.

$$\frac{dx}{dt} = r \cdot x + x^3 \tag{2}$$

(b) Draw the bifurcations as in Fig. 1 for the transcritical bifurcation.

$$\frac{dx}{dt} = r \cdot x - x^2 \tag{3}$$

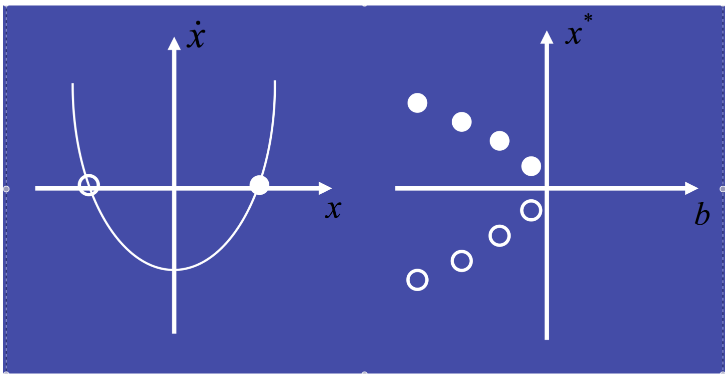

Figure 1: Saddle-node bifurcation diagram using the graphical method.

2. **Numerical Solution** *(3 points)*

   Solve (1, 2, 3) numerically using the Euler forward scheme:

$$\frac{dx}{dt} \rightarrow \frac{(x_{n+1} - x_n)}{\Delta t}$$

   and the right hand side is $f(x_n)$.

   (a) Write down the iteration for x_{n+1} for (1, 2, 3).
   (b) Plot the solutions for (1, 2, 3). Initial conditions x_0 shall be very close, but not identical to an unstable equilibrium point.

   Here is the solution for $\frac{dx}{dt} = A \cdot x$:

```
# ODE1.R
#demonstration of Euler forward method in 1st order ODE: dx/dt= A x

#constants
A<-  -0.5  #growth / decay rate
T<-  20    #integration time in time units
dt<-  .1     #step size in time units
x0<- 100   #inital value


n<-T/dt           #number of time steps (time / timestep)
t<-(0:(n-1))*dt   #create a vector of discrete timesteps
x<-vector()      #define an empty vector for the state variable y(t)
x[1]<-x0         #assign initial value


for (i in 1:(n-1))
{
        x[i+1]<-x[i]+dt*A*x[i]
}

plot(t,y,type="l") #plot the result against time

#additionaly plot the analytical solution in red
lines(t,Y0*exp(A*t),col="red")
```

Questions about Rayleigh-Bénard instability (*2 points*

Describe in words the Rayleigh-Bénard instability. The basic state possesses a steady-state solution in which there is no motion, and the temperature varies linearly with depth:

$$u = w = 0 \tag{4}$$

$$T_{eq} = T_0 + \left(1 - \frac{z}{H}\right) \Delta T \tag{5}$$

When this solution becomes unstable, ... (please continue)

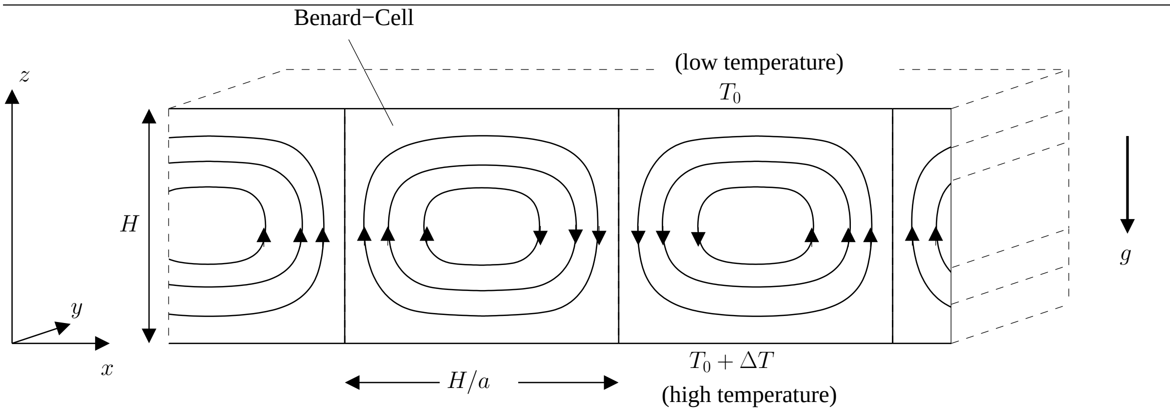

Figure 2: Geometry of the Rayleigh-Bénard system.

3. **Lorenz equations:** *(3 points)*

Consider the Lorenz equations which were derived from the Rayleigh-Bénard system (Fig. 2) in the lecture:

$$\dot{x} = \sigma(y - x) \quad (6)$$
$$\dot{y} = rx - xz - y \quad (7)$$
$$\dot{z} = xy - bz \quad (8)$$

with $\sigma, r, b > 0$. σ is the Prandtl number. Furthermore, Rayleigh number $R_a \sim \Delta T$, critical Rayleigh number R_c, and $r = R_a/R_c$.

(a) Evaluate the equilibrium points.
(b) Determine the stability of the $(0, 0, 0)-$equilibrium through linearization! Control parameter is r.
(c) Show the symmetry: The Lorenz equation has the following symmetry $(x, y, z) \to (-x, -y, z)$ independent on the parameters σ, r, b.

4. **Lorenz equations on the computer:** *(2 points)*

Solve the Lorenz equations numerically using the parameters $\sigma, r, b = 10, 28, 8/3$ and $\sigma, r, b = 10, 0.8, 8/3$. Initial conditions: $(x_0, y_0, z_0) = (1, 3, 5)$. Provide the 2 solutions with the associated short descriptions.

Notes on submission form of the exercises: *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00) to Fernanda Matos (Fernanda.Matos@awi.de), Ahmadreza Masoum (Ahmadreza.Masoum@awi.de).*