Dynamics 2 — Exercise 9, Summer semester 2019
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Due date: 1.07.2019 — 17.06.2019

1. **Laplace transformation of mixed layer model** *(5 points)*
Imagine that the temperature of the ocean mixed layer is governed by

$$\frac{dT}{dt} = -\lambda T + Q(t)\,, \tag{1}$$

where λ is the typical damping rate of a temperature anomaly and $Q(t)$ a forcing.

(a) Use the Laplace transformation to show

$$L(s) = \frac{Q(s) + T(0)}{s+\lambda} \quad . \tag{2}$$

where $Q(s) = \mathcal{L}\{Q(t)\}$

(b) Consider the special case $Q(t) = \exp(i\omega_0 t)$, then $Q(s) = \frac{1}{s-i\omega_0}$. The forcing and the temperature is of course a real number, but by representing it as a complex number we can simultaneously keep track of both phase components. Show that

$$L(s) = \frac{T(0) + Q(s)}{s+\lambda} = \frac{T(0)}{s+\lambda} + \frac{1}{(s+\lambda)}\frac{1}{(s-i\omega_0)} \tag{3}$$

and via the Laplace back-transformation and (3) of the exercise sheet 7 as well as (8) that

$$T(t) = \exp(-\lambda t)T(0) + \frac{[\exp(i\omega_0 t) - \exp(-\lambda t)]}{\lambda + i\omega_0} \quad . \tag{4}$$

(c) Calculate the real and imaginary part of (4).

(d) Take the real part. Show: At low frequencies, the output $T(t)$ is similar to the forcing $Q(t)$. At high frequencies it rolls off as $1/\omega$ (it is a low-pass filter) and is out of phase by $90°$.

(e) Instead of b), consider now the special case $Q(t) = c\cdot u(t)$ with $u(t)$ as unit step or the so-called Heaviside step function (step forcing). Show via Laplace transform that

$$\begin{aligned} < T(t) > &= \mathcal{L}^{-1}\{L(s)\}(t) = \mathcal{L}^{-1}\left\{\frac{< T(0) >}{s+\lambda} + \frac{c}{s}\cdot\frac{1}{s+\lambda}\right\} & (5)\\ &= T(0)\cdot\exp(-\lambda t) + \frac{c}{\lambda}(1-\exp(-\lambda t)) & (6)\end{aligned}$$

the equilibrium response

$$\Delta T = \lim_{t\to\infty} < T(t) > = \frac{c}{\lambda}\,. \tag{7}$$

Hint:

$$\mathcal{L}\{-\exp(-at) + \exp(-bt)\} = \frac{-1}{s+a} + \frac{1}{s+b} = \frac{a-b}{(s+a)}\frac{1}{(s+b)} \tag{8}$$

2. **Stochastic climate model** *(5 points)*

Imagine that the temperature of the ocean mixed layer of depth h is governed by

$$\frac{dT}{dt} = -\lambda T + Q\,, \tag{9}$$

where λ is the typical damping rate of a temperature anomaly, and Q is the air-sea fluxes due to weather systems are represented by a white-noise process with zero average $< Q >= 0$ and δ-correlated in time

$$Cov_Q(\tau) =< Q(t)Q(t+\tau) >= c \cdot \delta(\tau) \quad . \tag{10}$$

The Fourier transform of the auto-correlation function $Cov_Q(\tau)$ is called spectrum

$$S_Q(\omega) = \int_{\mathsf{R}} Cov_Q(\tau) e^{i\omega\tau}\, d\tau \quad = \int_{\mathsf{R}} c \cdot \delta(\tau) e^{i\omega\tau}\, d\tau \quad = \quad c \tag{11}$$

a) Solve Eq. (9) for the temperature response $T = \hat{T}(\omega)e^{-i\omega t}$ and hence show that:

$$\hat{T}(\omega) = \frac{\hat{Q}(\omega)}{(\lambda - i\omega)} \tag{12}$$

b) Show that it has a spectral density $\hat{T}(\omega)\,\hat{T}^*(\omega)$ is given by:

$$\hat{T}\,\hat{T}^* = \frac{\hat{Q}\,\hat{Q}^*}{(\lambda^2 + \omega^2)} \tag{13}$$

where $\hat{Q}^*$ is the complex conjugate.

c) Show that the spectrum of the T is

$$S_T(\omega) =< \hat{T}\,\hat{T}^* >= \frac{< \hat{Q}\,\hat{Q}^* >}{(\lambda^2 + \omega^2)} = \frac{c}{(\lambda^2 + \omega^2)}. \tag{14}$$

The brackets $< \cdots >$ denote the ensemble mean. Make a sketch of the spectrum S_T using a log-log plot. Why is it called red noise?

d) Show that the definition of the spectrum via $S(\omega) = \langle\, \hat{x}\hat{x}^* \,\rangle$ and the Fouriertransformation as it is used in (12) are equivalent.

Hint: Use the **ergodic hypothesis** where the ensemble average $S(\omega) = \langle\, \hat{x}\hat{x}^* \,\rangle$ can be expressed as the time average (see in the script **Covariance and spectrum**).

Notes on submission form of the exercises: *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date to to Yuchen Sun (yuchen.sun@awi.de).*