

1. **Ocean thermohaline circulation** (3 points)

Consider a geostrophic flow (u, v)

$$-fv = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} \quad (1)$$

$$fu = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} \quad (2)$$

The meridional overturning stream function $\Phi(y, z)$ is defined via

$$\Phi(y, z) = \int_0^z \frac{\partial \Phi}{\partial \tilde{z}} d\tilde{z} \quad (3)$$

$$\frac{\partial \Phi}{\partial \tilde{z}} = \int_{x_e}^{x_w} v(x, y, \tilde{z}) dx \quad (\text{zonally integrated transport}), \quad (4)$$

where x_e and x_w are the eastward and westward boundaries in the ocean basin (think e.g. of the Atlantic Ocean). Units of Φ are $m^3 s^{-1}$. At the surface $\Phi(y, 0) = 0$.

Calculate $\Phi(y, z)$ as a function of density ρ at the basin boundaries!

Use

$$v(x, y, z) = \frac{1}{\rho_0 f} \frac{\partial p}{\partial x}$$

and the hydrostatic approximation

$$\frac{\partial p}{\partial z} = -g\rho$$

2. **Laplace transform** (2 points)

is given by $\mathcal{L}\{x(t)\} = L(s) = \int_0^\infty e^{-st} x(t) dt \quad (5)$

a) Show that

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sL(s) - x(0) \quad (6)$$

through integration by parts.

b) Show furthermore that

$$\mathcal{L}\{\exp(-at)\} = \frac{1}{s+a} \quad (7)$$

3. Laplace transformation of mixed layer model (5 points)

Imagine that the temperature of the ocean mixed layer is governed by

$$\frac{dT}{dt} = -\lambda T + Q(t), \quad (8)$$

where λ is the typical damping rate of a temperature anomaly and $Q(t)$ a forcing.

(a) Use the Laplace transformation to show

$$L(s) = \frac{Q(s) + T(0)}{s + \lambda} \quad (9)$$

where $Q(s) = \mathcal{L}\{Q(t)\}$

(b) Consider the special case $Q(t) = \exp(i\omega_0 t)$, then $Q(s) = \frac{1}{s - i\omega_0}$. The forcing and the temperature is of course a real number, but by representing it as a complex number we can simultaneously keep track of both phase components. Show that

$$L(s) = \frac{T(0) + Q(s)}{s + \lambda} = \frac{T(0)}{s + \lambda} + \frac{1}{(s + \lambda)} \frac{1}{(s - i\omega_0)} \quad (10)$$

and via the Laplace back-transformation and (7) of the exercise above as well as (15) that

$$T(t) = \exp(-\lambda t)T(0) + \frac{[\exp(i\omega_0 t) - \exp(-\lambda t)]}{\lambda + i\omega_0} \quad (11)$$

- (c) Calculate the real and imaginary part of (11).
 (d) Take the real part. Show: At low frequencies, the output $T(t)$ is similar to the forcing $Q(t)$. At high frequencies it rolls off as $1/\omega$ (it is a low-pass filter) and is out of phase by 90° .
 (e) Instead of b), consider now the special case $Q(t) = c \cdot u(t)$ with $u(t)$ as unit step or the so-called Heaviside step function (step forcing). Show via Laplace transform that

$$\langle T(t) \rangle = \mathcal{L}^{-1}\{L(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{\langle T(0) \rangle}{s + \lambda} + \frac{c}{s} \cdot \frac{1}{s + \lambda}\right\} \quad (12)$$

$$= T(0) \cdot \exp(-\lambda t) + \frac{c}{\lambda} (1 - \exp(-\lambda t)) \quad (13)$$

the equilibrium response

$$\Delta T = \lim_{t \rightarrow \infty} \langle T(t) \rangle = \frac{c}{\lambda} \quad (14)$$

Hint:

$$\mathcal{L}\{-\exp(-at) + \exp(-bt)\} = \frac{-1}{s + a} + \frac{1}{s + b} = \frac{a - b}{(s + a)(s + b)} \quad (15)$$

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Exercise 6
13.5.2023
Due date: 20.5.2024

Notes on submission form of the exercises: *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00) to Hanna Knahl (hanna.knahl@awi.de), Alexander Thorneloe (alexander.thorn@awi.de).*