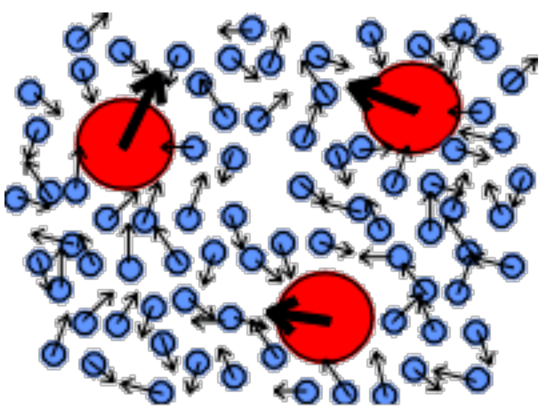
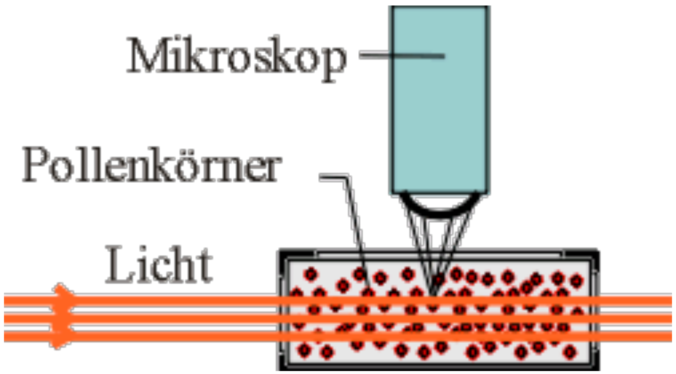
Stochastic climate model

(different time & length scales)

Brownian Motion: visible under the Mikroscope: Motion of particles



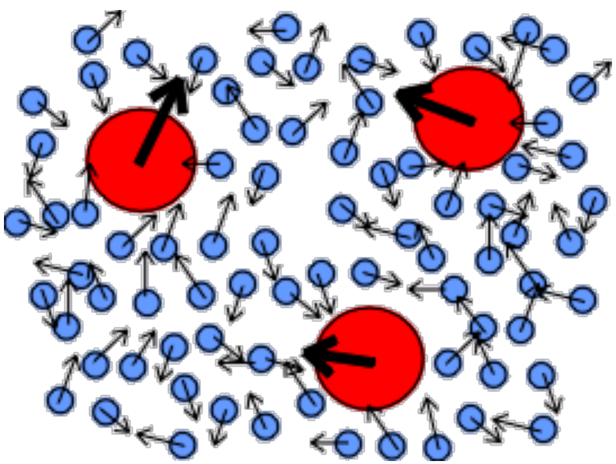


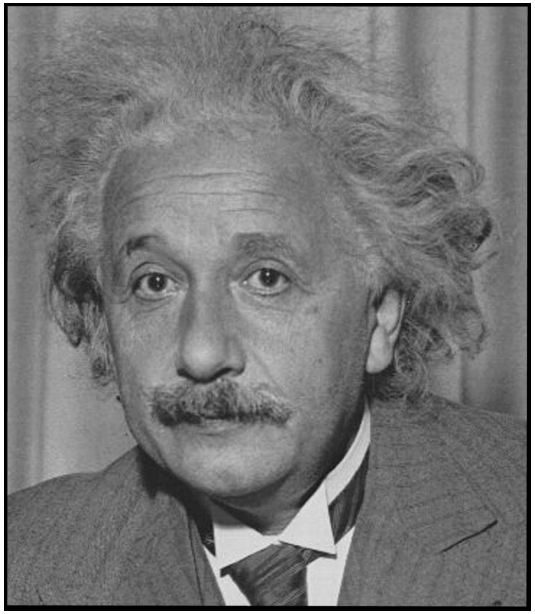
pulses, irregular
Living?
Pulses from all directions, random

Physics of the 20th century

- The matter the world is made of
- views: Elementary particles, quantum mechanics, relativity theory
- Limit of divisibility (Democritus, Aristotle): Matter is not a continuous whole: "The world cannot be composed of infinitely small particles".

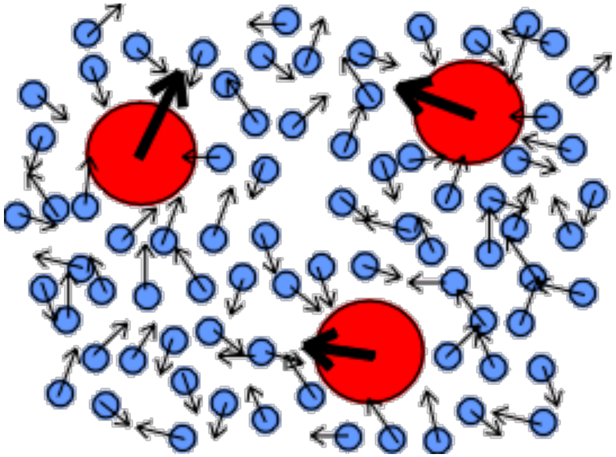
Brownian Motion





Einstein,Orstein, Uhlenbeck, Wiener, Fokker, Planck et al.

dx/dt = f(x) + g(x) dw/dt

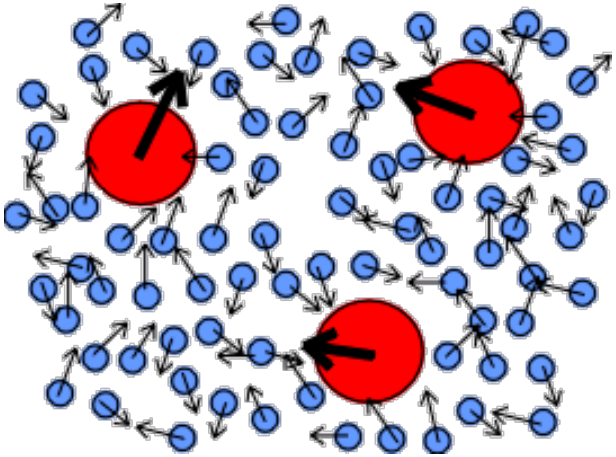


Brownian motion

is the random movement of particles, caused by their bombardment on all sides by molecules.

This motion can be seen in the behavior of pollen grains placed in a glass of water

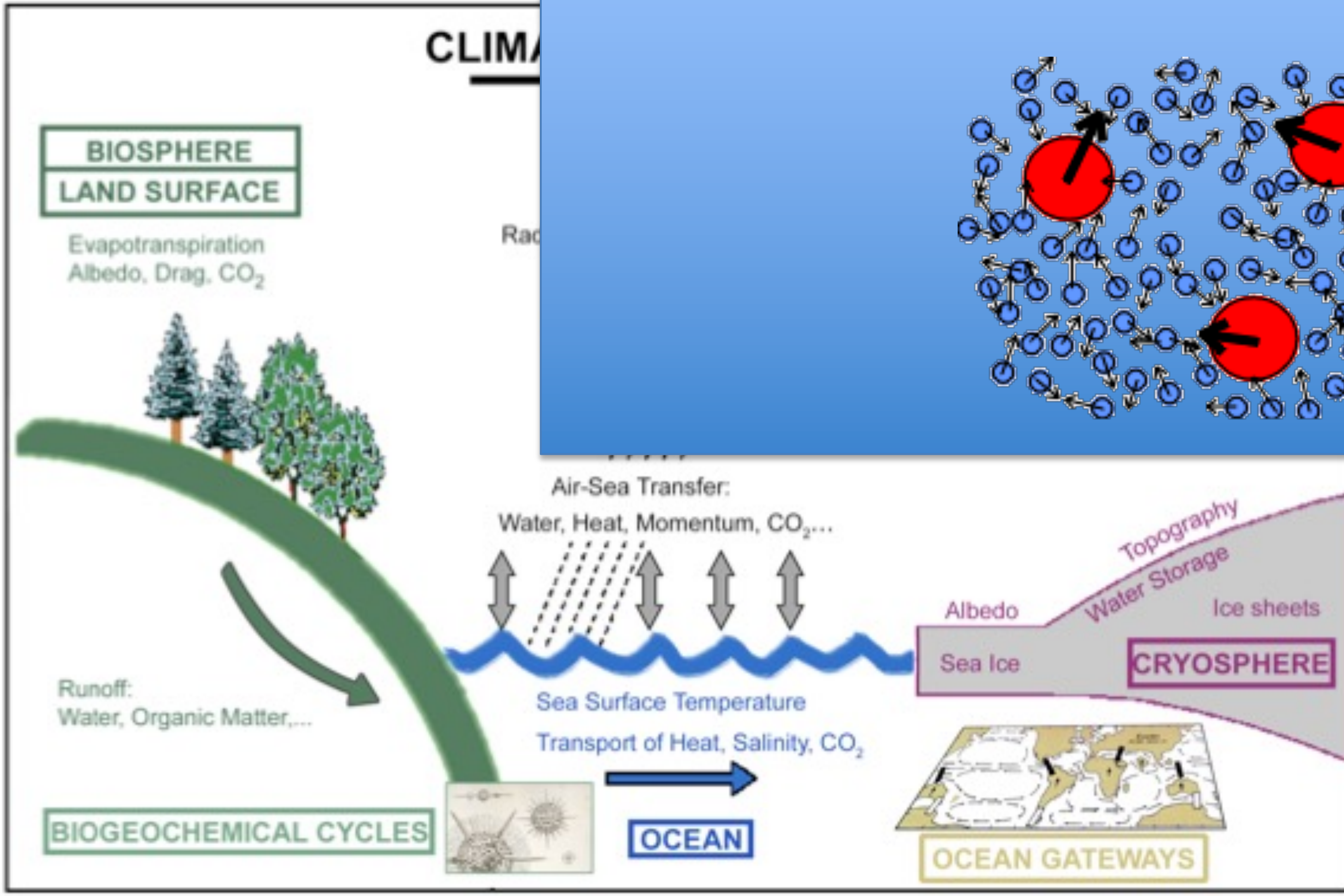
Because this motion often drives the interaction of time and spatial scales, it is important in several fields.



Following an idea of Hasselmann one can divide the climate dynamics into two parts. These two parts are the slowly changing climate part and rapidly changing weather part. **The** weather part can be modeled by a stochastic process such as white noise

Climate variability

- Brownian Particle: Climate
- Molecules: Weather



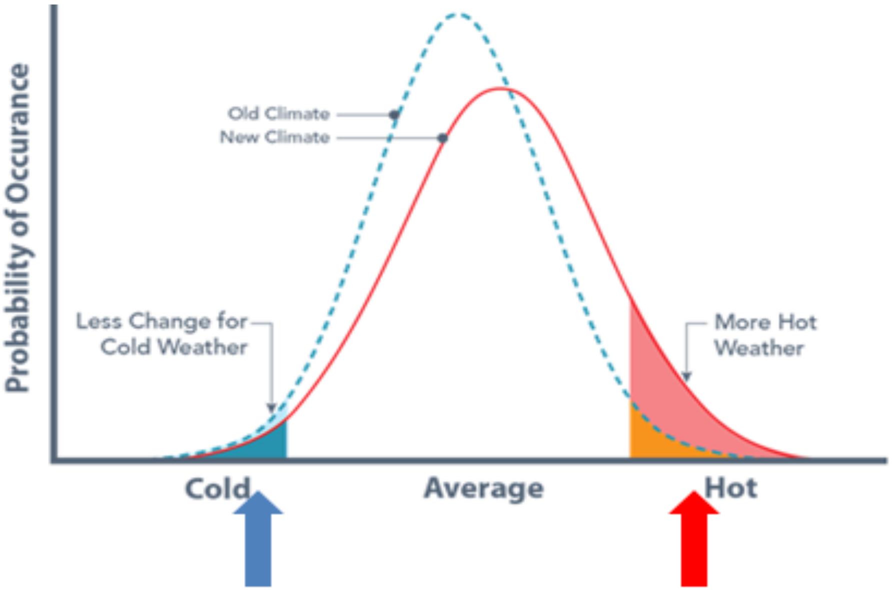
Distributions !

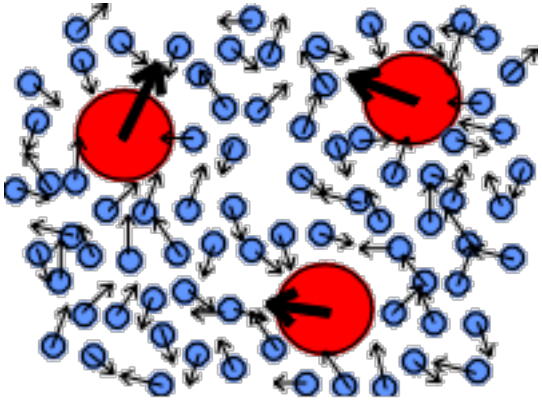
Climate

Brownsche Partikel: Klim

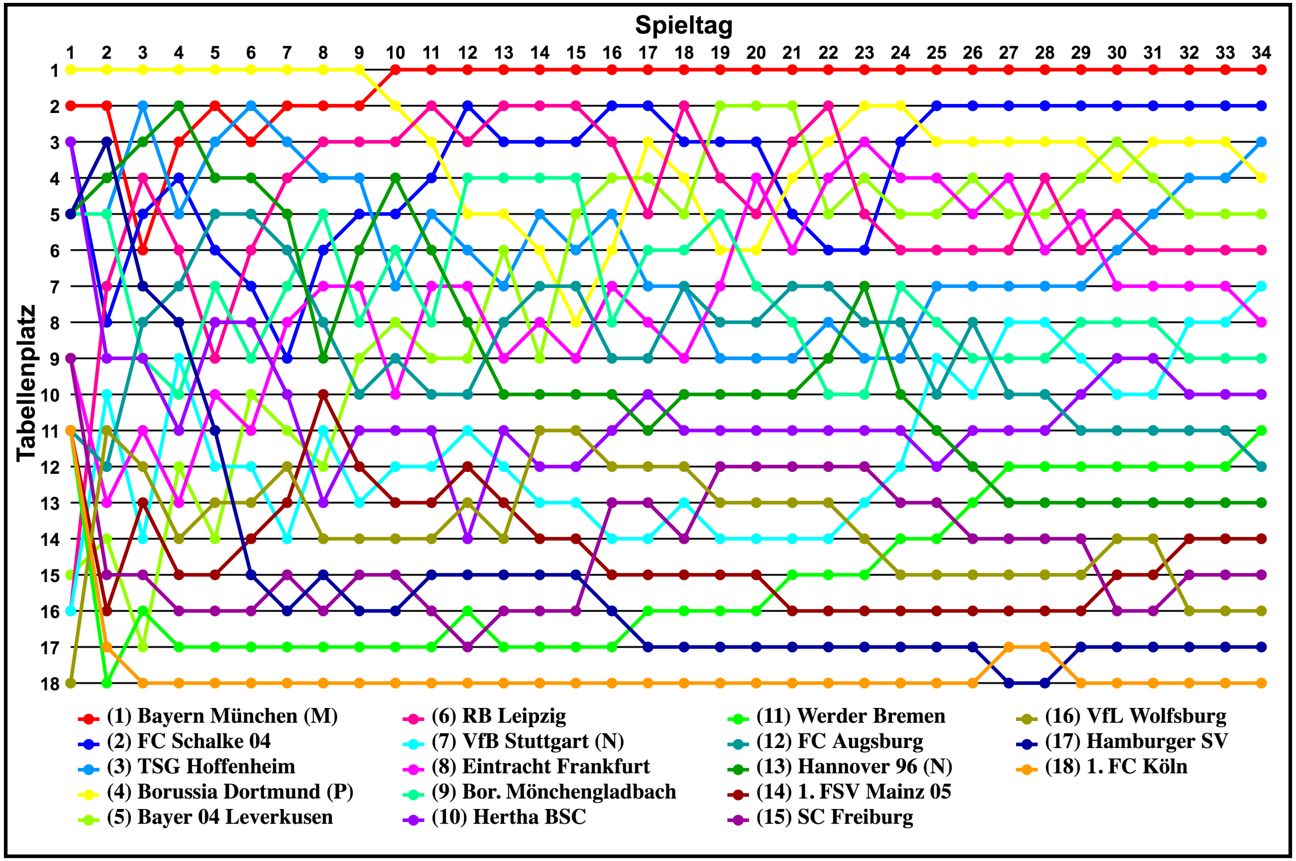
Moleküle: Wetter

Probabilities





Predictability



Coarse graining -> Stochastic

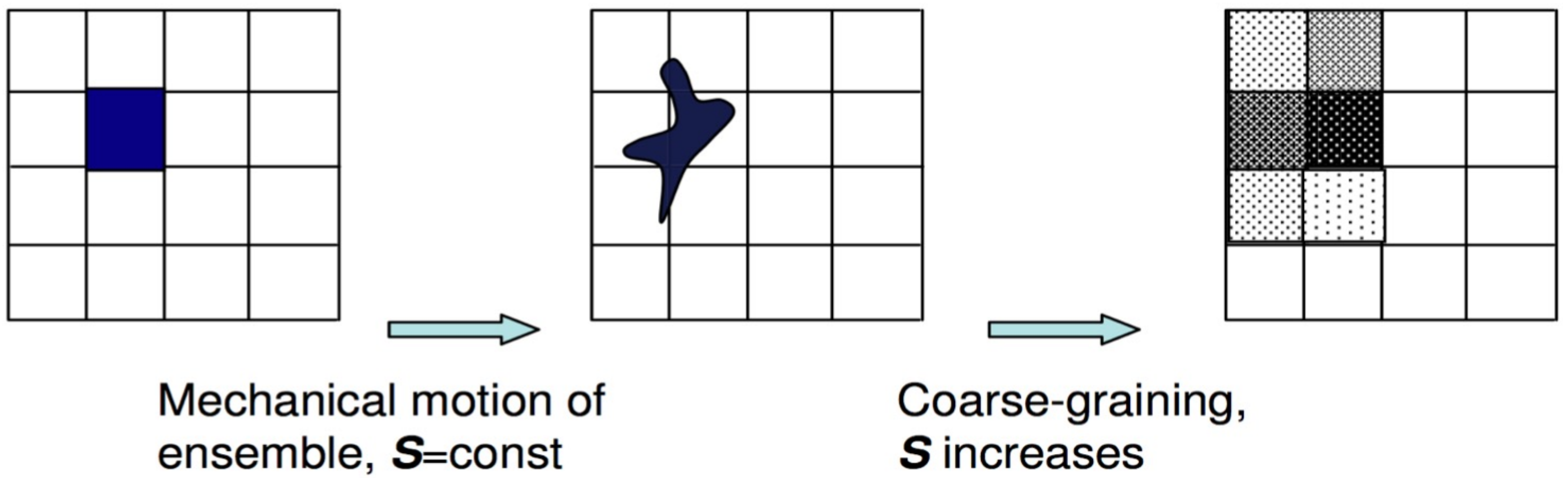
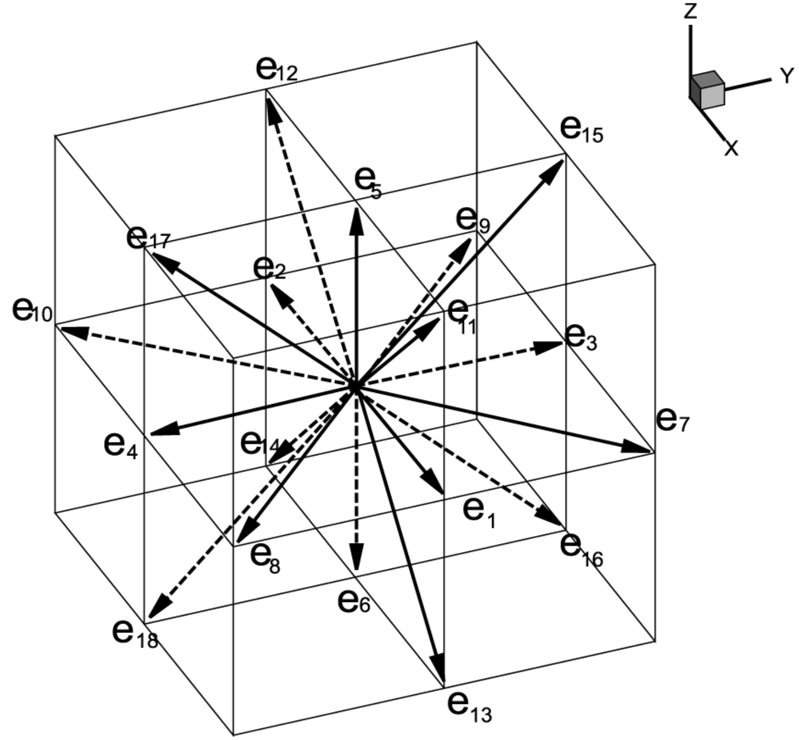


Figure 8.11: The Ehrenfests coarse-graining: two motion - coarse-graining cycles in 2D (values of probability density are presented by hatching density).

Lattice Boltzmann Method

- Simple "mesoscopic" rules yield complex behavior
- Recently established as CFD alternative in engineering
- Have been proven to simulate Navier-Stokes equations
- Velocity space discretized
- Explicit method, simple update rule:



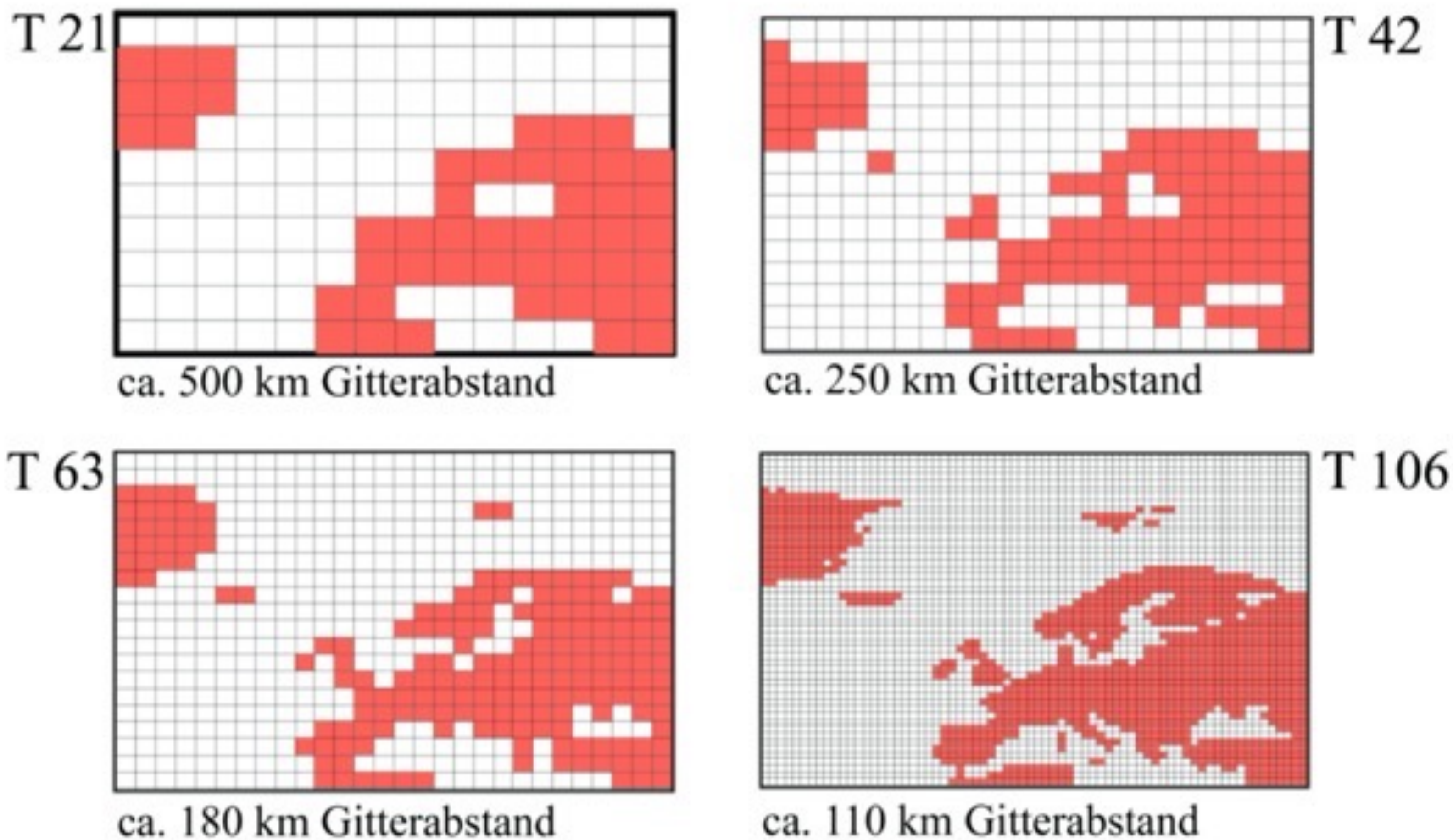
$$f_i(\vec{x}+\vec{e}_i, t+1) = f_i(\vec{x}, t) - \frac{f_i - f_i^{eq}}{\tau} + F_i$$

$$f_i^{eq} = \rho w_i \left[1 + 3(\vec{e}_i \cdot \vec{v}) + \frac{9}{2}(\vec{e}_i \cdot \vec{v})^2 - \frac{3}{2}\vec{v}^2\right]$$

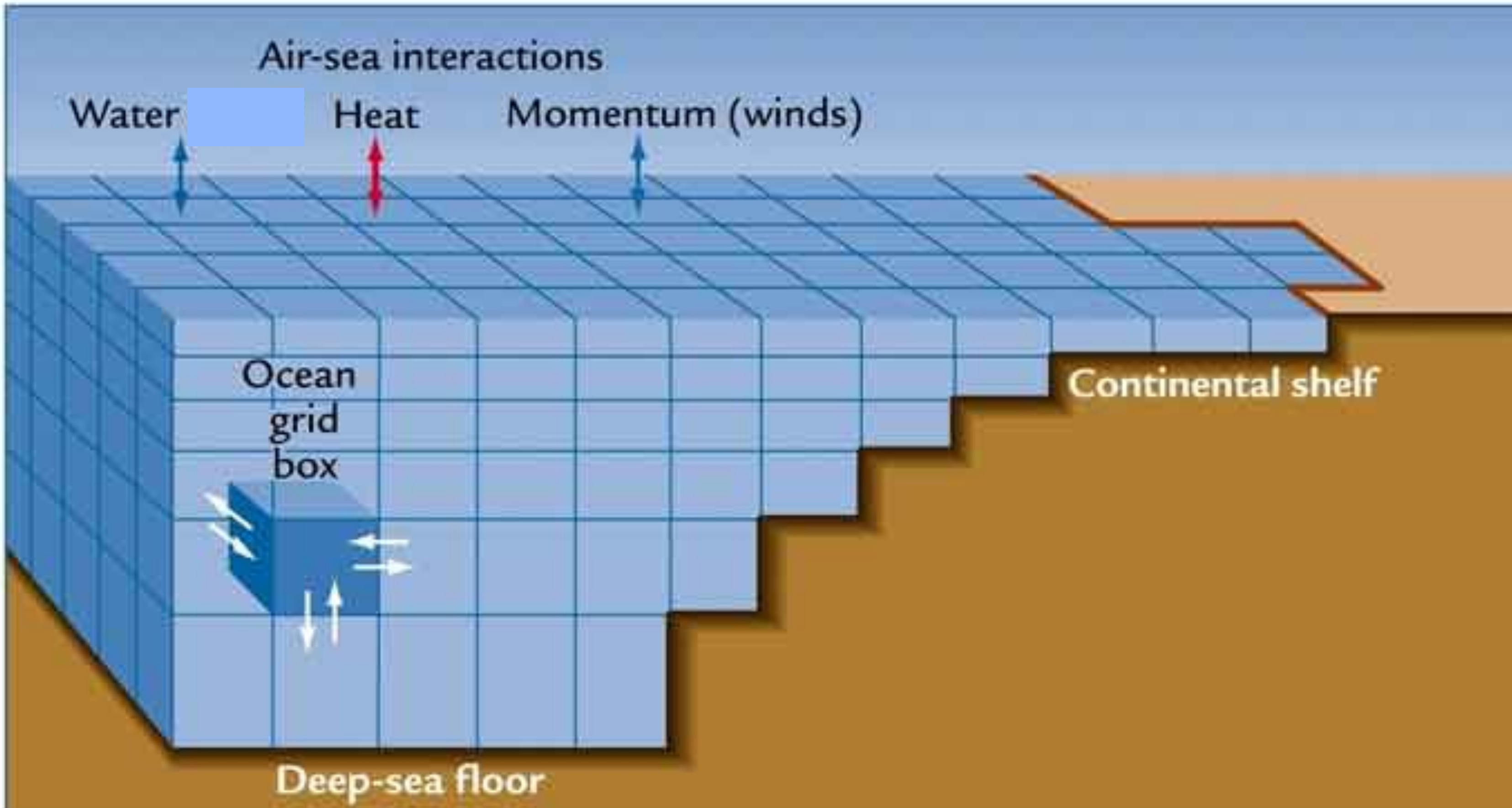
Force terms

Function of Fluid viscosity

Examples of Resolution
(global spectral model, zoom onto Europe)



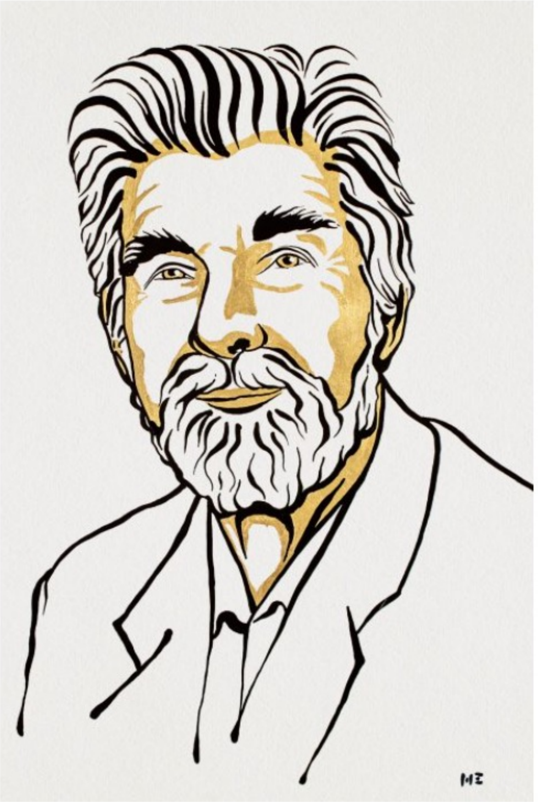
Ocean circulation models and boundary conditions



"for groundbreaking contributions to our understanding of complex systems"

Ill. Niklas Elmehed © Nobel Prize Outreach

Syukuro Manabe

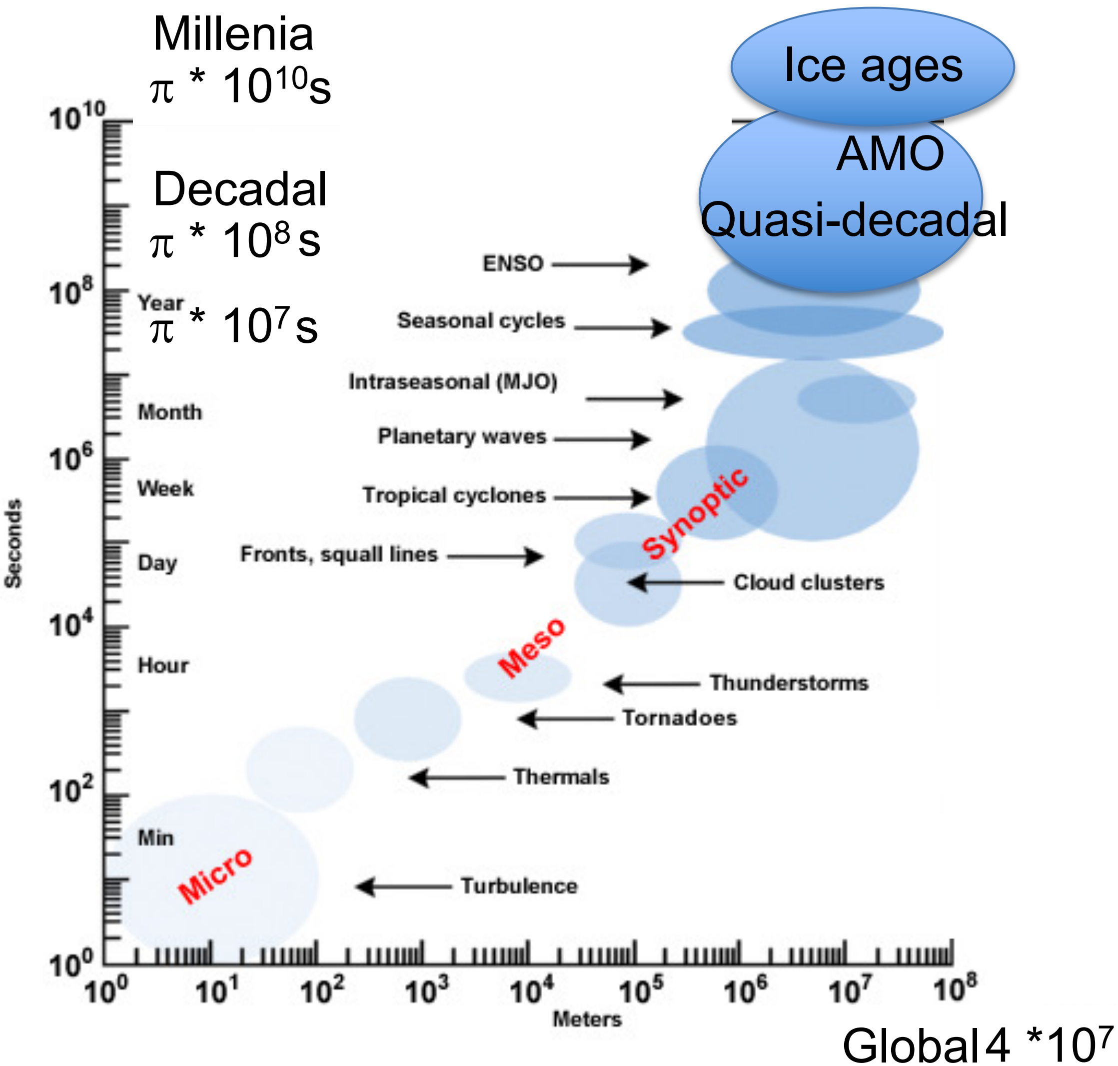


Ill. Niklas Elmehed © Nobel Prize Outreach

Klaus Hasselmann

"for the physical modelling of Earth's climate, quantifying variability and reliably predicting global warming"

Spatio-Temporal Scales



Spatial || temporal Scales

Optimal Fingerprints for the Detection of Time-dependent Climate Change

K. HASSELMANN

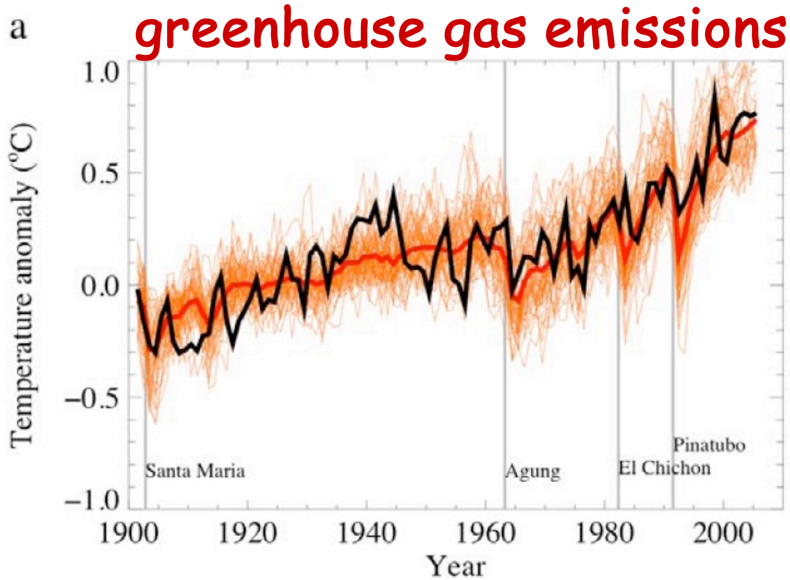
Max-Planck-Institut für Meteorologie, Hamburg, Germany

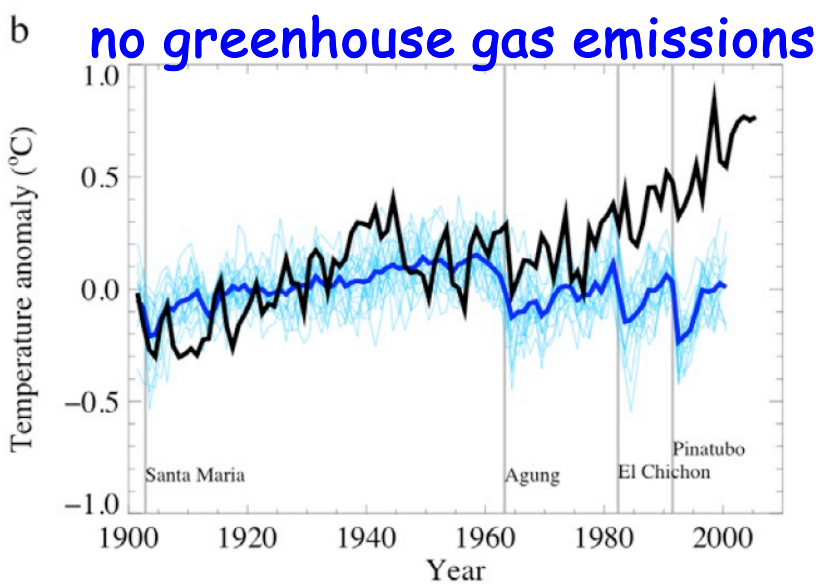
(Manuscript received 24 August 1992, in final form 17 March 1993)

$$f'_a = g'_a \sigma_a^{-2}. \quad (14)$$

The multiplication of the signal with the inverse of the covariance matrix is seen to weight the fingerprint components f'_a in the EOF frame relative to the signal components g'_a by the inverse σ_a^{-2} of the EOF variances, thereby slewing the fingerprint vector away from the EOF directions with high noise levels toward the low-noise directions.

Attribution (model world)

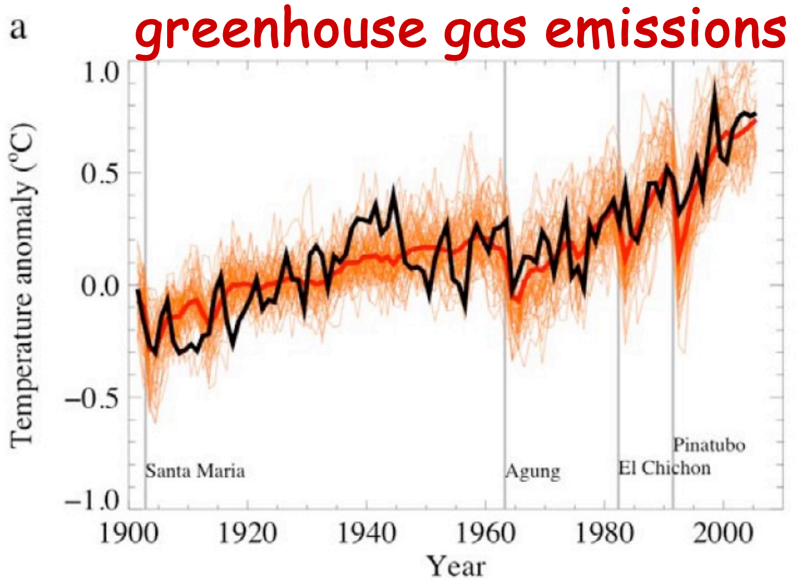




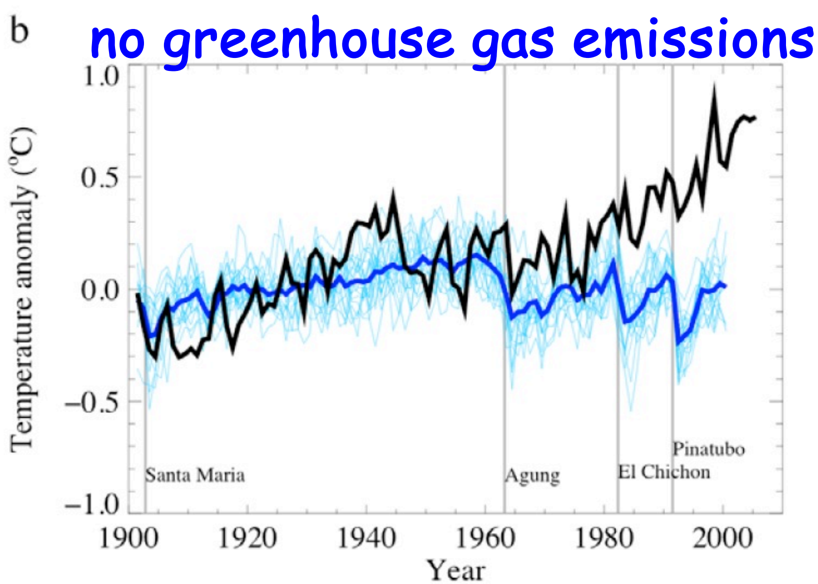
observed changes are consistent with
modeled response to external forcing,
inconsistent with alternative explanations

Nobel Price, 2021
Hasselmann

Attribution (model world)



observed changes are consistent with modeled response to external forcing, inconsistent with alternative explanations



Critics:

- **Time series too short**
- **Estimates of natural variability based only on models**

Stochastic climate model (Hasselmann, 1976)

$$\frac{dT}{dt} = -\lambda T + \text{Noise} + \text{Forcing}$$

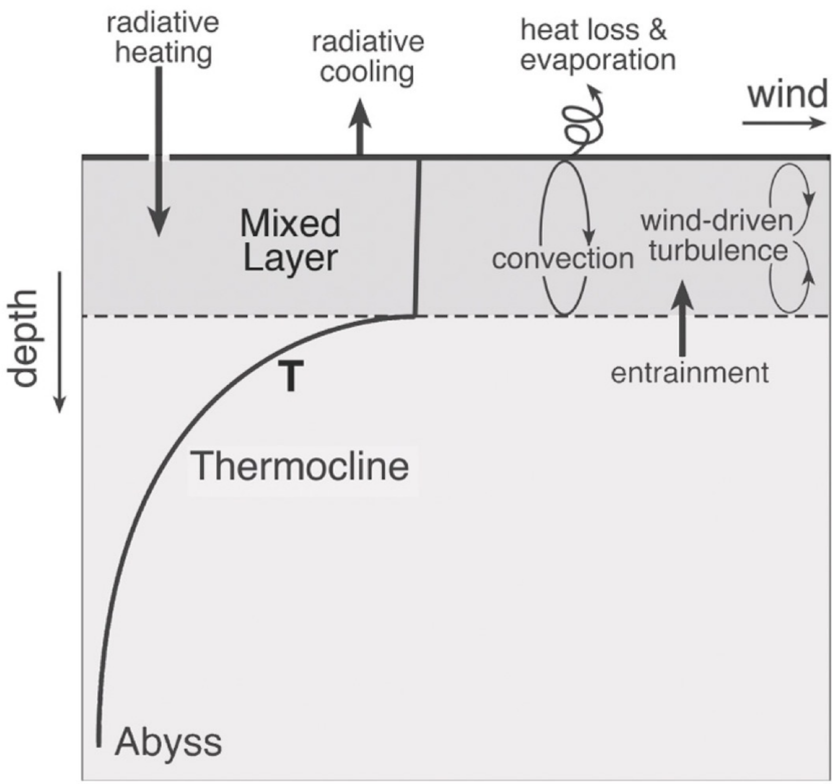
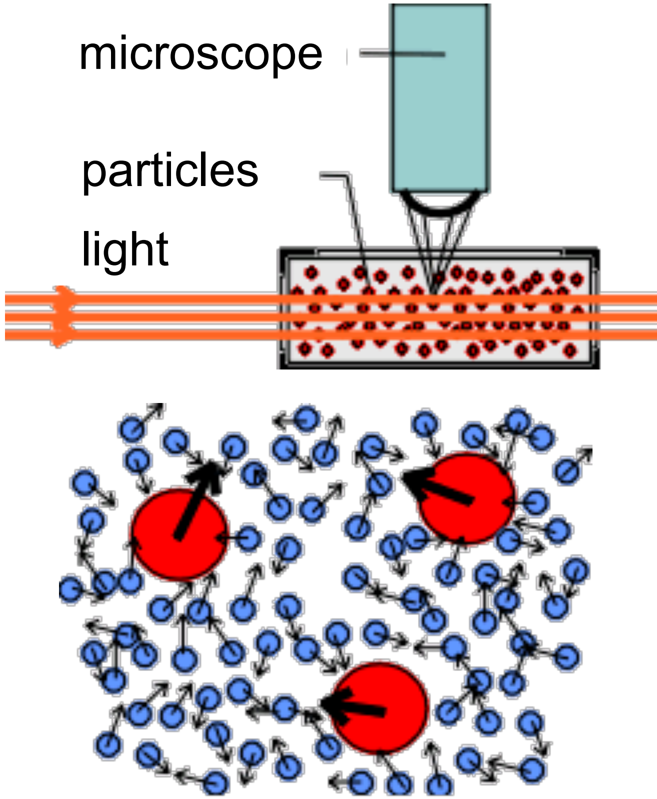
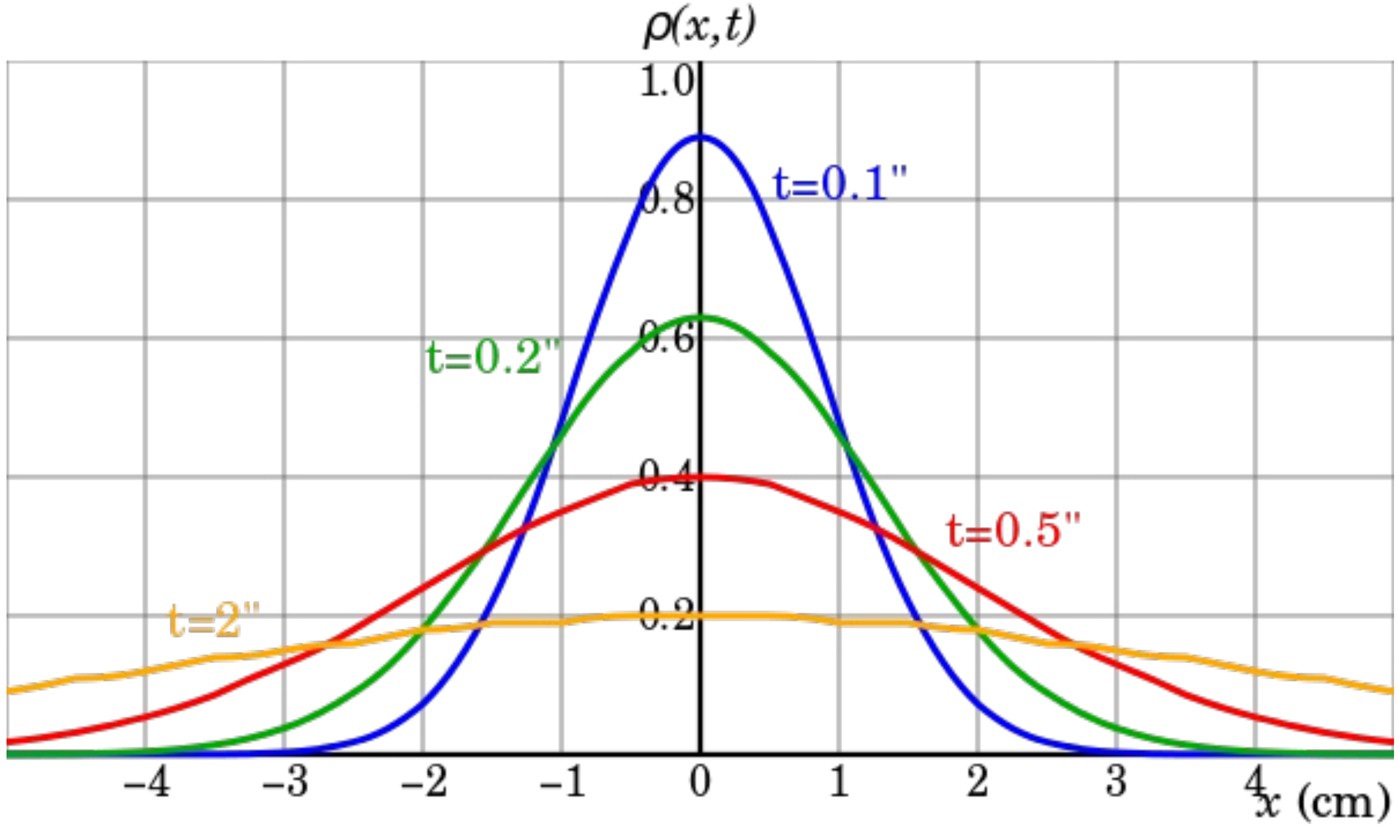


Figure 8.4: Schematic picture of mixed layer in the ocean.



Disorderly, random motion collision with molecules https://www.awi.de/fileadmin/user_upload/AWI/Forschung/Klimawissenschaft/Dynamik_des_Palaeoklimas/RandomSystems/index.html


ρ(x,t)
1.0
0.8
0.6
0.4
0.2
t=0.1"
t=0.2"
t=0.5"
t=2"
−4
−3
−2
−1
0
1
2
3
4
x (cm)

Diffusion of Brownian particles (Einstein)

$$\rho(x,t+\tau) = \rho(x,t) \cdot \int_{-\infty}^{+\infty} \phi(\Delta)\,\mathrm{d}\Delta + \frac{\partial \rho}{\partial x} \cdot \int_{-\infty}^{+\infty} \Delta \cdot \phi(\Delta)\,\mathrm{d}\Delta + \frac{\partial^2 \rho}{\partial x^2} \cdot \int_{-\infty}^{+\infty} \frac{\Delta^2}{2} \cdot \phi(\Delta)\,\mathrm{d}\Delta + \frac{\partial^3 \rho}{\partial x^3} \cdot \int_{-\infty}^{+\infty} \frac{\Delta^3}{3} \cdot \phi(\Delta)\,\mathrm{d}\Delta + \quad \ldots$$

The integral in the first term is equal to one by the definition of probability, and the second and other even terms (i.e. first and other odd moments) vanish because of space symmetry.

$$\rho(x,t+\tau) = \rho(x,t) \cdot 1 + 0 + \frac{\partial^2 \rho}{\partial x^2} \cdot \int_{-\infty}^{+\infty} \frac{\Delta^2}{2} \cdot \phi(\Delta)\,\mathrm{d}\Delta \quad + \quad 0 \quad + \quad \ldots$$

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2} \cdot \int_{-\infty}^{+\infty} \frac{\Delta^2}{2\,\tau} \cdot \phi(\Delta)\,\mathrm{d}\Delta + \text{higher order even moments}$$

Where the second moment of probability of displacement Δ, is interpreted as diffusivity

$$D = \int \frac{\Delta^2}{2\,\tau} \cdot \phi(\Delta)\,\mathrm{d}\Delta$$

Then the density of Brownian particles ρ at point x at time t satisfies the diffusion equation (ignoring higher order terms):

$$\frac{\partial \rho}{\partial t} = D \cdot \frac{\partial^2 \rho}{\partial x^2}$$

5.2 STOCHASTIC PROCESSES

It is now time for time to appear in our discussion of random systems. When it does, this becomes the study of *stochastic processes*. We will look at two ways to bring in time: the evolution of probability distributions for variables correlated in time, and stochastic differential equations.

If $x(t)$ is a time-dependent random variable, its Fourier transform

$$X(\nu) = \lim_{T\to\infty} \int_{-T/2}^{T/2} e^{i2\pi\nu t} x(t)\, dt \tag{5.27}$$

is also a random variable but its *power spectral density* $S(\nu)$ is not:

$$\begin{aligned} S(\nu) &= \langle |X(\nu)|^2 \rangle = \langle X(\nu) X^*(\nu) \rangle \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} e^{i2\pi\nu t} x(t)\, dt \int_{-T/2}^{T/2} e^{-i2\pi\nu t'} x(t')\, dt' \end{aligned} \tag{5.28}$$

(where X^* is the complex conjugate of X, replacing i with $-i$). The inverse Fourier transform of the power spectral density has an interesting form,

$$\begin{aligned} &\int_{-\infty}^{\infty} S(\nu) e^{-i2\pi\nu\tau}\, d\nu \\ &= \int_{-\infty}^{\infty} \langle X(\nu) X^*(\nu) \rangle e^{-i2\pi\nu\tau}\, d\nu \\ &= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \int_{-T/2}^{T/2} e^{i2\pi\nu t} x(t)\, dt \int_{-T/2}^{T/2} e^{-i2\pi\nu t'} x(t')\, dt'\, e^{-i2\pi\nu\tau}\, d\nu \end{aligned}$$

$$
\begin{aligned}
&= \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} e^{i2\pi\nu(t-t'-\tau)}\, d\nu\, x(t)x(t')\, dt\, dt' \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \delta(t-t'-\tau)x(t)x(t')\, dt\, dt' \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t)x(t-\tau)\, dt \\
&= \langle x(t)x(t-\tau)\rangle \quad ,
\end{aligned} \tag{5.29}
$$

found by using the Fourier transform of a *delta function*

$$
\int_{-\infty}^{\infty} e^{-i2\pi\nu t}\delta(t)\, dt = 1 \quad \Rightarrow \quad \delta(t) = \int_{-\infty}^{\infty} e^{i2\pi\nu t}\, dt \quad , \tag{5.30}
$$

where the delta function is defined by

$$
\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\, dx = f(x_0) \quad . \tag{5.31}
$$

This is the *Wiener–Khinchin* theorem. It relates the spectrum of a random process to its *autocovariance function*, or, if it is normalized by the variance, the *autocorrelation function* (which features prominently in time series analysis, Chapter 16).

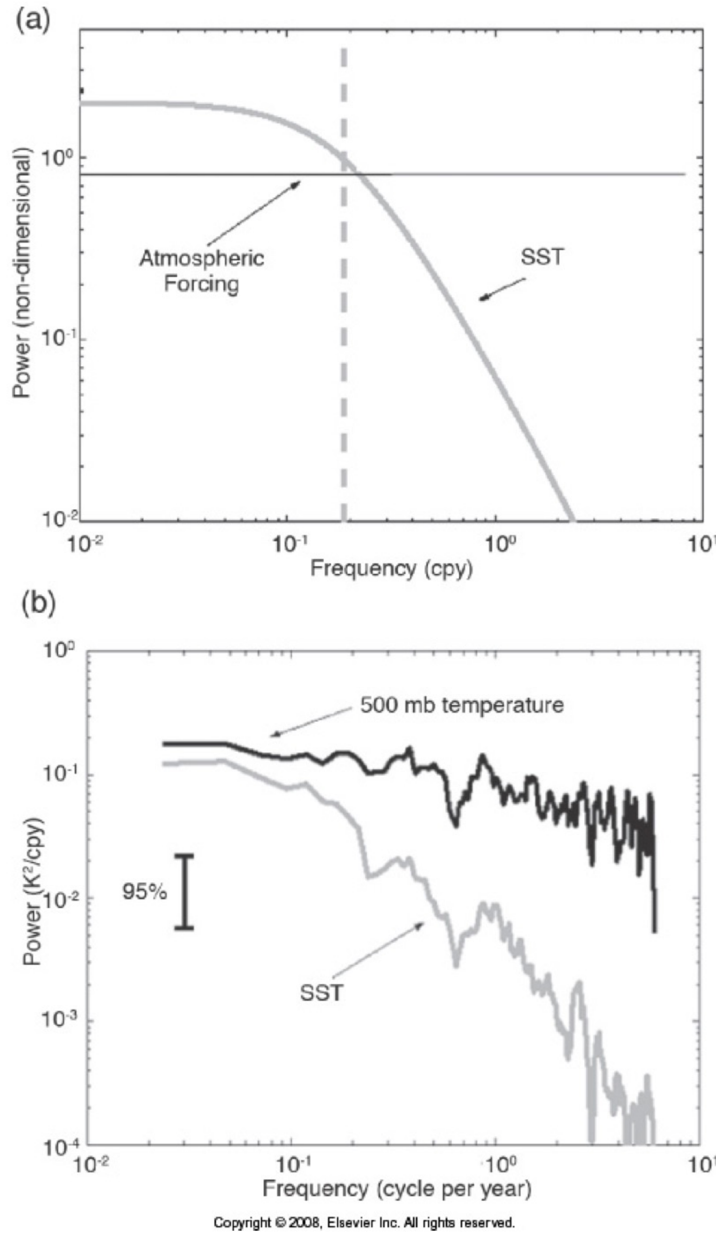


Figure 8.9: Powerspectrum of atmospheric temperature and sea surface temperature. Here $\mathbf{1/\lambda = 300}$ days from equation (8.43).

https://www.awi.de/fileadmin/user_upload/AWI/Forschung/Klimawissenschaft/Dynamik_des_Palaeoklimas/RandomSystems/index.html

https://www.awi.de/fileadmin/user_upload/AWI/Forschung/Klimawissenschaft/Dynamik_des_Palaeoklimas/BrownianMotion/index.html

Energy Balance Model (0D Linear EBM)

We solve the energy balance equation:

$$C\frac{dT(t)}{dt} = F(t) - \lambda T(t)$$

For a step-like forcing $F(t) = F_0 \cdot H(t)$, the solution using the Laplace transform is:

$$T(t) = \frac{F_0}{\lambda}\left(1 - e^{-\frac{\lambda}{C}t}\right) + T_0 \cdot e^{-\frac{\lambda}{C}t}$$

This model describes the delayed temperature response of the Earth's climate system due to thermal inertia.

Climate Sensitivity: CS= $\dfrac{F_0}{\lambda}$

📘 General Form of LRT:

$$R(t) = \int_0^t G(t - \tau) F(\tau) \, d\tau$$

- $F(t)$: forcing (e.g., radiative input)
- $G(t)$: **Green's function** (response to a delta impulse)
- $R(t)$: response (e.g., temperature anomaly)

$$\frac{dT(t)}{dt} = F(t) - \lambda T(t)$$

✅ Laplace Transform:

$$s\hat{T}(s) - T(0) = \hat{F}(s) - \lambda\hat{T}(s) \Rightarrow (s+\lambda)\hat{T}(s) = \hat{F}(s) + T(0)$$

If we assume $T(0) = 0$:

$$\hat{T}(s) = \frac{1}{s+\lambda}\hat{F}(s)$$

So the **transfer function** is:

$$\hat{G}(s) = \frac{1}{s+\lambda}$$

⏱️ Impulse Response Function:

Taking the inverse Laplace transform:

$$G(t) = e^{-\lambda t}$$

This is the **Green's function**: the system's response to a unit impulse of forcing at $t = 0$.

🧠 Interpretation in LRT Terms

Concept	Meaning in EBM with $C = 1$
Forcing $F(t)$	Radiative input
Response $T(t)$	Global mean temperature anomaly
Green's function $G(t)$	$e^{-\lambda t}$ (exponential decay)
Transfer function $\hat{G}(s)$	$\frac{1}{s+\lambda}$
Time constant	$\tau = \frac{1}{\lambda}$

✅ Example: Step Forcing

Let's solve for $T(t)$ when $F(t) = F_0 \cdot H(t)$, a step function:

Laplace of forcing:

$$\hat{F}(s) = \frac{F_0}{s}$$

Then:

$$\hat{T}(s) = \frac{F_0}{s(s+\lambda)} = \frac{F_0}{\lambda}\left(\frac{1}{s} - \frac{1}{s+\lambda}\right) \Rightarrow T(t) = \frac{F_0}{\lambda}\left(1 - e^{-\lambda t}\right)$$

📊 Interpretation

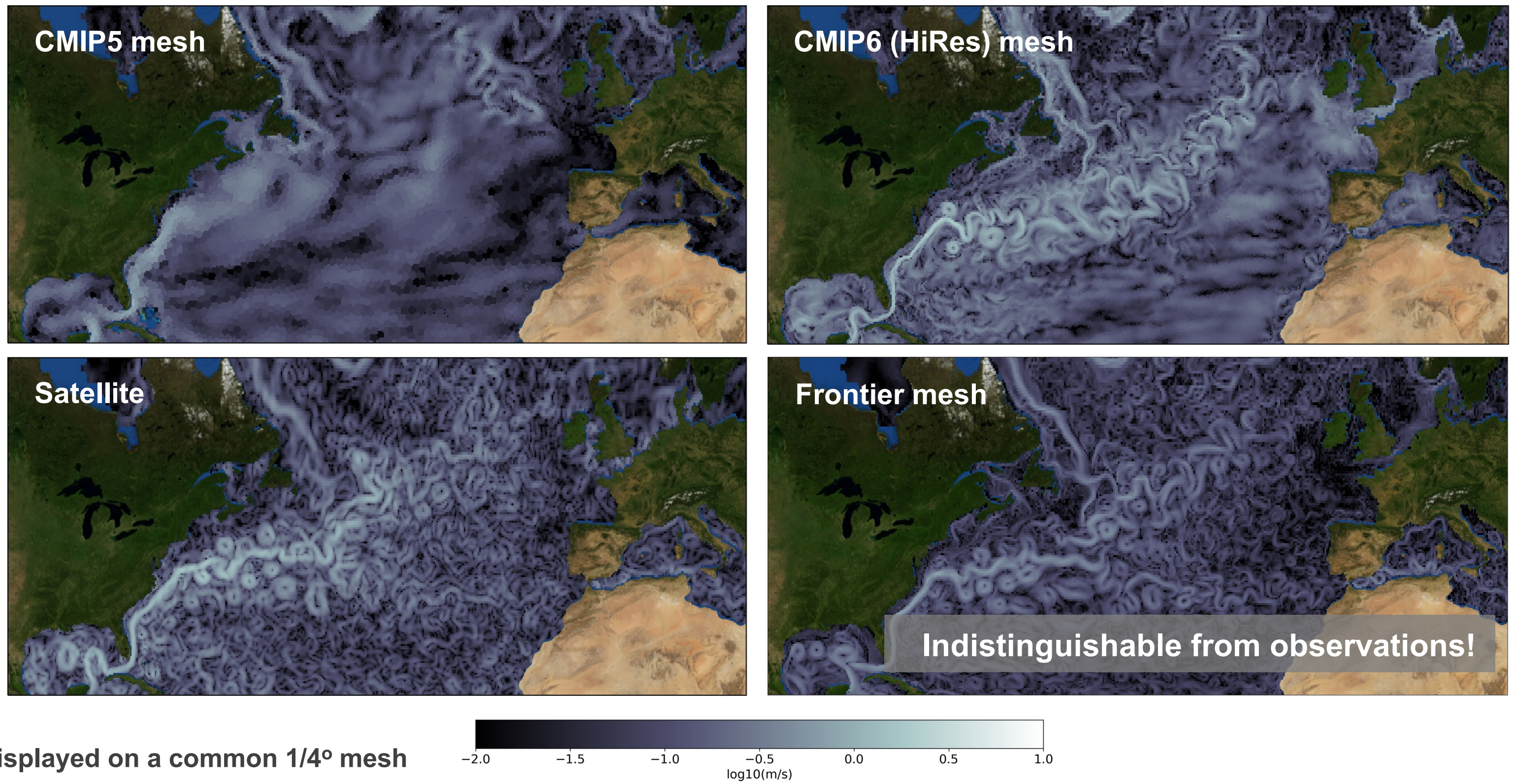
- Temperature gradually increases toward an **equilibrium** of $T_\infty = \frac{F_0}{\lambda}$
- The **adjustment timescale** is $\tau = \frac{1}{\lambda}$
- The full temperature evolution is a **convolution** of the forcing with the Green's function:

$$T(t) = \int_0^t e^{-\lambda(t-\tau)} F(\tau)\, d\tau$$

This leads to a **linear response**:

$$T(t) = \int_0^t G(t-\tau) F(\tau)\, d\tau$$

How realistic is the model?



Displayed on a common 1/4º mesh

Ocean velocity