1. **Laplace transform** *(2 points)*

$$\text{is given by} \qquad \mathcal{L}\left\{x(t)\right\} = L(s) = \int_0^\infty e^{-st} x(t) dt \tag{1}$$

Show that

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} \;=\; sL(s) - x(0) \tag{2}$$

through integration by parts. Show furthermore that

$$\mathcal{L}\left\{\exp(-at)\right\} \;=\; \frac{1}{s+a} \tag{3}$$

2. **Laplace transformation of mixed layer model** *(5 points)*
Imagine that the temperature of the ocean mixed layer is governed by

$$\frac{dT}{dt} = -\lambda T + Q(t)\,, \tag{4}$$

where λ is the typical damping rate of a temperature anomaly and $Q(t)$ a forcing.

(a) Use the Laplace transformation to show

$$L(s) = \frac{Q(s) + T(0)}{s+\lambda} \quad . \tag{5}$$

where $Q(s) = \mathcal{L}\left\{Q(t)\right\}$

(b) Consider the special case $Q(t) = \exp(i\omega_0 t)$, then $Q(s) = \frac{1}{s - i\omega_0}$. The forcing and the temperature is of course a real number, but by representing it as a complex number we can simultaneously keep track of both phase components. Show that

$$L(s) = \frac{T(0) + Q(s)}{s+\lambda} = \frac{T(0)}{s+\lambda} + \frac{1}{(s+\lambda)}\frac{1}{(s-i\omega_0)} \tag{6}$$

and via the Laplace back-transformation and (3, 8) that

$$T(t) = \exp(-\lambda t)T(0) + \frac{[\exp(i\omega_0 t) - \exp(-\lambda t)]}{\lambda + i\omega_0} \quad . \tag{7}$$

(c) Calculate the real and imaginary part of (7).

(d) Take the real part. Show: At low frequencies, the output $T(t)$ is similar to the forcing $Q(t)$. At high frequencies it rolls off as $1/\omega$ (it is a low-pass filter) and is out of phase by 90°.

Hint:

$$\mathcal{L}\{-\exp(-at) + \exp(-bt)\} = \frac{-1}{s+a} + \frac{1}{s+b} = \frac{a-b}{(s+a)}\frac{1}{(s+b)} \tag{8}$$

3. **Stochastic climate model** *(3 points)*

Imagine that the temperature of the ocean mixed layer of depth h is governed by

$$\frac{dT}{dt} = -\lambda T + Q\,, \tag{9}$$

where $Q = \frac{Q_{net}}{\gamma_O}$ is the heat flux, the heat capacity $\gamma_O = c_p \rho h$, and λ is the typical damping rate of a temperature anomaly. The air-sea fluxes due to weather systems are represented by a white-noise process $Q = \hat{Q}_\omega e^{i\omega t}$ where $\hat{Q}_\omega$ is the amplitude of the random forcing at frequency ω. $\hat{Q}^*$ is the complex conjugate.

a) Solve Eq. 9 for the temperature response $T = \hat{T}_\omega e^{i\omega t}$ and hence show that:

$$\hat{T}_\omega = \frac{\hat{Q}_\omega}{(\lambda + i\omega)} \tag{10}$$

b) Show that it has a spectral density $\hat{T}_\omega \hat{T}^*_\omega$ is given by:

$$\hat{T}\,\hat{T}^* = \frac{\hat{Q}\,\hat{Q}^*}{(\lambda^2 + \omega^2)} \tag{11}$$

and the spectrum

$$S(\omega) =< \hat{T}\,\hat{T}^* >= \frac{1}{(\lambda^2 + \omega^2)}. \tag{12}$$

The brackets $< \cdots >$ denote the ensemble mean. Make a sketch of the spectrum using a log-log plot and show that fluctuations with a frequency greater than λ are damped.

Hint: Use the **ergodic hypothesis** where the ensemble average $S(\omega) = \langle\, \hat{x}\hat{x}^* \,\rangle$ can be expressed as the time average (see in the script **Covariance and spectrum**):

$$\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} dt \quad \hat{x}\hat{x}^* \quad . \tag{13}$$

4. **Climate sensitivity and variability in the Stochastic Climate Model** *(4 points)*

Imagine that the temperature of the ocean mixed layer of depth h is governed by

$$\frac{dT}{dt} = -\lambda T + Q + f(t)\,, \tag{14}$$

where the air-sea fluxes due to weather systems are represented by a white-noise process with zero average $< Q >= 0$ and δ-correlated in time $< Q(t)Q(t+\tau) >= \delta(\tau)$. The function $f(t)$ is a time dependent deterministic forcing. Assume furthermore that $f(t) = c \cdot u(t)$ with $u(t)$ as unit step or the so-called Heaviside step function.

a) Show via Laplace transform that

$$< T(t) > \;=\; \mathcal{L}^{-1}\{L(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{< T(0) >}{s+\lambda} + \frac{c}{s}\cdot\frac{1}{s+\lambda}\right\} \tag{15}$$

$$= \; T(0)\cdot\exp(-\lambda t) \,+\, \frac{c}{\lambda}\,(1-\exp(-\lambda t)) \tag{16}$$

b) Show that the equilibrium response is

$$\Delta T = \lim_{t\to\infty} < T(t) >= \frac{c}{\lambda}\,. \tag{17}$$

c) Calculate the spectrum of (14) for $f(t) = 0$! What is the relationship of the dissipation (through λ) and the fluctuations (through the spectrum $S(\omega)$) ?

<u>Notes on submission form of the exercises:</u> *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date to Anna Pagone (anna.pagone@awi.de).*