1. **Stochastic climate model** *(5 points)(1,1,3)*

   Imagine that the temperature of the ocean mixed layer of depth h is governed by

   $$\frac{dT}{dt} = -\lambda T + Q\,, \tag{1}$$

   where λ is the typical damping rate of a temperature anomaly, and Q is the air-sea fluxes due to weather systems are represented by a white-noise process with zero average $< Q >= 0$ and δ-correlated in time

   $$Cov_Q(\tau) =< Q(t)Q(t+\tau) >= c \cdot \delta(\tau) \quad . \tag{2}$$

   The Fourier transform of the auto-correlation function $Cov_Q(\tau)$ is called spectrum

   $$S_Q(\omega) = \int_{\mathsf{R}} Cov_Q(\tau) e^{i\omega\tau}\, d\tau \quad = \int_{\mathsf{R}} c \cdot \delta(\tau) e^{i\omega\tau}\, d\tau \quad = \quad c \tag{3}$$

   a) Solve Eq. (1) for the temperature response $T = \hat{T}(\omega)e^{-i\omega t}$ and hence show that:

   $$\hat{T}(\omega) = \frac{\hat{Q}(\omega)}{(\lambda - i\omega)} \tag{4}$$

   b) Show that it has a spectral density $\hat{T}(\omega)\,\hat{T}^*(\omega)$ is given by:

   $$\hat{T}\,\hat{T}^* = \frac{\hat{Q}\,\hat{Q}^*}{(\lambda^2 + \omega^2)} \tag{5}$$

   where $\hat{Q}^*$ is the complex conjugate.

   c) Show that the spectrum of the T is

   $$S_T(\omega) =< \hat{T}\,\hat{T}^* >= \frac{< \hat{Q}\,\hat{Q}^* >}{(\lambda^2 + \omega^2)} = \frac{c}{(\lambda^2 + \omega^2)}. \tag{6}$$

   The brackets $< \ldots >$ denote the ensemble mean. Make a sketch of the spectrum S_T using a log-log plot.

2. **Covariance and spectrum** *(3 points)*
   Show that the definition of the spectrum via $S(\omega) = \langle\, \hat{x}\hat{x}^*\, \rangle$ and the Fouriertransformation as it is used in (3) are equivalent.

Hint: Use the **ergodic hypothesis** where the ensemble average $S(\omega) = \langle \hat{x}\hat{x}^* \rangle$ can be expressed as the time average

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \quad \hat{x}\hat{x}^* \tag{7}$$

where

$$x(t) = \frac{1}{2\pi} \int_{\mathsf{R}} \hat{x}(\omega) e^{-i\omega t} \, d\omega \tag{8}$$

3. **Some R exercises** *(5 points)(1,1,1,2)*
   (a) Create a vector t "t<-seq(-2*pi,2*pi,by=0.01)"
   Plot several functions in one window $(\sin(t), \cos(t), \exp(\frac{t}{5}), (\frac{t}{5})^2)$. Try some of the plot arguments: Set ylim, label the axes, set a different colour for each function, vary the line width. Plot the figure. For help try "?plot" or "?plot.default"
   (b) Set up a vector a of length 20 and create vector b with a linear relationship to a (e.g. $a = 3b + 7$). Calculate the correlation("cor(a,b)").
   (c) Set up two random vectors a,b of length 20 and calculate the correlation. Repeat this procedure several times to get a feeling for the correlation coefficient. Vary the length of vector a and b (sample number) and discuss how the correlation coefficient changes (10,100,1000).
   (d) Repeat the experiment from the previous task 100 times by using a loop. Create before the loop an empty vector ("cor.val<-vector()") and save the correlation of a and b in this vector (e.g. "cor.val[i]<-cor(a,b)") for each realisation. Compute the mean value and plot the histogram of cor.val. What happens with the histogram when the length of a and b is varied (10,100)? Save two different histograms as a figure and explain the difference between them.

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# Important R-commands
rnorm(N) # create vector with N normal distribution random numbers
cor(a,b) # calculates the correlation coefficient
hist(a)  # histogram of vector a
mean(a)  # mean value of vector a
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Notes on submission form of the exercises: *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00) to Smit Doshi (Smit.Doshi@awi.de), Dr. Qiyun Ma (Qiyun.Ma@awi.de).*