Mathematics Introductory Course for the Master Programme "Environmental Physics"

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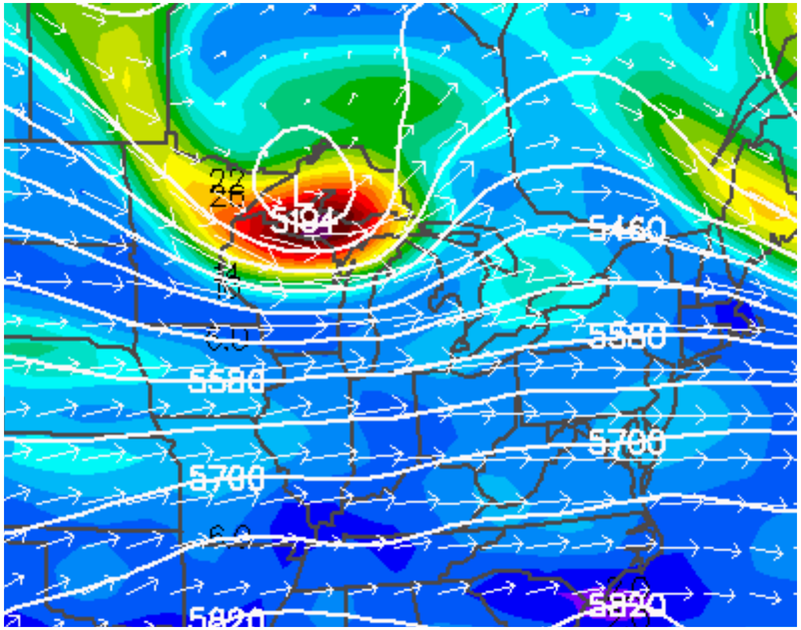
General: vector calculus (Gerrit Lohmann)
Geophysics (Klaus Grosfeld)
Applications (Thomas Laepple)

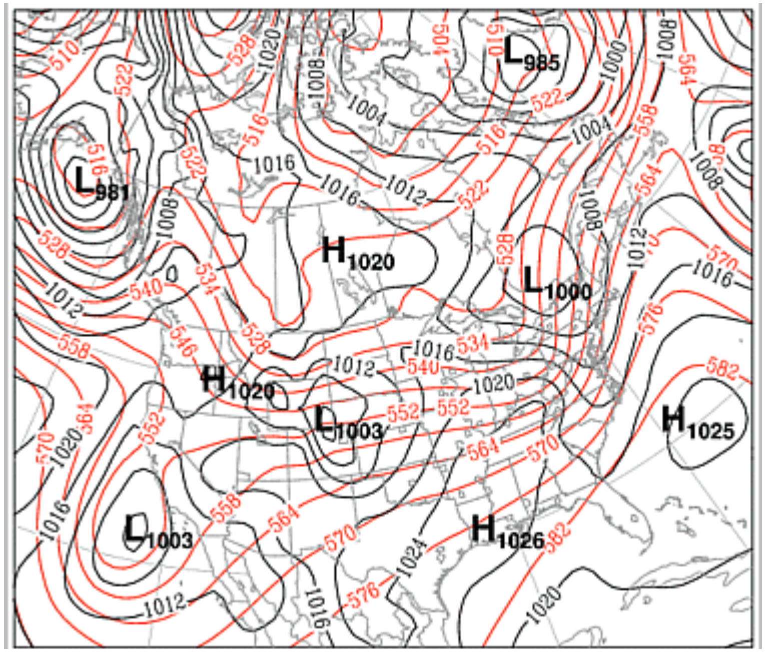
Introduction into "R"
Numerical solutions of vector field problems
The Stokes' Theorem
Worksheet: Exercise 1 , Exercise 2 , Exercise 4 , commands

http://www.awi.de/en/go/paleo/lectures/mathematics_introductory_course/

What is dynamics?

- Definition: The study of atmospheric and oceanic motions, with emphasis on the physical laws that govern such motions.





Basic Laws

- Conservation of mass (continuity equation)
- Conservation of energy (1st law of thermodynamics)
- Newton’s 1st Law (no resultant force → no change in motion)
- Newton’s 2nd Law (rate of change of motion of a body is proportional to resultant force acting on it)
- Conservation of angular momentum
- Newton’s Law of Gravitation
- Ideal Gas Law (equation of state: atmosphere)

Fundamental Mathematical Concepts and Operations

- Fundamental state variables such as wind speed, temperature and pressure are functions of (i.e., depend upon) the independent variables (x, y, z, t).
- For example, atmospheric pressure can be expressed as a function of space and time:

$$P = P(x, y, z, t)$$

Expansion of Total Derivative

If $f = f(x, y, z, t)$ then

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

with $u \equiv \dfrac{dx}{dt}, \quad v \equiv \dfrac{dy}{dt}, \quad w \equiv \dfrac{dz}{dt}$

u = west-east component of fluid velocity
v = south-north component of fluid velocity
w = vertical component of fluid velocity

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\overset{u}{\boxed{\frac{dx}{dt}}} + \frac{\partial f}{\partial y}\overset{v}{\boxed{\frac{dy}{dt}}} + \frac{\partial f}{\partial z}\overset{w}{\boxed{\frac{dz}{dt}}}$$

Euler’s relation (expansion of total derivative):

$$\underset{\text{A}}{\frac{df}{dt}} = \underset{\text{B}}{\frac{\partial f}{\partial t}} + \underset{\text{C}}{u\frac{\partial f}{\partial x}} + \underset{\text{D}}{v\frac{\partial f}{\partial y}} + \underset{\text{E}}{w\frac{\partial f}{\partial z}}$$

Term A: Total rate of change of *f* following the fluid motion

Term B: Local rate of change of *f* at a fixed location

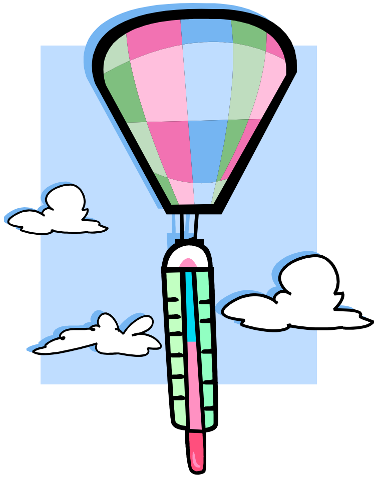
Term C: Advection of *f* in *x* direction by the *x*-component flow

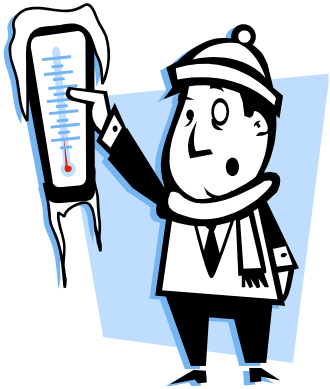
Term D: Advection of *f* in *y* direction by the *y*-component flow

Term E: Advection of *f* in *z* direction by the *z*-component flow

Total Derivative vs. Local Derivative

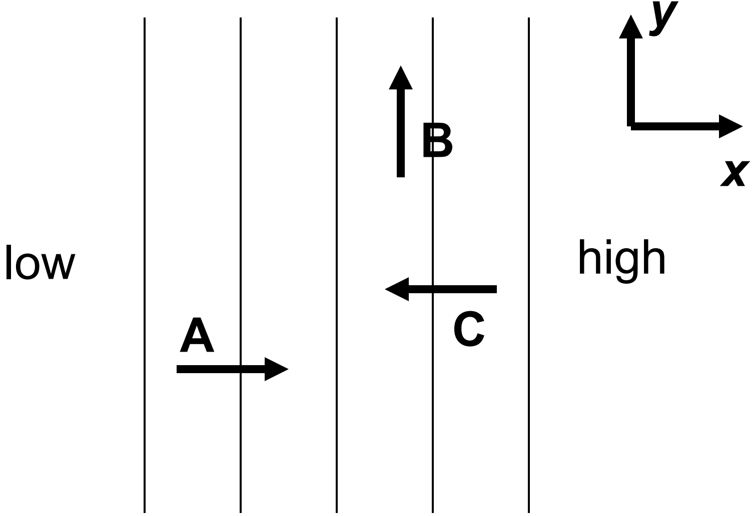
Total derivative is the temporal rate of change following the fluid motion. Example: A thermometer measuring changes as a balloon floats through the atmosphere.





Local derivative is the temporal rate of change at a fixed point. Example: An observer measures changes in temperature at a weather station.

Advection Terms



Assume that thin lines are contours of a scalar quantity *f* and thick arrows indicate the fluid motion. We wish to evaluate the advection term $u\frac{\partial f}{\partial x}$

At point A: $u > 0, \frac{\partial f}{\partial x} > 0 \rightarrow u\frac{\partial f}{\partial x} > 0$ ➔ **Transport from low to high: "negative advection of *f*"**

At point B: $u = 0, \frac{\partial f}{\partial x} > 0 \rightarrow u\frac{\partial f}{\partial x} = 0$ ➔ **"neutral advection of *f*"**

At point C: $u < 0, \frac{\partial f}{\partial x} > 0 \rightarrow u\frac{\partial f}{\partial x} < 0$ ➔ **Transport from high to low: "positive advection of *f*"**

Uses of the gradient operator

If $f = f(x,y,z,t)$ is a scalar function, then

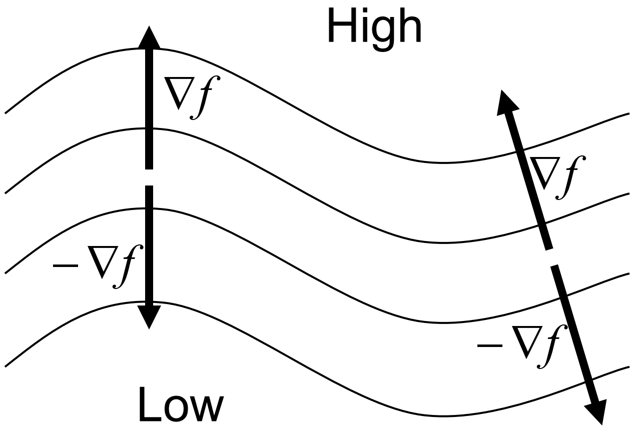
$$\nabla_3 f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

indicates gradient is computed in 3 dimensions

horizontal gradient

$\nabla_2 f$

vertical gradient



∇f is a vector that points in the direction of most rapid increase of f at a given point, and ∇f points from low to high values of f.

Calculate gradient of a function!

- $F(x,y,z)= 3x^2 -6 xy + y^2 z$

Euler’s relation (expansion of total derivative):

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z}$$

$$u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} = \vec{V} \cdot \nabla f$$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \vec{V} \cdot \nabla f$$

Divergence of a Vector

$$div\ \vec{A} = \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

(Divergence is a scalar quantity.)

Example (divergence of the wind):

$$\nabla \cdot \vec{V} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Laplacian of a Scalar

$$\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

(The Laplacian is a scalar quantity.)

Calculate divergence operator !

- $\vec{\mathbf{F}}(x,y,z)= (x^2 y, x z, - y z^2)$

Calculate Laplace operator !

$g(x,y,z)= x^2 * y * \cos(x) + x z - y z^2 - \ln(z)$

Continuum Mechanics

- we do not treat the fluid as a collection of individual molecules.
- we treat the fluid as a continuum in which a “point” is a volume element that is *very small compared to the total fluid volume* but still contains a *very large number of molecules.*
- These volume elements are commonly called *“air/water parcels”* or *“air/water particles.”*
- The properties of these volume elements describe the state of the system.

Newton’s Second Law

- motions are governed by Newton’s second law of motion, which states that the rate of change of momentum of an object equals the sum of all the forces acting.

$$\frac{d}{dt}(mv)=\sum F$$

$$m\frac{dv}{dt}=\sum F$$

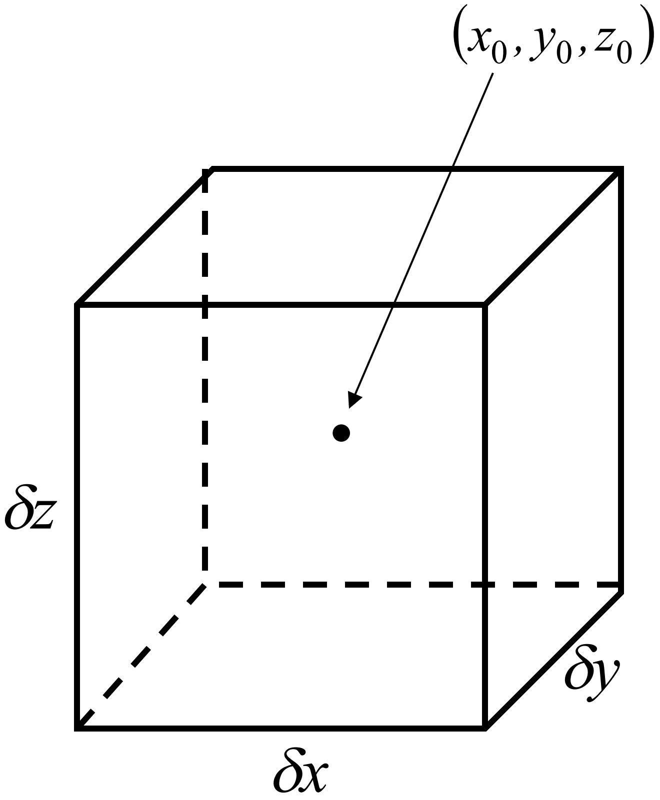
$$ma=\sum F$$

Pressure Gradient Force

We consider a very small volume element of air

$$\delta V = \delta x \, \delta y \, \delta z$$

that is centered at the point $\left(x_0, y_0, z_0\right)$

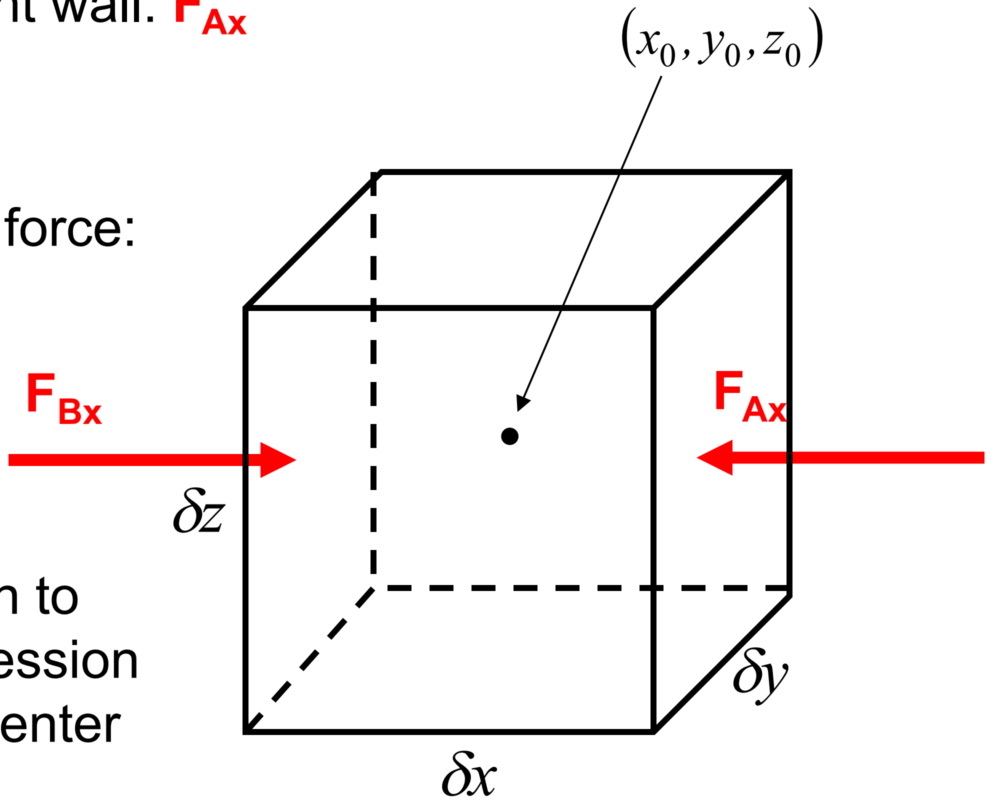


Pressure force exerted on left wall: **F_{Bx}**

Pressure force exerted on right wall: **F_{Ax}**

Net x component of pressure force:

$F_x = F_{Ax} + F_{Bx}$



Method: Use Taylor expansion to develop a mathematical expression for the pressure force at the center of this fluid element.

Taylor series expansion:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + higher\ order\ terms$$

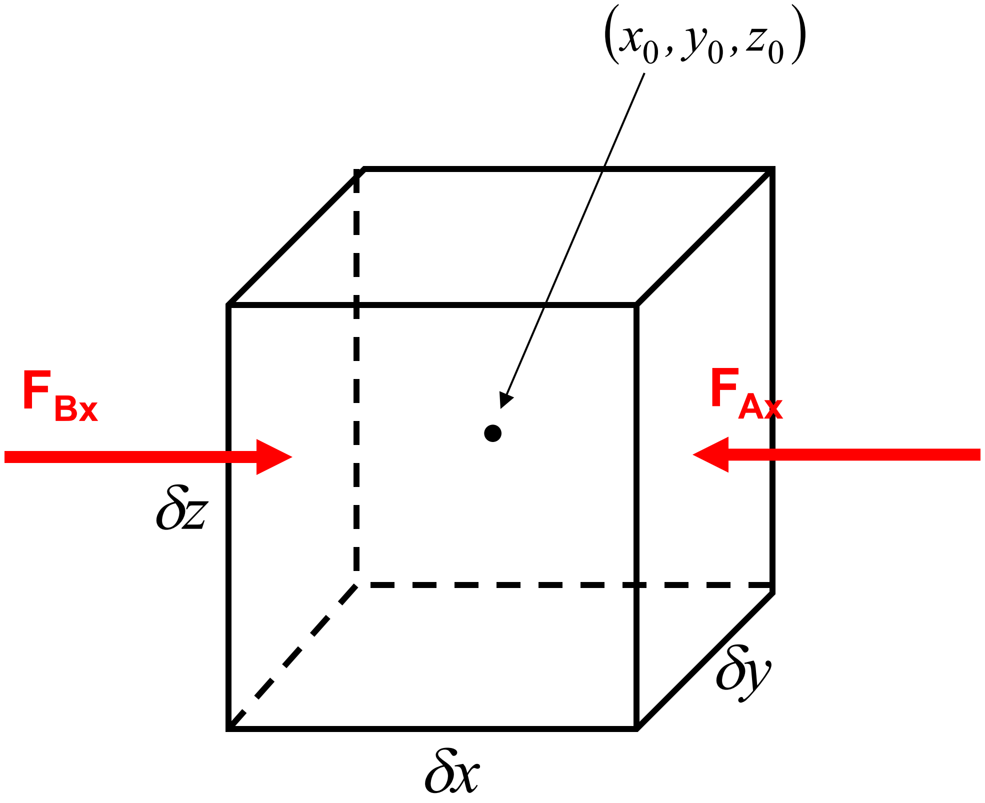
In this case (neglecting higher order terms):

$$p(x) = p(x_0) + \frac{\partial p}{\partial x}(x - x_0)$$

Therefore, we can express the pressure forces as

$$F_{Ax} = -\left(p_0 + \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\ \delta z$$

$$F_{Bx} = +\left(p_0 - \frac{\partial p}{\partial x}\frac{\delta x}{2}\right)\delta y\ \delta z$$



$$F_{Ax} = -\left(p_0 + \frac{\partial p}{\partial x} \frac{\delta x}{2} \right) \delta y \; \delta z$$

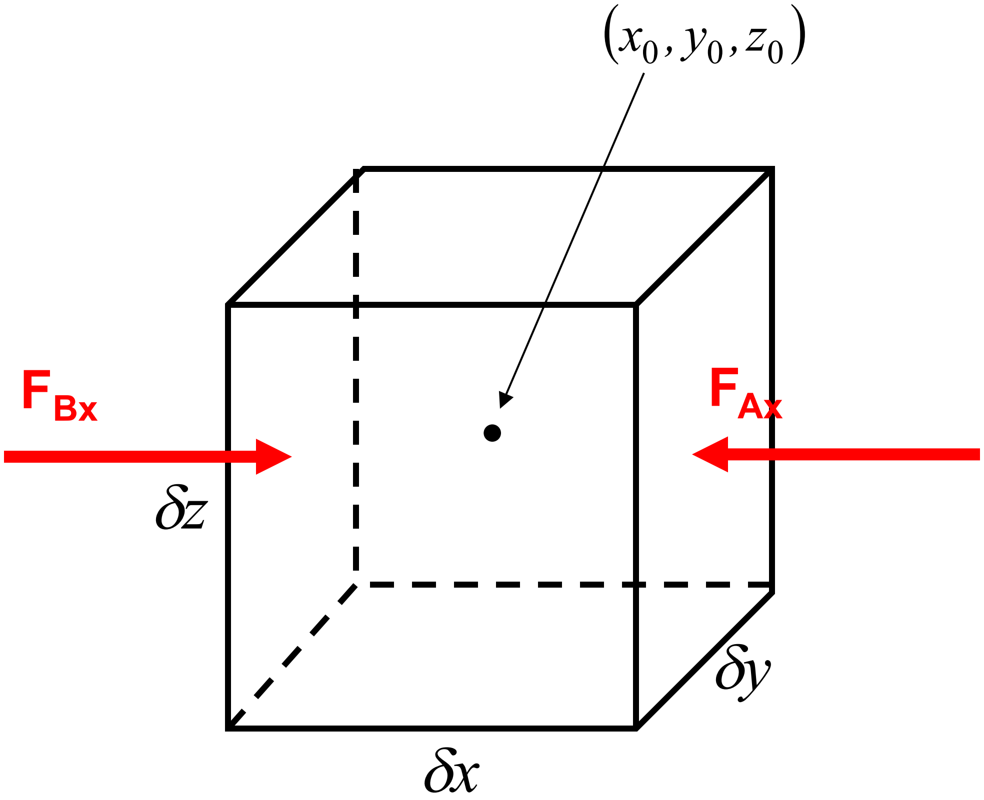
$$F_{Bx} = +\left(p_0 - \frac{\partial p}{\partial x} \frac{\delta x}{2} \right) \delta y \; \delta z$$

$$F_x = F_{Ax} + F_{Bx} = -\frac{\partial p}{\partial x} \delta x \; \delta y \; \delta z$$

mass = density x volume

$$m = \rho \; \delta x \; \delta y \; \delta z$$

$$\frac{F_x}{m} = -\frac{1}{\rho} \frac{\partial p}{\partial x}$$



$$\frac{F_x}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

In the same manner, it can be shown that

$$\frac{F_y}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{F_z}{m} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

In vector form:

$$\frac{\vec{F}}{m} = -\frac{1}{\rho}\nabla p$$

Note: Pressure force is proportional to *gradient* of pressure.

Divergence Theorem

One of the most important theorems in vector analysis is known as the Divergence Theorem. This is essentially just an application of the fundamental theorem of calculus

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b)-f(a)$$

This enables us to express the integral of the quantity df/dx along an interval in terms of the values of f itself at the endpoints of that interval. Now suppose we are given a scalar function f(x,y,z) throughout a region V enclosed by a surface S, and we want to evaluate the integral of the quantity f over this entire region. This can be written as

$$\int_V \frac{\partial f}{\partial x}\,dV = \iiint_{z\,y\,x} \frac{\partial f}{\partial x}\,dx\,dy\,dz$$

where the three integrals are evaluated over suitable ranges to cover the entire region V.

Divergence theorem examples

Compute $\iint_S \mathbf{F} \cdot d\mathbf{S}$ where

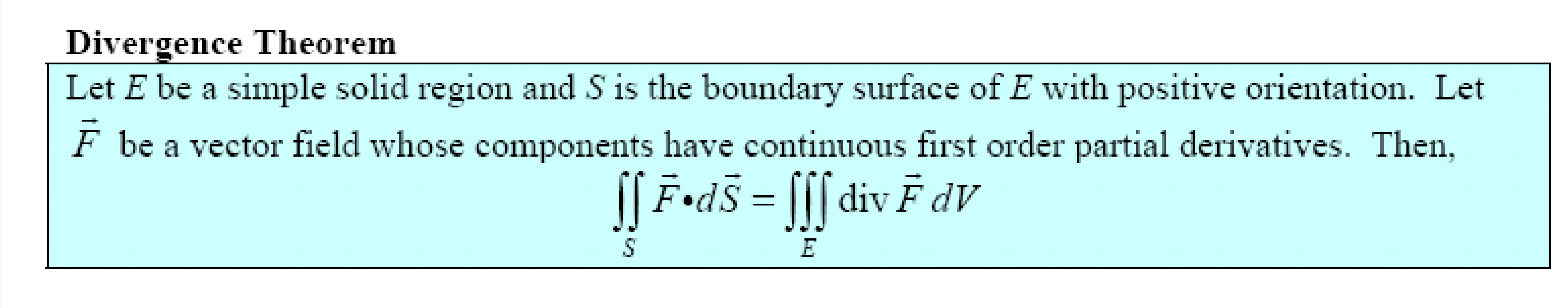
$$\mathbf{F} = (3x + z^{77}, y^2 - \sin x^2 z, xz + ye^{x^5})$$

and S is surface of box

$$0 \le x \le 1, \quad 0 \le y \le 3, \quad 0 \le z \le 2.$$

Use outward normal **n**.

Convert 2 to 3 dim. surfaces



Divergence Theorem

Let E be a simple solid region and S is the boundary surface of E with positive orientation. Let $\vec{F}$ be a vector field whose components have continuous first order partial derivatives. Then,

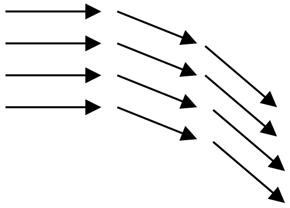
$$\iint_S \vec{F} \bullet d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV$$

The Divergence Theorem relates relates volume integrals to surface integrals of vector fields.

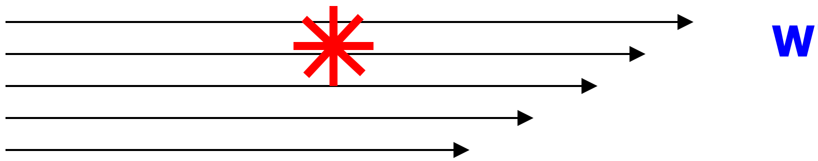
Vorticity

Vorticity is the microscopic measure of spin and rotation in a fluid.

Vorticity is defined as the curl of the velocity: $\nabla \times \vec{V}$



Wind direction varies → clockwise spin



Wind speed varies → clockwise spin

Absolute vorticity (inertial reference frame): $\vec{\omega}_a \equiv \nabla \times \vec{V}_a$

Relative vorticity (relative to rotating earth): $\vec{\omega} \equiv \nabla \times \vec{V}$

Expansion of relative vorticity into Cartesian components:

$$\nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

$$\nabla \times \vec{V} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right)\hat{k}$$

For large scale dynamics, the vertical component of vorticity is most important. The vertical components of absolute and relative vorticity in vector notation are:

$$\zeta = \hat{k} \cdot \left(\nabla \times \vec{V} \right)$$ relative vorticity

$$\eta = \hat{k} \cdot \left(\nabla \times \vec{V}_a \right)$$ absolute vorticity

From now on, vorticity implies the vertical component (unless otherwise stated.)

The absolute vorticity is equal to the relative vorticity plus the earth’s vorticity. Since the earth’s vorticity is

$$\hat{k}\cdot\left(\nabla\times\vec{V}_e\right)=2\Omega\, sin\,\phi=f$$

then

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \quad \text{and} \quad \eta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+f=\zeta+f$$

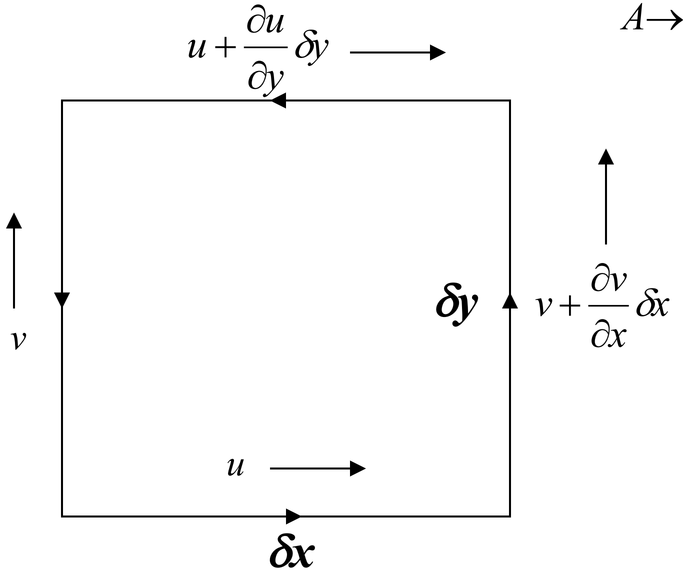
For large scale circulations, a typical magnitude for vorticity is

$$\zeta\approx\frac{U}{L}=10^{-5}\ s^{-1}$$

Circulation and Vorticity

The relationship between relative vorticity ζ and circulation can be seen by considering the following expression, in which we will define the relative vorticity as the circulation about a closed contour in the horizontal plane divided by the area enclosed by that contour, in the limit as the area approaches zero.

$$\zeta = \lim_{A\to 0}\left(\oint \vec{V}\cdot d\vec{l}\,\right)A^{-1}$$



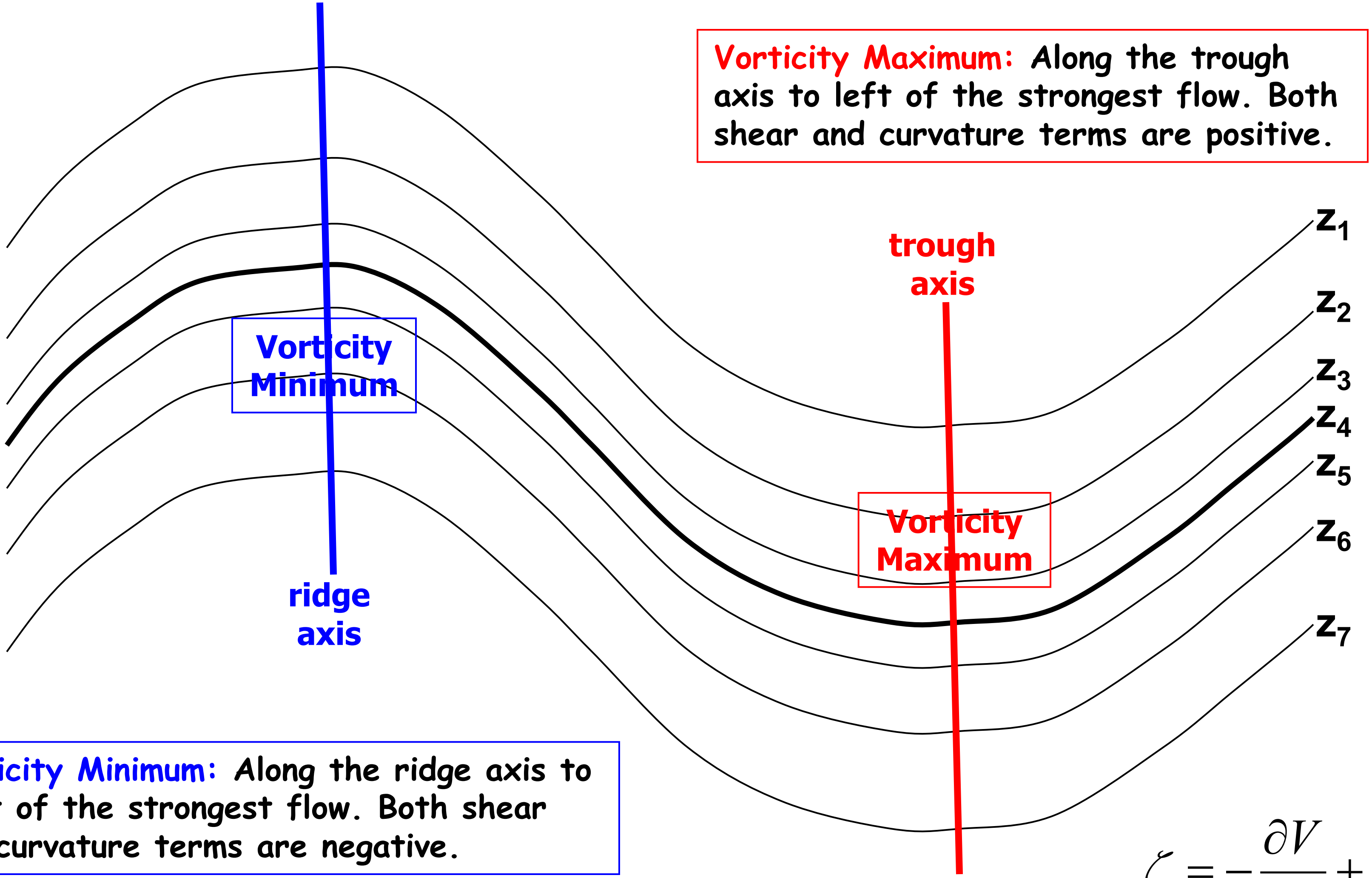
Evaluating $\vec{V}\cdot d\vec{l}$ for each side of the rectangle yields the circulation:

$$\delta C = u\,\delta x + \left(v + \frac{\partial v}{\partial x}\delta x\right)\delta y - \left(u + \frac{\partial u}{\partial y}\delta y\right)\delta x - v\,\delta y$$

$$\delta C = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\delta x\delta y \rightarrow \frac{\delta C}{\delta A} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \zeta$$

Vorticity On The Weather Map

Vorticity Maximum: Along the trough axis to left of the strongest flow. Both shear and curvature terms are positive.



Vorticity Minimum: Along the ridge axis to right of the strongest flow. Both shear and curvature terms are negative.

$$\zeta = -\frac{\partial V}{\partial n} + \frac{V}{R_s}$$

Potential Vorticity

Adiabatic flow can be described by Kelvin’s circulation theorem:

$$\frac{d}{dt}\left(C + 2\Omega\,\delta A \sin\phi\right) = 0$$

where C is evaluated for a closed loop encompassing the area δA on an isentropic surface.

The vertical component of vorticity is given by $\zeta = \lim_{A \to 0} \frac{C}{\delta A}$,

thus if the isentropic surface is approximately horizontal, for an infinitesimal parcel of air:

$$\frac{d}{dt}\left(\delta A\left(\zeta_\theta + f\right)\right) = 0 \rightarrow \delta A\left(\zeta_\theta + f\right) = const$$

relative vorticity on an isentropic surface

Coriolis parameter

The Vorticity Equation

To understand the processes that produce changes in vorticity, we would like to derive an expression that includes the time derivative of vorticity:

$$\frac{d}{dt}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=\ldots$$

Recall that the momentum equations are of the form

$$\frac{du}{dt}=\ldots$$ x-component momentum equation

$$\frac{dv}{dt}=\ldots$$ y-component momentum equation

Thus we will begin our derivation by taking

$$\frac{\partial}{\partial x}[\text{y-component momentum equation}]-\frac{\partial}{\partial y}[\text{x-component momentum equation}]$$

$$\frac{\partial}{\partial x}[\text{y-component momentum equation}]-\frac{\partial}{\partial y}[\text{x-component momentum equation}]=$$

$$\frac{\partial}{\partial x}\left[\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}\right]-\frac{\partial}{\partial y}\left[\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}\right]$$

$$\underline{\frac{\partial}{\partial x}\frac{\partial v}{\partial t}}+\underline{u\frac{\partial^2 v}{\partial x^2}}+\underline{\frac{\partial v}{\partial x}\frac{\partial u}{\partial x}}+\underline{v\frac{\partial^2 v}{\partial x\partial y}}+\underline{\frac{\partial v}{\partial y}\frac{\partial v}{\partial x}}+\underline{w\frac{\partial^2 v}{\partial x\partial z}}+\underline{\frac{\partial v}{\partial z}\frac{\partial w}{\partial x}}+\underline{f\frac{\partial u}{\partial x}}+u\cancel{\frac{\partial f}{\partial x}}=-\cancel{\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}}+\underline{\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}\right)}$$

$$-\quad\underline{\frac{\partial}{\partial y}\frac{\partial u}{\partial t}}+\underline{u\frac{\partial^2 u}{\partial x\partial y}}+\underline{\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}}+\underline{v\frac{\partial^2 u}{\partial y^2}}+\underline{\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}}+\underline{w\frac{\partial^2 u}{\partial y\partial z}}+\underline{\frac{\partial u}{\partial z}\frac{\partial w}{\partial y}}-\underline{f\frac{\partial v}{\partial y}}-\underline{v\frac{\partial f}{\partial y}}=-\cancel{\frac{1}{\rho}\frac{\partial^2 p}{\partial x\partial y}}+\underline{\frac{1}{\rho^2}\left(\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)}$$

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}+\boxed{u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}+\boxed{v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}+\boxed{w\frac{\partial}{\partial z}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)}+\boxed{\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}+\boxed{f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}$$

$$+\boxed{\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)}+\boxed{v\frac{\partial f}{\partial y}}=\boxed{\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)}$$

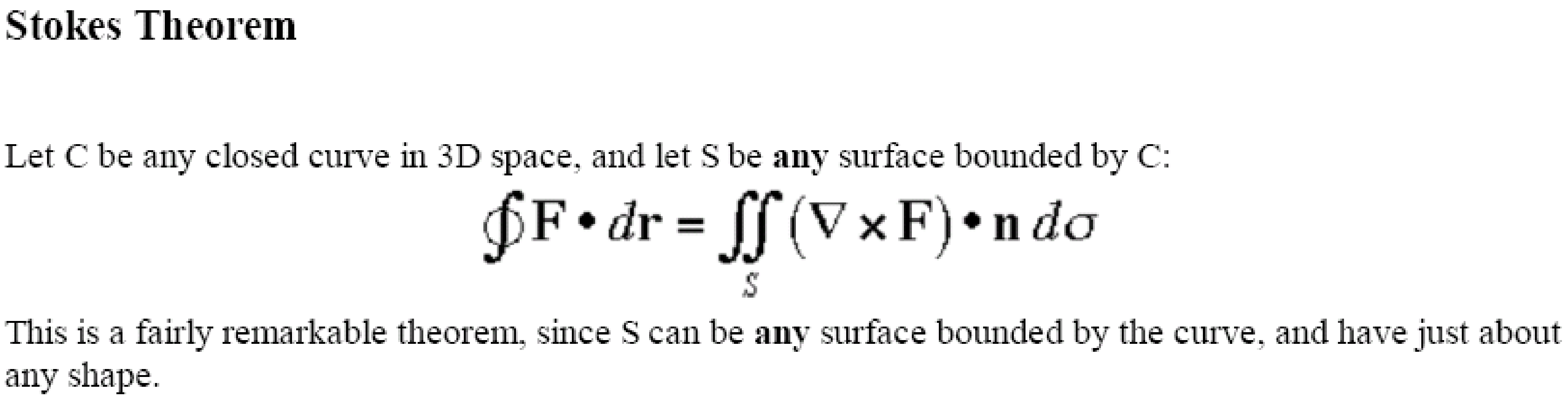
$$\boxed{\frac{df}{dt}=\cancel{\frac{\partial f}{\partial t}}+u\cancel{\frac{\partial f}{\partial x}}+v\frac{\partial f}{\partial y}+w\cancel{\frac{\partial f}{\partial z}}}$$

$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial y}+w\frac{\partial \zeta}{\partial z}+\zeta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+f\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+v\frac{\partial f}{\partial y}=\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)$$

$$\boxed{\frac{d}{dt}(\zeta+f)=-(\zeta+f)\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\left(\frac{\partial w}{\partial x}\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}\frac{\partial u}{\partial z}\right)+\frac{1}{\rho^2}\left(\frac{\partial p}{\partial y}\frac{\partial \rho}{\partial x}-\frac{\partial p}{\partial x}\frac{\partial \rho}{\partial y}\right)}$$

vorticity equation

Stokes Theorem



Stokes Theorem

Let C be any closed curve in 3D space, and let S be **any** surface bounded by C:

$$\oint \mathbf{F} \bullet d\mathbf{r} = \iint_{S} (\nabla \times \mathbf{F}) \bullet \mathbf{n}\, d\sigma$$

This is a fairly remarkable theorem, since S can be **any** surface bounded by the curve, and have just about any shape.

Stokes' Theorem

Consider a vector field $\mathbf{B}(\overline{r})$ where:

$$\mathbf{B}(\overline{r}) = \nabla \times \mathbf{A}(\overline{r})$$

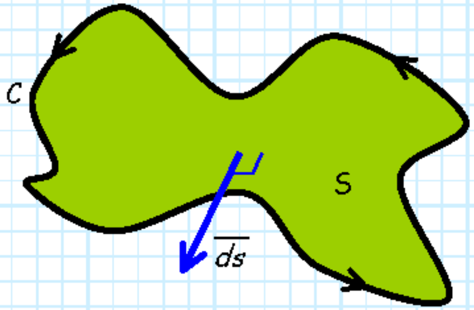
Say we wish to integrate this vector field over an **open** surface S:

$$\iint_S \mathbf{B}(\overline{r}) \cdot \overline{ds} = \iint_S \nabla \times \mathbf{A}(\overline{r}) \cdot \overline{ds}$$

We can likewise evaluate this integral using **Stokes' Theorem**:

$$\boxed{\iint_S \nabla \times \mathbf{A}(\overline{r}) \cdot \overline{ds} = \oint_C \mathbf{A}(\overline{r}) \cdot \overline{d\ell}}$$

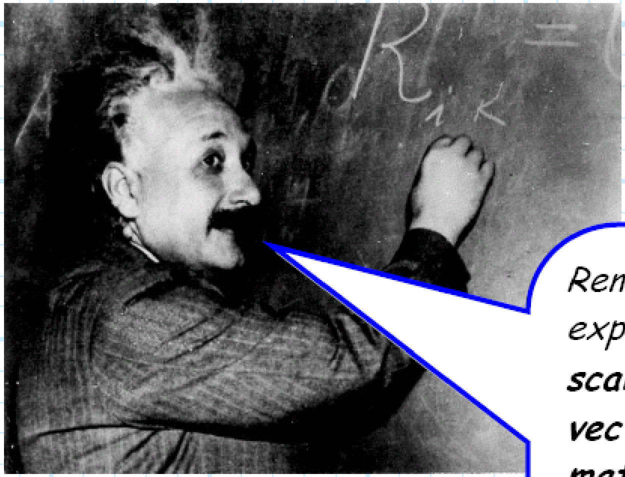
In this case, the contour C is a **closed** contour that **surrounds** surface S. The direction of C is defined by $\overline{ds}$ and the **right – hand rule**. In other words C rotates **counter clockwise** around $\overline{ds}$. E.G.,



A Gallery of Vector Fields

To help **understand** how a vector field relates to its mathematical representation using base vectors, carefully examine and consider these **examples**, plotted on either the ***x-y* plane** (i.e, the plane with all points whose coordinate z=0) or the ***x-z* plane** (i.e, the plane with all points whose coordinate y=0).

Spend some **time** studying each of these examples, until **you** see how the **math** relates to the vector field **plot** and vice versa.



*Remember, **vector fields**—expressed in terms of **scalar components** and **base vectors**—are the **mathematical language** that we will use to describe much of* Dynamics *-you must learn how to **speak and interpret** this language!*