1. **Ocean thermohaline circulation** *(3 points)*

Consider a geostrophic flow (u, v)

$$-fv = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{1}$$

$$fu = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \quad . \tag{2}$$

The meridional overturning stream function $\Phi(y, z)$ is defined via

$$\Phi(y,z) = \int_0^z \frac{\partial \Phi}{\partial \tilde{z}}\, d\tilde{z} \tag{3}$$

$$\frac{\partial \Phi}{\partial \tilde{z}} = \int_{x_e}^{x_w} v(x,y,\tilde{z})\, dx \quad \text{(zonally integrated transport)}, \tag{4}$$

where x_e and x_w are the eastward and westward boundaries in the ocean basin (think e.g. of the Atlantic Ocean). Units of Φ are $m^3 s^{-1}$. At the surface $\Phi(y, 0) = 0$.

Calculate $\Phi(y, z)$ as a function of density ρ at the basin boundaries!

Use

$$v(x,y,z) = \frac{1}{\rho_0 f}\frac{\partial p}{\partial x}$$

and the hydrostatic approximation

$$\frac{\partial p}{\partial z} = -g\rho$$

2. **Laplace transform** *(2 points)*

is given by $$\mathcal{L}\left\{x(t)\right\} = L(s) = \int_0^\infty e^{-st} x(t) dt \tag{5}$$

a) Show that

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sL(s) - x(0) \tag{6}$$

through integration by parts.
b) Show furthermore that

$$\mathcal{L}\left\{\exp(-at)\right\} = \frac{1}{s+a} \tag{7}$$

3. **Laplace transformation of mixed layer model** *(5 points)*
Imagine that the temperature of the ocean mixed layer is governed by

$$\frac{dT}{dt} = -\lambda T + Q(t)\,, \tag{8}$$

where λ is the typical damping rate of a temperature anomaly and $Q(t)$ a forcing.

(a) Use the Laplace transformation to show

$$L(s) = \frac{Q(s) + T(0)}{s + \lambda} \quad . \tag{9}$$

where $Q(s) = \mathcal{L}\left\{Q(t)\right\}$

(b) Consider the special case $Q(t) = \exp(i\omega_0 t)$, then $Q(s) = \frac{1}{s - i\omega_0}$. The forcing and the temperature is of course a real number, but by representing it as a complex number we can simultaneously keep track of both phase components. Show that

$$L(s) = \frac{T(0) + Q(s)}{s + \lambda} = \frac{T(0)}{s + \lambda} + \frac{1}{(s + \lambda)}\frac{1}{(s - i\omega_0)} \tag{10}$$

and via the Laplace back-transformation and (7) of the exercise above as well as (15) that

$$T(t) = \exp(-\lambda t)T(0) + \frac{[\exp(i\omega_0 t) - \exp(-\lambda t)]}{\lambda + i\omega_0} \quad . \tag{11}$$

(c) Calculate the real and imaginary part of (11).

(d) Take the real part. Show: At low frequencies, the output $T(t)$ is similar to the forcing $Q(t)$. At high frequencies it rolls off as $1/\omega$ (it is a low-pass filter) and is out of phase by $90°$.

(e) Instead of b), consider now the special case $Q(t) = c \cdot u(t)$ with $u(t)$ as unit step or the so-called Heaviside step function (step forcing). Show via Laplace transform that

$$< T(t) > \;=\; \mathcal{L}^{-1}\{L(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{< T(0) >}{s + \lambda} + \frac{c}{s} \cdot \frac{1}{s + \lambda}\right\} \tag{12}$$

$$= T(0) \cdot \exp(-\lambda t) + \frac{c}{\lambda}\left(1 - \exp(-\lambda t)\right) \tag{13}$$

the equilibrium response

$$\Delta T = \lim_{t \to \infty} < T(t) >= \frac{c}{\lambda}\,. \tag{14}$$

Hint:

$$\mathcal{L}\left\{-\exp(-at) + \exp(-bt)\right\} = \frac{-1}{s + a} + \frac{1}{s + b} = \frac{a - b}{(s + a)}\frac{1}{(s + b)} \tag{15}$$

Notes on submission form of the exercises: *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00) to Fernanda Matos (Fernanda.Matos@awi.de), Ahmadreza Masoum (Ahmadreza.Masoum@awi.de).*