1. **Taylor-Proudman Theorem** (*4 points*)

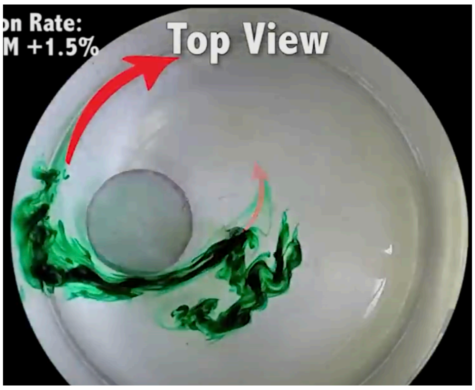

Figure 1: Taylor-Proudman Theorem: Experiment in a rotating tank.

The influence of vorticity due to Earth's rotation is most striking for geostrophic flow of a fluid with constant density ρ_0 on a plane with constant rotation $f = f_0$. The components of the geostrophic and hydrostatic pressure equations are:

$$-f\ v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x} \tag{1}$$

$$f\ u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y} \tag{2}$$

$$g = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} \tag{3}$$

Show that the flow is two-dimensional and does not vary in the vertical direction by showing that

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \tag{4}$$

Hint: Taking the z derivative of (1) and using (3) gives: $-f_0 \frac{\partial v}{\partial z} = \quad \ldots$

The *Taylor-Proudman Theorem* applies to a homogeneous, rotating, inviscid fluid. The theorem places strong constraints on the flow.

2. **Fourier transform** *(8 points)*

The Fourier transform decomposes a function of time (e.g., a signal) into the frequencies that make it up, similarly to how a musical chord can be expressed as the amplitude (or loudness) of its constituent notes. The Fourier transform of a function of time itself is a complex-valued function of frequency, whose absolute value represents the amount of that frequency present in the original function, and whose complex argument is the phase offset of the basic sinusoid in that frequency. The Fourier transform is called the frequency domain representation of the original signal. The term Fourier transform refers to both the frequency domain representation and the mathematical operation that associates the frequency domain representation to a function of time (see also `https://en.wikipedia.org/?title=Fourier_transform`).

The Fourier transformation of x is defined as

$$\hat{x}(\omega) = \int_{\mathsf{R}} x(t) e^{i\omega t}\, dt \tag{5}$$

and is denoted as a hat in the following.

And the inverse Fourier transformation of x is defined as

$$x(t) = \frac{1}{2\pi} \int_{\mathsf{R}} \hat{x}(\omega) e^{-i\omega t}\, d\omega \tag{6}$$

Tasks: Calculate the Fourier transformation of

(a) $x(t + a)$ (time shift).
(b) $x(t * a)$ (Scaling in the time domain).
(c) $\frac{d}{dt}x(t)$ (time derivative).
(d) $x(t) = \exp(-at^2)$ (Gaussian).
(e) $x(t) = \delta(t)$ where the δ distribution is defined through the operator on any function y: $y(t_0) = \int_{\mathsf{R}} y(t)\delta(t - t_0)\ dt$
(f) Calculate the Fourier transformation of a the periodic function $x(t) = \sin(\omega_0 t)$. Remember that $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$.

3. **Laplace transform** *(8 points)*

The Laplace and Fourier transforms are related. The Laplace transform $L(s)$ is also used for the solution of differential equations and the analysis of filters (`https://en.wikipedia.org/wiki/Laplace_transform`). We introduce the complex variable $s = -i\omega$.

The Laplace transform is given by

$$\mathcal{L}\left\{x(t)\right\} = L(s) = \int_0^\infty e^{-st} x(t) dt \tag{7}$$

a) Show that integration by parts leads to

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sL(s) - x(0) \tag{8}$$

b) Show that

$$\mathcal{L}\left\{\exp(-at)\right\} = \frac{1}{s+a} \tag{9}$$

c) Imagine that the temperature anomaly of an ocean mixed layer is governed by

$$\frac{dT}{dt} = -\lambda T + Q(t)\,, \tag{10}$$

where λ is the typical damping rate of a temperature anomaly and $Q(t)$ a forcing. Use the Laplace transformation to show

$$L(s) = \frac{Q(s) + T(0)}{s+\lambda} \quad . \tag{11}$$

where $Q(s) = \mathcal{L}\left\{Q(t)\right\}$

d) Consider the special case $Q(t) = \exp(i\omega_0 t)$, then $Q(s) = \frac{1}{s - i\omega_0}$ Show that

$$L(s) = \frac{T(0) + Q(s)}{s+\lambda} = \frac{T(0)}{s+\lambda} + \frac{1}{(s+\lambda)}\frac{1}{(s-i\omega_0)} \tag{12}$$

<u>Notes on submission form of the exercises:</u> *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00 pm) to Alessandro Gagliardi (Alessandro.Gagliardi@awi.de), Georg Huettner (Georg.Huettner@awi.de).*