Chapter 3

Atmosphere and Ocean Dynamics

3.1 Pseudo forces and the Coriolis effect

A pseudo force on an object arises when the frame of reference used to describe the object's motion is accelerating compared to a non-accelerating frame. It acts on all masses whose motion is described using a non-inertial frame of reference, such as a rotating reference frame. The inertial frame is the Sun and not the Earth.[1] Assuming Newton's second law in the form F = ma, pseudo forces are always proportional to the mass m. The surface of the Earth is a rotating reference frame. To solve classical mechanics problems exactly in an Earth-bound reference frame, three pseudo forces must be introduced, the Coriolis force, the centrifugal force (described below) and the Euler force. The Euler force is typically ignored because the variations in the angular velocity

[1] Galilean invariance or Galilean relativity states that the laws of motion are the same in all inertial frames. Galileo Galilei first described this principle in 1632 in his Dialogue Concerning the Two Chief World Systems using the example of a ship travelling at constant velocity, without rocking, on a smooth sea; any observer doing experiments below the deck would not be able to tell whether the ship was moving or stationary. Galilean relativity can be shown as follows. Consider two inertial frames S and S' . A physical event in S will have position coordinates $r = (x, y, z)$ and time t; similarly for S' . By the second axiom above, one can synchronize the clock in the two frames and assume $t = t'.$ Suppose S' is in relative uniform motion to S with velocity v. Consider a point object whose position is given by $r'(t) = r(t)$ in S. We see that $r'(t) = r(t) - vt.$ and acceleration is identical in the two frames $a'(t) = \frac{d^2}{dt^2} r'(t) = \frac{d^2}{dt^2} r(t) = a(t).$ A side remark: All approximations of the dynamical equations shall be Galilean invariant. In numerical examples, the lack of invariance for unresolved solutions is because the truncation error is not Galilean invariant. While advanced methods reduce the truncation error, none of them eliminate it entirely, and therefore formally solutions will still violate Galilean invariance at the level of the truncation error.

of the rotating Earth surface are usually insignificant. Both of the other pseudo forces are weak compared to most typical forces in everyday life, but they can be detected under careful conditions. For example, Foucault was able to show the Coriolis force that results from the Earth's rotation using the Foucault pendulum (see Exercise 20). If the Earth were to rotate a thousand times faster (making each day only ≈ 86 seconds long), people could easily get the impression that such fictitious forces are pulling on them, as on a spinning carousel.

In the rotating framework, we have the Coriolis and centrifugal forces which stem from the rotating framework. We derive from the simple relation for the time derivative in the inertial system (i) to the Earth system (e)

$$(d_t \boldsymbol{A})_i = (d_t \boldsymbol{A})_e + \boldsymbol{\Omega} \times \boldsymbol{A} \tag{3.1}$$

where the $\times$ symbol represents the cross product operator. For the case $\boldsymbol{A} = \boldsymbol{r}$, it follows for the velocity

$$\boldsymbol{v}_i = \boldsymbol{v}_e + \boldsymbol{\Omega} \times \boldsymbol{r} \tag{3.2}$$

and the relation for the acceleration (case $\boldsymbol{A} = \boldsymbol{v}_i$)

$$\begin{aligned} \boldsymbol{a}_i &= (d_t \boldsymbol{v}_i)_e + \boldsymbol{\Omega} \times \boldsymbol{v}_i \\ &= d_t \boldsymbol{v}_e + \boldsymbol{\Omega} \times \boldsymbol{v}_e + \boldsymbol{\Omega} \times (\boldsymbol{v}_e + \boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{a}_e + 2\boldsymbol{\Omega} \times \boldsymbol{v}_e + \boldsymbol{\Omega} \times \boldsymbol{\Omega} \times \boldsymbol{r} \end{aligned} \tag{3.3}$$

At a given rate of rotation of the observer, the magnitude of the Coriolis acceleration of the object is proportional to the velocity of the object and also to the sine of the angle between the direction of movement of the object and the axis of rotation. In the following the subscript $_e$ is dropped, since we are only interested in the dynamics in the rotating Earth system. The forces in the rotating system are thus the forces in the inertial system plus the Coriolis and centrifugal forces:

$$\boldsymbol{F} = \boldsymbol{F}_i + \boldsymbol{F}_C + \boldsymbol{F}_{cf} \tag{3.4}$$

where

$$\boldsymbol{F}_C = -2\, m\, \boldsymbol{\Omega} \times \boldsymbol{v}. \tag{3.5}$$

$\boldsymbol{\Omega}$ is the angular velocity vector which has magnitude equal to the rotation rate ω and is directed along the axis of rotation of the rotating reference frame. The formula implies that the Coriolis acceleration is perpendicular both to the direction of the velocity of the moving mass and to the frame's rotation axis.

The centrifugal term is equal to

$$\boldsymbol{F}_{cf} = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = -\omega^2 \boldsymbol{R}, \tag{3.6}$$

where $\boldsymbol{r}$ is the space vector and $\boldsymbol{R}$ the component of $\boldsymbol{r}$ perpenticular to the axis of rotation. This term can be absorbed into the gravitation is then called gravity. One can introduce the gravitational potential

$$\phi = gz - \frac{\omega^2 R^2}{2} = gz - \frac{\omega^2 (a+z)^2 \cos^2(\varphi)}{2} \simeq gz - \frac{\omega^2 a^2 \cos^2(\varphi)}{2} \quad . \tag{3.7}$$

where a is the Earth radius and φ the latitude. The combined vector $\nabla \phi$ shows only minor modification with respect to the vertical coordinate defined by the gravitation. In practice, the gravity is used for the vertical coordinate.

Exercise 19 – Earth's curvature

1. The highest building on the campus of the University of Bremen is the so-called drop tower with a hight of h=110 metres (Fig. 3.1 upper panel). How far one can look onto the horizon under good weather conditions?

Hint: Denote this distance by d. Remember the Earth's radius $a = 6378km$ and apply Pythogoras!

2. Why is the rule-of-thumb

$$d = \sqrt{2ha}$$

a good approximation? (For h=10m this means d=11 km.) When h is in m, d in km, the formula can be written as

$$d = 3.5\sqrt{\frac{h}{\mathrm{m}}}\ \text{km}.$$

3. The town Bremerhaven where the Alfred Wegener Institute is located lies about 60 km north of Bremen. How big must a tower in Bremen be in order to see the coast in Bremerhaven? (Fig. 3.1 lower panel).

Figure 3.1: Upper panel: Drop tower in Bremen. Lower panel: Harbor in Bremerhaven, ca. 60 km north of Bremen.

3.2 Pendulum and Earth rotation

The simple pendulum is another mechanical system that exhibits periodic motion. It consists of a particle-like bob of mass m suspended by a light string of length L that is fixed at the upper end, as shown in Fig. 3.2. The motion occurs in the vertical plane and is driven by the force of gravity. We shall show that, provided the angle Θ is small (less than about $10°$), the motion is that of a simple harmonic oscillator. The forces acting on the bob are the force $\boldsymbol{T}$ exerted by the string and the

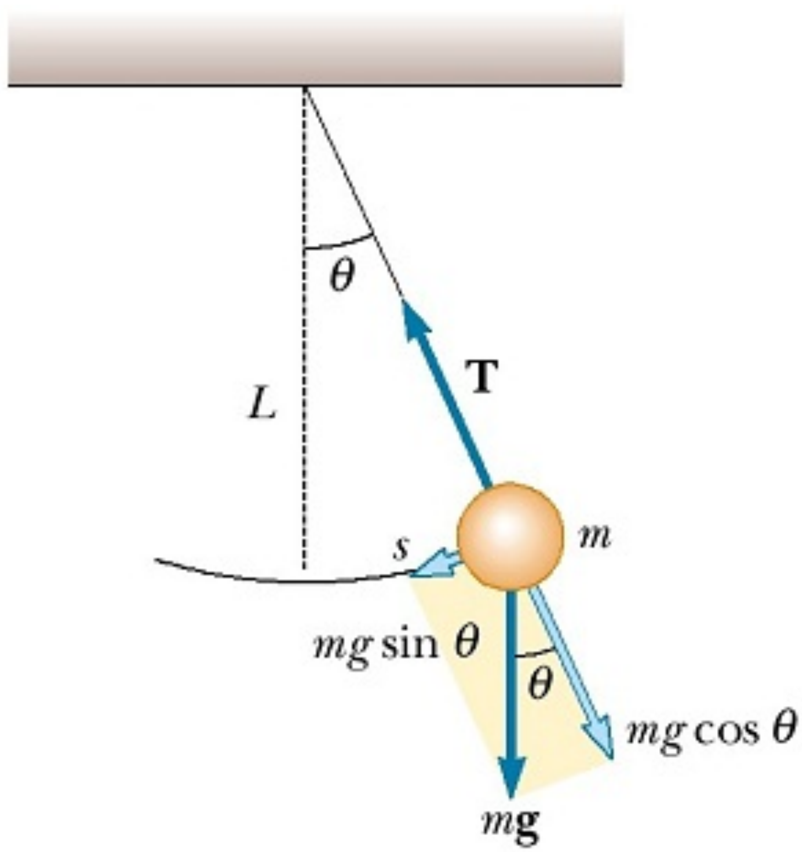

Figure 3.2: When Θ is small, a simple pendulum oscillates in simple harmonic motion about the equilibrium position $\Theta = 0$. The restoring force is $mg \sin \Theta$, the component of the gravitational force tangent to the arc.

gravitational force mg. The tangential component of the gravitational force, $mg \sin \Theta$, always acts toward $\Theta = 0$, opposite the displacement. Therefore, the tangential force is a restoring force, and we can apply Newton's second law for motion in the tangential direction:

$$F = -mg\sin\Theta = m\frac{d^2 s}{dt^2} \tag{3.8}$$

where s is the bob's displacement measured along the arc and the minus sign indicates that the tangential force acts toward the equilibrium (vertical) position. Because $s = L\Theta$ and L is constant, this equation reduces to the equation of motion for the simple pendulum.

$$\frac{d^2\Theta}{dt^2} = -\frac{g}{L}\sin\Theta \tag{3.9}$$

If we assume that Θ is small, we can use the approximation $\sin\Theta = \Theta$, thus the equation of motion for the simple pendulum becomes equation of motion for the simple pendulum

$$\frac{d^2\Theta}{dt^2} = -\frac{g}{L}\Theta \tag{3.10}$$

with solution

$$\Theta = \Theta_0\cos(\omega t) \tag{3.11}$$

where $\omega = \sqrt{\frac{g}{L}}$ is the angular frequency.

The period and frequency of a simple pendulum depend only on the length of the string and the acceleration due to gravity. Because the period is independent of the mass, we conclude that all simple pendulums that are of equal length and are at the same location (so that g is constant) oscillate with the same period. The simple pendulum can be used as a timekeeper because its period depends only on its length and the local value of g. It is also a convenient device for making precise measurements of the free-fall acceleration. Such measurements are important because variations in local values of g can provide information on the location of oil and of other valuable underground resources.

Rule of thumb for pendulum length

It is useful to have a Rule of thumb for the period of the motion, the time for a complete oscillation (outward and return) is

$$T = 2\pi\sqrt{\frac{L}{g}} \quad \text{can be expressed as} \quad L = \frac{g}{\pi^2}\frac{T^2}{4}. \tag{3.12}$$

If SI units are used (i.e. measure in metres and seconds), and assuming the measurement is taking place on the Earth's surface, then $g \approx 9.81 m/s^2$, and $g/\pi^2 \approx 1$ (0.994 is the approximation to 3 decimal places). Therefore, a relatively reasonable approximation for the length and period are,

$$\begin{aligned} L &\approx \frac{T^2}{4}, \\ T &\approx 2\sqrt{L} \end{aligned} \tag{3.13}$$

where T is the number of seconds between two beats (one beat for each side of the swing), and L is measured in metres.

Full problem without the approximation

If we consider the full problem without the approximation, the period is modified according to

$$T = 4\sqrt{\frac{L}{g}}K(k), \quad k = \sin\frac{\theta_0}{2} \tag{3.14}$$

where K is the complete elliptic integral of the first kind defined by

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-k^2\sin^2 u}}\, du\,. \tag{3.15}$$

For comparison of the approximation to the full solution, consider the period of a pendulum of length 1 m on Earth at initial angle 10 degrees is

$$4\sqrt{\frac{1\ \mathrm{m}}{g}}\,K\left(\sin\frac{10^\circ}{2}\right) \approx 2.0102\ \mathrm{s}. \tag{3.16}$$

The linear approximation gives

$$2\pi\sqrt{\frac{1\ \mathrm{m}}{g}} \approx 2.0064\ \mathrm{s}. \tag{3.17}$$

The difference between the two values, less than 0.2%,is much less than that caused by the variation of g with geographical location.

Foucault pendulum

Exercise 20 – Foucault pendulum

The Foucault pendulum was the brainchild of the French physicist Leon Foucault. It was intended to prove that Earth rotates around its axis. Let us denote x, y the pendulum bob coordinates as seen by an observer on Earth. L is the length of the pendulum string and Θ is the pendulum angle. The pendulum moves, according to the restoring force from gravity. The string tension components can be expressed using small angle approximations, which also considerably simplify the problem, making it two-dimensional. The string tension due to the gravity force is

$$F_g = mg\begin{pmatrix}\sin\Theta\\ \sin\Theta\\ \cos\Theta\end{pmatrix} \approx mg\begin{pmatrix}x/L\\ y/L\\ 1-z/L\end{pmatrix} \quad .$$

Figure 3.3: Foucault's famous pendulum in the Pantheon, Paris. What keeps it moving? Air resistance would normally stop the pendulum after a few hours – so an iron collar is installed on the wire surrounded by an electromagnet that attracts the collar as the bob swings out, then shuts off automatically as it swings back, thus, keeping pendulum going. The magnet is turned on and off by a switch which is activated when the support wire interrupts a beam of light shining across its path. Similar idea is followed by the Bremen Foucault's pendulum in our department.

Then, the horizontal dynamics can be described as

$$\ddot{x} = f\dot{y} - \frac{g}{L}x \tag{3.18}$$

$$\ddot{y} = -f\dot{x} - \frac{g}{L}y \tag{3.19}$$

where $f = 2\Omega \sin\varphi$.

1. Show the analytic solution to the Foucault pendulum problem introducing the complex number $\xi = x + i \cdot y$. Furthermore, call $\omega = \sqrt{\frac{g}{L}}$ is the angular frequency. Then,

$$\ddot{\xi} + if\dot{\xi} + \omega^2\xi = 0 \tag{3.20}$$

With the ansatz

$$\xi = H(t) \cdot \exp\left(-\frac{if}{2}t\right) \tag{3.21}$$

we obtain an equation for H

$$\ddot{H} + \left(\omega^2 + \frac{f^2}{4}\right) H = 0 \tag{3.22}$$

$$H(t) = C \exp\left[\pm it\sqrt{\omega^2 + \frac{f^2}{4}}\right] \tag{3.23}$$

and therefore

$$\xi = C \exp\left[it\left(-\frac{f}{2} \pm \sqrt{\omega^2 + \frac{f^2}{4}}\right)\right] \approx C \exp\left[it\left(-\frac{f}{2} \pm \omega\right)\right] \tag{3.24}$$

where C is a complex integration constant. The pendulum swing has a natural frequency (also called pulsation) $\omega = \sqrt{g/L}$, which depends on the length of the pendulum string.[2] Looking at the last term in (3.29): At either the North Pole or South Pole, the plane of oscillation of a pendulum remains fixed relative to the distant masses of the universe while Earth rotates underneath it, taking one day to complete a rotation (frequency $\Omega = 2\pi/24h$). So, relative to Earth, the plane of oscillation of a pendulum at the North Pole undergoes a full clockwise rotation during one day, a pendulum at the South Pole rotates counterclockwise.[3]

When a Foucault pendulum is suspended at the equator, the plane of oscillation remains fixed relative to Earth. At other latitudes, the plane of oscillation precesses relative to Earth with a frequency $f/2 = \Omega \sin\varphi$ proportional to the sine of the latitude, where latitudes north and south of the equator are defined as positive and negative, respectively. For example, a Foucault pendulum at $30°$ S, viewed from above by an earthbound observer, rotates counterclockwise $360°$ in two days.

2. For Foucault's famous pendulum in Paris: The plane of the pendulum's swing rotated clockwise $11°$ per hour, making a full circle in 32.7 hours. What is the time period in Bremen, Germany?

3. Display the solution and compare it with the numerical solution with the following initial condition:

```
g = 9.81             # acceleration of gravity (m/s^2)
L = 67               # pendulum length (m) for the experiment in Paris
initial_x = L/100 # initial x coordinate (m)
initial_y = 0        # initial y coordinate (m)
initial_u = 0        # initial x velocity (m/s)
initial_v = 0        # initial y velocity (m/s)
Omega=2*pi/86400     # Earth's angular velocity of rotation (rad/s)
phi=49/180*pi        # 49 deg latitude in (rad)   for Paris 1851
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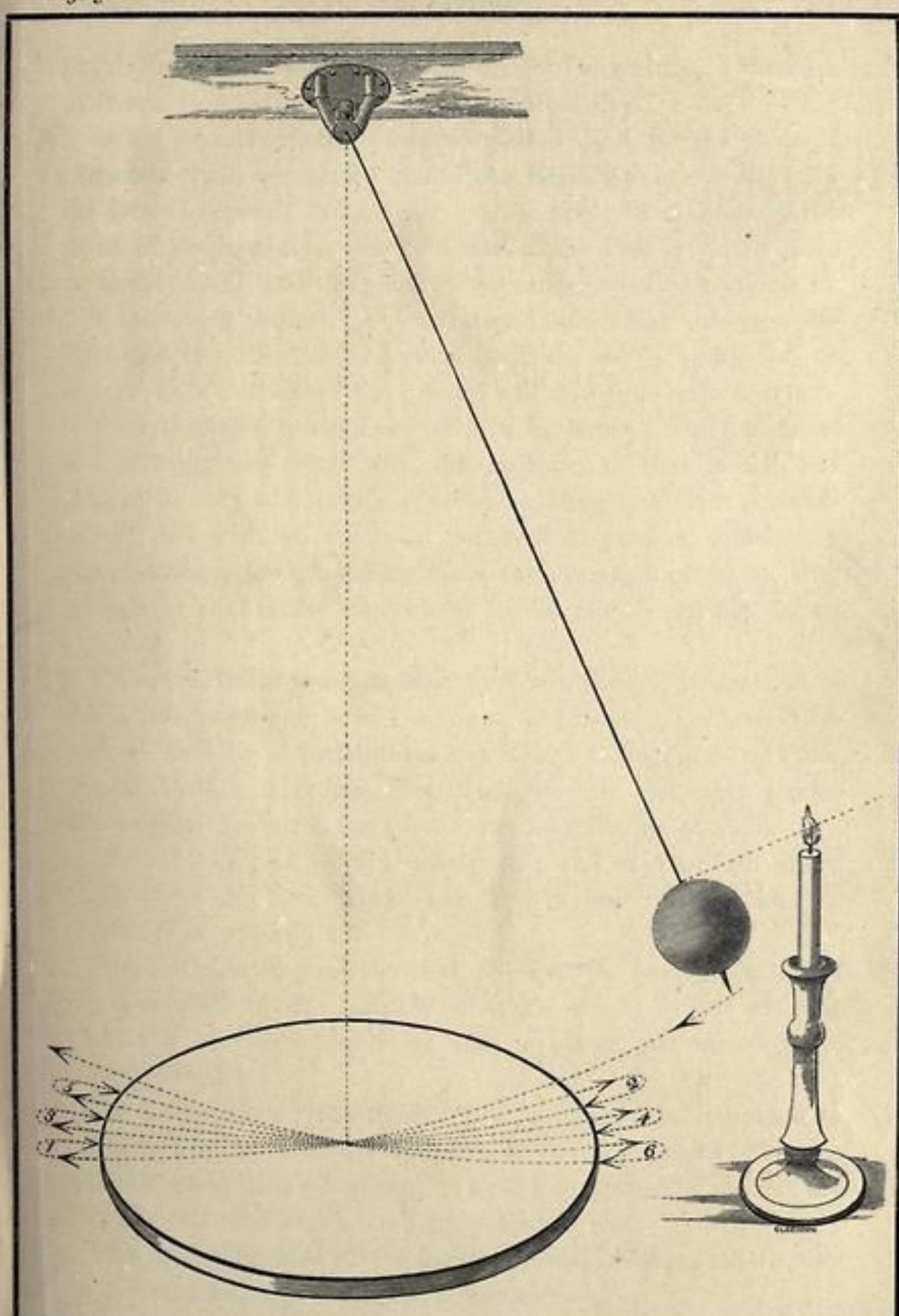
[2] For Foucault's famous pendulum: he suspended a 28 kg brass-coated lead bob with a 67 meter long wire from the dome of the Pantheon in Paris (about $49°$N). The natural frequency is $\sqrt{g/L} = 0.381/s$ related to a time period of 16 s.

[3] for the South Pole, there was indeed an experiment [Baker and Blackburn, 2005].

Fig. 56. *Plate* VI.



FOUCAULT'S PENDULUM EXPERIMENT TO SHOW THE EARTH'S AXIAL ROTATION.

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Figure 3.4: Foucault's pendulum experiment.

Exercise 21 – Foucault pendulum 2

The horizontal dynamics of the Foucault pendulum with length L is given by

$$\ddot{x} = f\dot{y} - \frac{g}{L}x \tag{3.25}$$

$$\ddot{y} = -f\dot{x} - \frac{g}{L}y \tag{3.26}$$

with $f = 2\Omega \sin\varphi$. The length is typically on the order of 1-10 m.

a) Show that the solution is given by

$$x = x_0 \cos\omega^* t \tag{3.27}$$

$$y = x_0 \sin\omega^* t \tag{3.28}$$

$$\text{with} \quad \omega^* = \left(-\frac{f}{2} + \sqrt{\omega^2 + \frac{f^2}{4}}\right) \tag{3.29}$$

where x_0 is the initial condition, and $\omega = \sqrt{g/L}$.

b) Show that $\omega^2 >> \frac{f^2}{4}$ and that

$$\omega^* \approx -\frac{f}{2} + \omega \quad . \tag{3.30}$$

c) Explain that the natural frequency (also called pulsation) ω can be used to measure gravity.

d) Show that the precession cycle can be used to determine the latitude! Discuss the special cases equator and South Pole!

3.3 Scaling of the dynamical equations

As we will see now, the Coriolis effect is one of the dominating forces for the large-scale dynamics of the oceans and the atmosphere. It is convenient to work in the rotating frame of reference of the Earth. The equation can be scaled by a length-scale L, determined by the geometry of the flow, and by a characteristic velocity U. Starting from (1.9), we can estimate the relative contributions in units of m/s^2 in the horizontal momentum equations:

$$\underbrace{\frac{\partial \mathbf{v}}{\partial t}}_{U/T\sim 10^{-8}} + \underbrace{\mathbf{v}\cdot\nabla\mathbf{v}}_{U^2/L\sim 10^{-8}} = \underbrace{-\frac{1}{\rho}\nabla p}_{\delta \mathrm{P}/(\rho \mathrm{L})\sim 10^{-5}} + \underbrace{2\Omega\times\mathbf{v}}_{\mathrm{f_0 U}\sim 10^{-5}} + \underbrace{fric}_{\nu U/H^2\sim 10^{-13}} \tag{3.31}$$

where fric denotes the contributions of friction due to eddy stress divergence (usually $\sim \nu\nabla^2\mathbf{v}$). Typical values are given in Table 3.3. The values have been taken for the **ocean.** You may repeat the estimate for the atmosphere using Table 3.3.

It is useful to think about the orders of magnitude: Because of the continuity equation $U/L \sim W/H$ and since the horizonatal scales are orders of magnitude larger than the vertical ones, the vertical velocity is very small relative to the horizontal. For small scale motion (like small-scale ocean convection or cumuls clouds) the horizontal length scale is of the same order as the vertical one and therefore the vertical motion is in the same order of magnitude as the horizontal motion. The timescales are related to $T \sim L/U \sim H/W$.

It is already useful to think about the relative importance of the different terms in the momentum balance (3.31). The Rossby number Ro is the ratio of inertial (the left hand side) to Coriolis (second term on the right hand side) terms

$$Ro = \frac{(U^2/L)}{(fU)} = \frac{U}{fL} \quad . \tag{3.32}$$

It is used in the oceans and atmosphere, where it characterizes the importance of Coriolis accelerations arising from planetary rotation. It is also known as the Kibel number. Ro is small when the flow is in a so-called geostrophic balance. This will be the subject in the next paragraphs.

	Quantity	Atmosphere	Ocean
horizontal velocity	U	$10\ \ ms^{-1}$	$10^{-1}\,ms^{-1}$
vertical velocity	W	$10^{-1}\,ms^{-1}$	$10^{-4}\,ms^{-1}$
horizontal length	L	$10^{6}\,m$	$10^{6}\,m$
vertical length	H	$10^{4}\,m$	$10^{3}\,m$
horizonal Pressure changes	δP (horizontal)	$10^{3}\,Pa$	$10^{4}\,Pa$
mean pressure	P_0	$10^{5}\,Pa$	$10^{7}\,Pa$
time scale	T	$10^{5}\,s$	$10^{7}\,s$
gravity (gravitation+centrifugal)	g	$10\,ms^{-2}$	$10\,ms^{-2}$
Earth radius	a	$10^{7}\,m$	$10^{7}\,m$
Coriolis parameter at 45°N	$f_0 = 2\Omega\sin\varphi_0$	$10^{-4}\,s^{-1}$	$10^{-4}\,s^{-1}$
2nd Coriolis parameter at 45°N	$f_1 = 2\Omega\cos\varphi_0$	$10^{-4}\,s^{-1}$	$10^{-4}\,s^{-1}$
density	ρ	$1\ \ kgm^{-3}$	$10^{3}\,kgm^{-3}$
viscosity (turbulent)	ν	$10^{-5}\,kgm^{-3}$	$10^{-6}\,kgm^{-3}$

Table 3.1: Table shows the typical scales in the atmosphere and ocean system. Using these orders of magnitude, one can derive estimates of the different terms in (3.31).

Exercise 22 – Non-dimensional system

a) Write down the non-dimensional version of (3.31) ! What are the characteristic numbers?

b) Use Table 3.3 to estimate the order of magnitude of the characteristic numbers !

c) Compare the procedure to exercise 8.

3.4 The coordinate system

The equations have to be solved on a proper coordinate system. Consider a location with latitude φ on a sphere that is rotating around the north-south axis. A local coordinate system is set up with the x axis horizontally due east, the y axis horizontally due north and the z axis vertically upwards. The axis of rotation is then expressed by a y-component $\sim \cos\varphi$ and a z-component $\sim \sin\varphi$. The rotation vector expressed in this local coordinate system is

$$\boldsymbol{\Omega} = \Omega \begin{pmatrix} 0 \\ \cos\varphi \\ \sin\varphi \end{pmatrix} . \tag{3.33}$$

Likewise, the components of the velocity vector are listed in the order East (u), North (v) and Upward (w):

$$\boldsymbol{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} , \tag{3.34}$$

and Coriolis acceleration is therefore in this coordinate system

$$\boldsymbol{a}_C = -2\boldsymbol{\Omega} \times \boldsymbol{v} = 2\,\Omega \begin{pmatrix} v\sin\varphi - w\cos\varphi \\ -u\sin\varphi \\ u\cos\varphi \end{pmatrix} . \tag{3.35}$$

In the following, $f = 2\Omega\sin\varphi$ is called the Coriolis parameter, $f^{(2)} = 2\Omega\cos\varphi$ is called the second Coriolis parameter.

When considering atmospheric or oceanic dynamics, the vertical velocity is small and therefore the vertical component of the Coriolis acceleration is small compared to gravity (see table 3.3 and the following paragraph). For such cases, only the horizontal (East and North) components matter.

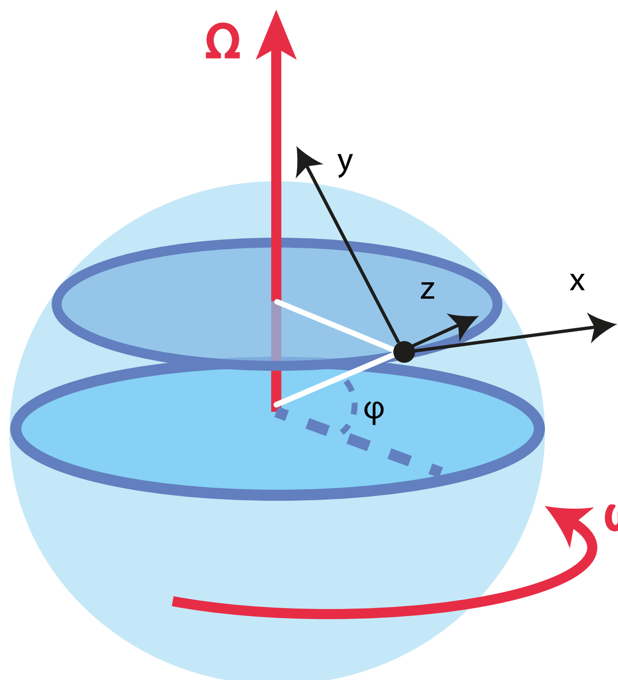

Figure 3.5: Coordinate system at a local latitude φ with x-axis east, y-axis north and z-axis upward (that is, radially outward from center of sphere). $(x, y, z) = (a\lambda\cos\varphi, a\varphi, z)$ where (λ, φ, z) denote longitude, latitude, hight. a is the Earth radius. Ω is the the Earth rotation and equal to $2\pi/(24h)$. Note that the axis of rotation has a y- and z-component in this coordinate system (see text for details).

If we further assume $v = 0,$ it can be seen immediately that (for positive φ) a movement to the east results in an acceleration to south. Similarly, for $u = 0,$ it is seen that a movement due north results in an acceleration due east. In general, observed horizontally, looking along the direction of the movement causing the acceleration, the acceleration always is turned $90°$ to the right on the Northern Hemisphere (left on the Southern Hemisphere) and of the same size regardless of the horizontal orientation.

Exercise 23 – Calculate the Double Vector Product

Examine the double vector product $\Omega \times (\Omega \times r)$ with vectors $\Omega = (0, 0, \omega)$, $r = (x, y, z)$.

Solution

$$\begin{aligned}
\Omega \times (\Omega \times \mathrm{r}) &= \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \left(\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right) \\
&= \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} \times \begin{pmatrix} -\omega y \\ \omega x \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} -\omega^2 y \\ -\omega^2 x \\ 0 \end{pmatrix} = -\|\Omega\|^2 \mathrm{R}
\end{aligned}$$

with $\mathrm{R} = (x, y, 0)^T$ and $\|\Omega\|^2 = \omega^2$.

Exercise 24 – Some questions about the stmosphere

1. Consider the heat diffusion-advection equation

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} + u \frac{\partial T}{\partial x}$$

   and determine the time evolution with initial conditions

   a) $T(x, 0) = \exp(-x^2/a)$ with a = constant.

   b) $T(x, 0) = T_0$ for $x \geq 0$ and $T(x, 0) = 0$ elsewhere.

   Discuss the special cases $k = 0$ (no diffusion) and $u = 0$ (no advection).

2. A tornado rotates with constant angular velocity ω. Show that the surface pressure at the center of the tornado is given by:

$$p = p_0 \exp\bigl(-\omega^2 {r_0}^2/(2RT)\bigr)$$

   where p_0 is the surface pressure at the distance r_0 from the center and T is the temperature (assumed constant). [Hint: What are the dominant forces? Pressure gradient and centrifugal force.]

   If temperature is 288K, pressure at 100m from the center is 10^2 kPa, and wind speed at 100m from the center is 100m/s, what is the central pressure?

3. Suppose a 1kg parcel of dry air is rising at a constant vertical velocity. If the parcel is being heated by radiation at a rate of $10^{-1} W/kg$, what must the speed of rise be in order to maintain the parcel at a constant temperature. [Hint: Energy equation.]

4. Show that for an atmosphere with an adiabatic lapse rate (i.e. constant potential temperature), the geopotential $Z(z) := \Phi(z)/g_0$ is given by

$$Z = H_\Theta[1 - (p/p_0)^a]$$

   where p_0 is the pressure at $Z = 0$ and $H_\Theta = c_p\Theta/g_0$ is the total geopotential in the atmosphere. $a = R/c_p$.

Exercise 25 – Some simple repetition questions

1. Please write down the equation of state for the ocean and atmosphere!
2. What is the hydrostatic approximation in the momentum equations?
3. Please clarify: On the Northern Hemisphere, particles tend to go to the right or left relative to the direction of motion due to the Coriolis force?

3.5 Geostrophy

The momentum equations (3.31) can be also written in the coordinate system (Fig. 3.5) above as

$$\frac{\partial u}{\partial t} + \mathrm{v}\cdot\nabla u - \frac{uv\tan\varphi}{a} - \frac{uw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - f^{(2)}w + \nu\nabla^2 u \tag{3.36}$$

$$\frac{\partial v}{\partial t} + \mathrm{v}\cdot\nabla v - \frac{u^2\tan\varphi}{a} - \frac{vw}{a} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \qquad + \nu\nabla^2 v \tag{3.37}$$

complemented by the dynamics for the vertical component w :

$$\underbrace{\frac{\partial w}{\partial t}}_{W/T\sim 10^{-11}} + \underbrace{\mathrm{v}\cdot\nabla w}_{UW/L\sim 10^{-11}} - \underbrace{\frac{u^2+v^2}{a}}_{U^2/a\sim 10^{-9}} = \underbrace{-\frac{1}{\rho}\frac{\partial p}{\partial z}}_{\mathrm{P}_0/(\rho\mathrm{H})\sim 10} + \underbrace{g}_{\sim 10} + \underbrace{f^{(2)}u}_{\sim 10^{-5}} + \underbrace{\nu\partial_z^2 w}_{\nu W/H^2\sim 10^{-16}} \tag{3.38}$$

As boundary conditions, equations (3.36, 3.37) are complemented by the horizontal wind stresses $\partial_z\tau_{xz}$ and $\partial_z\tau_{yz}$ at the ocean surface, respectively.

$$\frac{\partial u}{\partial t} + \mathrm{v}\cdot\nabla u + \ldots = -\frac{1}{\rho}\frac{\partial p}{\partial x} + fv - f^{(2)}w + \nu\nabla^2 u + \frac{1}{\rho}\partial_z\tau_{xz} \tag{3.39}$$

$$\frac{\partial v}{\partial t} + \mathrm{v}\cdot\nabla v + \ldots = -\frac{1}{\rho}\frac{\partial p}{\partial y} - fu \qquad + \nu\nabla^2 v + \frac{1}{\rho}\partial_z\tau_{yz} \tag{3.40}$$

It should be noted that due to sperical coordinates (see Fig. 3.5), one has metric terms, e.g. on the left hand sides of (3.36,3.37,3.38): $-\frac{uv\tan\varphi}{a} - \frac{uw}{a}$, $\frac{u^2\tan\varphi}{a} - \frac{vw}{a}$, and $\frac{u^2+v^2}{a}$, respectively. In the geostropic approximation, one can drop these terms.[4]

A small Rossby number signifies a system which is strongly affected by Coriolis forces, and a large Rossby number signifies a system in which inertial forces dominate. For example, in tornadoes, the Rossby number is large ($\approx 10^3$), in atmospheric low-pressure systems it is low ($\approx 0.1 - 1$), but depending on the phenomena can range over several orders of magnitude ($\approx 10^{-2} - 10^2$).[5] Using the values in table 3.3, Ro in oceanic systems is of the order of 10^{-3}.

[4]Task: Calculate the order of magnitude of the metric terms in (3.36,3.37) by using table 3.3.

[5]As a result, in tornadoes the Coriolis force is negligible, and balance is between pressure and centrifugal forces (called cyclostrophic balance). This balance also occurs at the outer eyewall of a tropical cyclone.

When the Rossby number is large (either because f is small, such as in the tropics and at lower latitudes; or because L is small, that is, for small-scale motions such as flow in a bathtub; or for large speeds), the effects of planetary rotation are unimportant and can be neglected. Repeating: When the Rossby number is small, then the effects of planetary rotation are large and the net acceleration is comparably small allowing the use of the so-called geostrophic approximation: The force balance is largely between the pressure gradient force acting towards the low-pressure area and the Coriolis force acting away from the center of the low pressure in equation (3.31). By scaling arguments, one can derive the geostrophic horizontal flow components (u_g, v_g) as:

$$u_g = -\frac{1}{f\rho}\frac{\partial p}{\partial y} \tag{3.41}$$

$$v_g = \frac{1}{f\rho}\frac{\partial p}{\partial x} \tag{3.42}$$

The validity of this approximation depends on the local Rossby number. It is invalid at the equator, because $f = 2\Omega \sin \varphi$ is equal to zero there, and therefore generally not used in the tropics.

Equations (3.41,3.42) show that large-scale motions in the atmosphere and ocean tend to occur perpendicular to the pressure gradient, instead of flowing down the gradient. This circulation is called geostrophic flow. On a non-rotating planet, fluid would flow along the straightest possible line, quickly eliminating pressure gradients.[6]

[6]Task: Think how the geostrophy can be derived in the inertial system with a fixed reference frame, e.g. the Sun. The final result shall be independent on the reference system used!

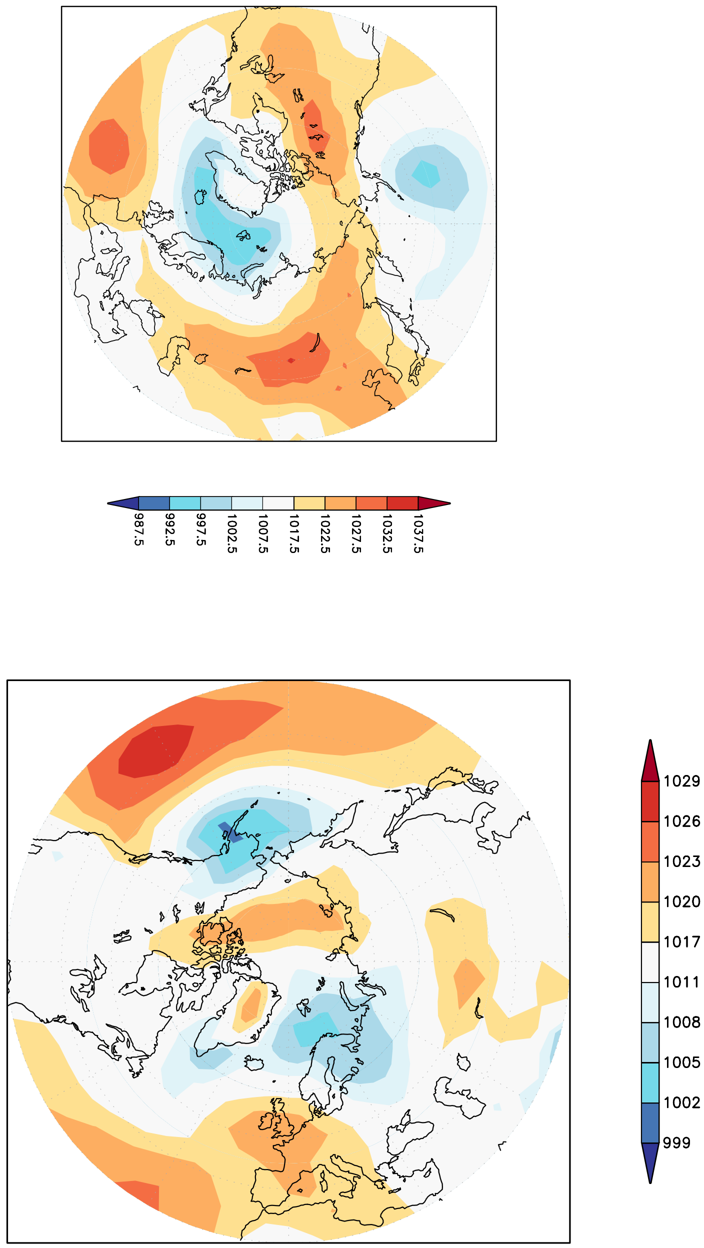

Figure 3.6: Sea level pressure (hPa) field for February (upper) and April (lower) 2015. In February, the circulation is characterized by a low pressure over the Greenland-Iceland-Norwegian Sea, and a surrounded high pressure. In April, the circulation was dominated by a high pressure over northern France and the subtropical Atlantic and Pacific Oceans, a low pressure over Scandianavia and the Aleutian Islands. Task: Draw the direction of large-scale motions in the atmosphere using the geostrophic balance (3.41,3.42). Data are from Trenberth and Paolino (1980).

3.6 Geostrophic Stream Lines and Stream Function

At each instant in time, we can represent a flow field by a vector velocity at each point in space. The instantaneous curves that are everywhere tangent to the direction of the vectors are called the *stream lines* of the flow. If the flow is unsteady, the pattern of stream lines change with time. The trajectory of a fluid particle, the path followed by a Lagrangian drifter, is called the path line in fluid mechanics. The path line is the same as the stream line for steady flow, and they are different for an unsteady flow. We can simplify the description of two-dimensional, incompressible flows by using the *stream function* ψ defined by:

$$u \equiv \frac{\partial \psi}{\partial y}, \qquad v \equiv -\frac{\partial \psi}{\partial x}, \tag{3.43}$$

The stream function is often used because it is a scalar from which the vector velocity field can be calculated. This leads to simpler equations for some flows.

The volume rate of flow between any two stream lines of a steady flow is $d\psi$, and the volume rate of flow between two stream lines $\psi 1$ and ψ_2 is equal to $\psi_1 - \psi_2$. To see this, consider an arbitrary line $dx = (dx, dy)$ between two stream lines (Fig. 3.7). The volume rate of flow between the stream lines is:

$$v\ dx + (-u)\ \ dy = -\frac{\partial \psi}{\partial x}\, dx - \frac{\partial \psi}{\partial y}\, dy = -d\psi \tag{3.44}$$

and the volume rate of flow between the two stream lines is numerically equal to the difference in their values of ψ.

Now, lets apply the concepts to satellite-altimeter maps of the oceanic topography. One can show that

$$u_s = -\frac{g}{f}\frac{\partial \eta}{\partial y}, \qquad v_s = -\frac{g}{f}\frac{\partial \eta}{\partial x}, \tag{3.45}$$

where g is gravity, f is the Coriolis parameter, and η is the height of the sea surface above a level

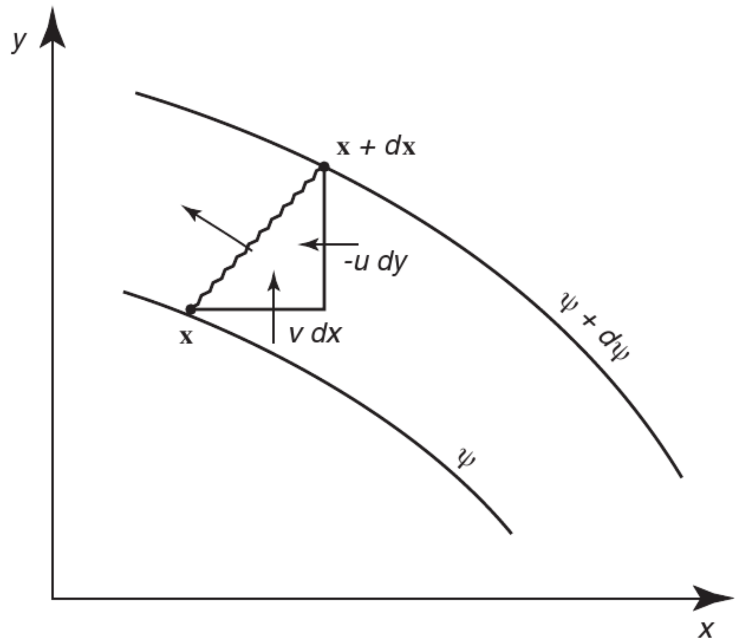

Figure 3.7: Volume transport between stream lines in a two-dimensional, steady flow. After Kundu (1990: 68).

surface. Comparing 3.45 with 3.43 it is clear that

$$\psi = -\frac{g}{f}\,\eta \tag{3.46}$$

and the sea surface is a stream function scaled by g/f. The lines of constant height are stream lines, and flow is along the lines. The surface geostrophic transport is proportional to the difference in height, independent of the distance between the stream lines. The transport is relative to transport at the 1000 decibars surface, which is roughly one kilometer deep.

In addition to the stream function, oceanographers use the mass-transport stream function Ψ defined by:

$$M_x \equiv \frac{\partial \Psi}{\partial y}, \qquad M_y \equiv -\frac{\partial \Psi}{\partial x} \tag{3.47}$$

3.7 Conservation of vorticity

In simple words, vorticity is the rotation of the fluid. The rate of rotation can be defined in various ways. Consider a bowl of water sitting on a table in a laboratory. The water may be spinning in the bowl. In addition to the spinning of the water, the bowl and the laboratory are rotating because they are on a rotating earth. The two processes are separate and lead to two types of vorticity.

Everything on earth, including the ocean, the atmosphere, and bowls of water, rotates with the earth. This rotation is the *planetary vorticity* f. It is twice the local rate of rotation of earth:

$$f \equiv 2\,\Omega \sin\varphi \quad \left(\frac{1}{s}\right) = 2\ \sin\varphi \quad \left(\frac{cycles}{day}\right) \tag{3.48}$$

Planetary vorticity is also called the Coriolis parameter. It is greatest at the poles where it is twice the rotation rate of earth. Note that the vorticity vanishes at the equator and that the vorticity in the Southern Hemisphere is negative because φ is negative.

The ocean and atmosphere do not rotate at exactly the same rate as the Earth. They have some rotation relative to Earth due to currents and winds. *Relative vorticity* ζ is the vorticity due to currents in the ocean.[7] Mathematically it is:

$$\zeta \equiv \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \tag{3.49}$$

where we have assumed that the flow is two-dimensional.

For a rigid body rotating at rate Ω, $\zeta = 2\ \Omega$. Of course, the flow does not need to rotate as a rigid body to have relative vorticity. Vorticity can also result from shear. For example, at a north/south western boundary in the ocean, $u = 0$, $v = v\,(x)$ and $\zeta = \partial v\,(x)\,/\partial x$.

ζ is usually much smaller than f. To make an estimate for ζ : It is greatest at the edge of fast currents such as the Gulf Stream. To obtain some understanding of the size of ζ, consider the edge

[7] ζ is the vertical component of the threedimensional vorticity vector ω, and it is sometimes written ω_z. ζ is positive for counter-clockwise rotation viewed from above. This is the same sense as Earth's rotation in the Northern Hemisphere. One could use ω_z for relative vorticity, but ω is also commonly used to mean frequency in radians per second.

of the Gulf Stream off Cape Hatteras where the velocity decreases by $1\,ms^{-1}$ in 100km at the boundary. The curl of the current is approximately

$$\zeta = \frac{\partial v}{\partial x} = \frac{1\,ms^{-1}}{100\,km} = \quad 0.14 \quad \frac{cycles}{day} = \quad 1 \quad \frac{cycle}{week} \quad = 1.62 \cdot 10^{-6}\,\frac{1}{s}. \tag{3.50}$$

Hence even this large relative vorticity is still almost seven times smaller than f (compare 3.48). A more typical value of relative vorticity, such as the vorticity of eddies, is a cycle per month. The sum of the planetary and relative vorticity is called absolute vorticity:

$$\text{Absolute Vorticity} \equiv (\zeta + f) \tag{3.51}$$

We can obtain an equation for absolute vorticity in the ocean by manipulating the equations of motion for frictionless flow. We begin with:

$$\frac{Du}{Dt} - f\;v = -\frac{1}{\rho}\frac{\partial p}{\partial x} \tag{3.52}$$

$$\frac{Dv}{Dt} + f\;u = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{3.53}$$

If we expand the substantial derivative, and if we subtract $\partial/\partial y$ of (3.52) from $\partial/\partial x$ of (3.53) to eliminate the pressure terms, we obtain

$$\begin{aligned}\frac{D}{Dt}\left(\partial_x v - \partial_y u\right) \quad &+ \quad (\partial_x u \partial_x v + \partial_x v \partial_y v) - (\partial_y u \partial_x u + \partial_y v \partial_y u) \\ &+ \quad f\,(\partial_x u + \partial_y v) + v\,\partial_y f \;= 0\end{aligned} \tag{3.54}$$

Using $\frac{D}{Dt}f = v\,\partial_y f$:

$$\begin{aligned}\frac{D}{Dt}\zeta \quad &+ \quad \partial_x v\,(\partial_x u + \partial_y v) - \partial_y u\,(\partial_x u + \partial_y v) \\ &+ \quad f\,(\partial_x u + \partial_y v) + \frac{D}{Dt}f \;= 0\end{aligned} \tag{3.55}$$

this yields

$$\frac{D}{Dt}(\zeta + f) + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad . \tag{3.56}$$

Exercise 26 – Non-dimensional system of the vorticity dynamics

a) For cnstant depth, derive the the non-dimensional version of the vorticity equation

$$\frac{D}{Dt}(\zeta + f) = \nu \nabla^2 \zeta \quad .$$

Hint: Repeat exercise 4. b) What are the characteristic numbers?

c) Estimate the order of magnitude of the characteristic numbers for the atmosphere and ocean ! You can use Table 3.3 and other references.

3.7.1 Potential vorticity equation $(\zeta + f)/h$

3.7.2 Examples for conservation of Vorticity

The rotation rate of a column of fluid changes as the column is expanded or contracted. This changes the vorticity through changes in ζ. To see how this happens, consider barotropic, geostrophic flow in an ocean with depth $h\,(x, y, t)$, where h is the distance from the sea surface to the bottom. That is, we allow the surface to have topography (Fig. 3.8). Integrating the continuity equation from the bottom to the top of the ocean gives:

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) \int_b^{b+h} dz + w|_b^{b+h} = 0 \tag{3.57}$$

where b is the topography of the bottom, and h is the depth of the water. Notice that $\partial u/\partial x$ and $\partial v/\partial y$ are independent of z because they are barotropic, and the terms can be taken outside the integral. The boundary conditions require that flow at the surface and the bottom be along the

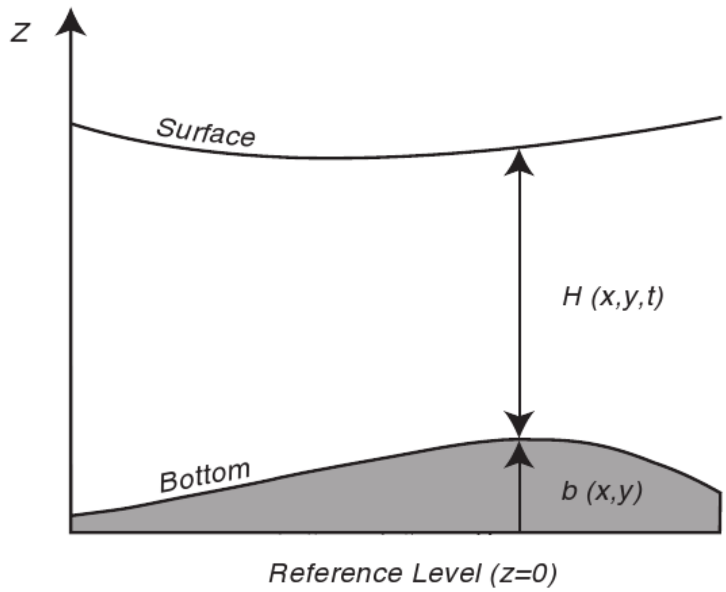

Figure 3.8: Sketch of fluid flow used for deriving conservation of potential vorticity. Here $H = h$. After Cushman-Roisin (1994: 55).

surface and the bottom. Thus the vertical velocities at the top and the bottom are:

$$w|_{b+h} = D_t(b+h) = \frac{\partial(b+h)}{\partial t} + u\frac{\partial(b+h)}{\partial x} + v\frac{\partial(b+h)}{\partial y} \tag{3.58}$$

$$w|_b = D_b = u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} \tag{3.59}$$

where we used $\partial b/\partial t = 0$ because the bottom does not move, and $\partial h/\partial z = 0$. Substituting (3.58) and (3.59) into (3.57) we obtain

$$\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{1}{h}\frac{Dh}{Dt} = 0 \tag{3.60}$$

Substituting this into (3.56) gives:

$$\frac{D}{Dt}(\zeta + f) - \frac{(\zeta + f)}{h}\frac{Dh}{Dt} = 0 \tag{3.61}$$

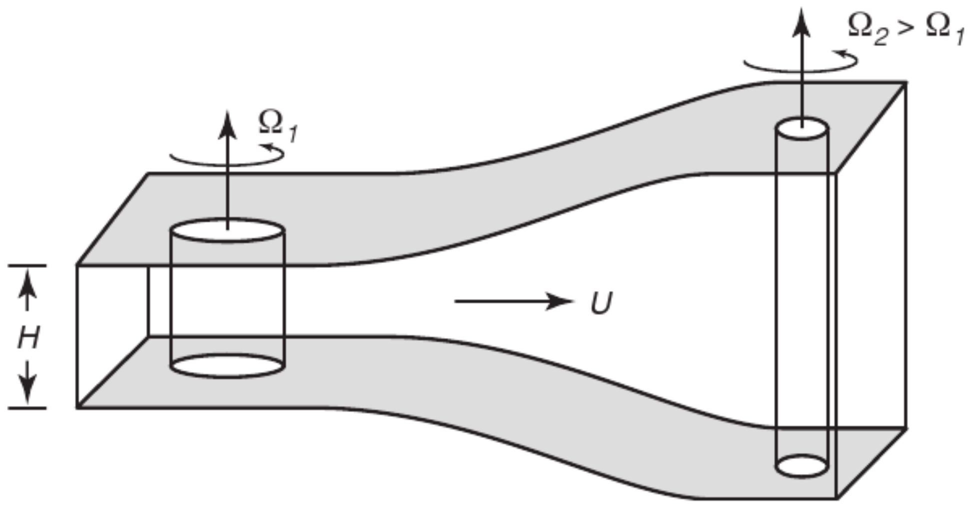

Figure 3.9: Sketch of the production of relative vorticity by change in the height of a fluid column. As the vertical fluid column moves from left to right, vertical stretching reduces the moment of inertia of the column, causing it to spin faster.

which can be rewritten as

$$\frac{1}{h}\frac{D}{Dt}(\zeta + f) - (\zeta + f)\frac{D_t h}{h^2} = 0 \tag{3.62}$$

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0 \quad . \tag{3.63}$$

The quantity within the parentheses must be constant. It is called *potential vorticity* $\mathbf{\Pi}$. Potential vorticity is conserved along a fluid trajectory:

$$\text{Potential Vorticity} = \mathbf{\Pi} \equiv \frac{\zeta + f}{h} \tag{3.64}$$

The angular momentum of any isolated spinning body is conserved. The spinning body can be an eddy in the ocean or the earth in space. If the spinning body is not isolated, that is, if it is linked to another body, then angular momentum can be transferred between the bodies. The conservation of potential vorticity couples changes in depth, relative vorticity, and changes in latitude. All three interact:

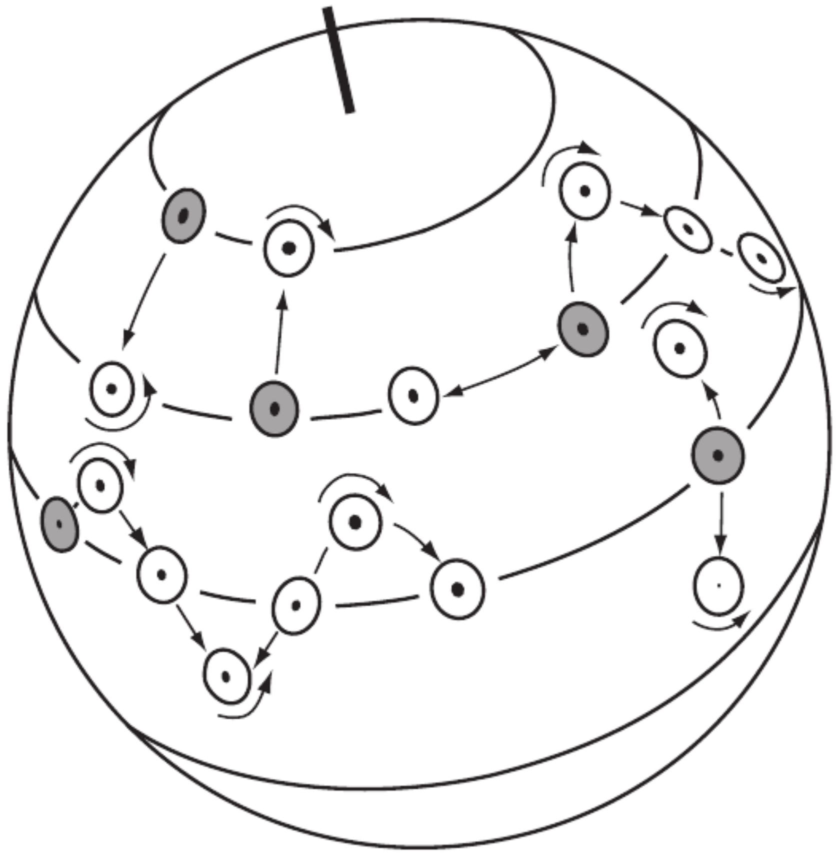

Figure 3.10: Angular momentum tends to be conserved as columns of water change latitude. This changes the relative vorticity of the columns. After von Arx (1962).

- Changes in the depth h of the flow results in change of the relative vorticity. The concept is analogous with the way figure skaters decrease their spin by extending their arms and legs. The action increases their moment of inertia and decreases their rate of spin (Fig. 3.9).

- Changes in latitude require a corresponding change in ζ. As a column of water moves equatorward, f decreases, and ζ must increase (Fig. 3.10). If this seems somewhat mysterious, von Arx (1962) suggests we consider a barrel of water at rest at the north pole. If the barrel is moved southward, the water in it retains the rotation it had at the pole, and it will appear to rotate counterclockwise at the new latitude where f is smaller.

3.7.3 Potential vorticity conservation $(\zeta + f)/h$: Implications

The concept of conservation of potential vorticity has far reaching consequences, and its application to fluid flow in the ocean gives a deeper understanding of ocean currents.

Flow Tends to be Zonal

In the ocean f tends to be much larger than ζ and thus $f/h =$ constant. This requires that the flow in an ocean of constant depth be zonal. Of course, depth is not constant, but in general, **currents tend to be east-west rather than north-south**. Wind makes small changes in ζ, leading to a small meridional component of the flow (see Fig. 3.10).

Topographic Steering

Barotropic flows are diverted by sea floor features. Consider what happens when a flow that extends from the surface to the bottom encounters a sub-sea ridge (Fig. 3.11). As the depth decreases, $\zeta + f$ must also decrease, which requires that f decrease, and the flow is turned toward the equator. This is called topographic steering. If the change in depth is sufficiently large, no change in latitude will be sufficient to conserve potential vorticity, and the flow will be unable to cross the ridge. This is called topographic blocking.

Streamfunction f/h

In the ocean, f tends to be much larger than ζ and

$$\frac{D}{Dt}\left(\frac{f}{h}\right) = 0 \tag{3.65}$$

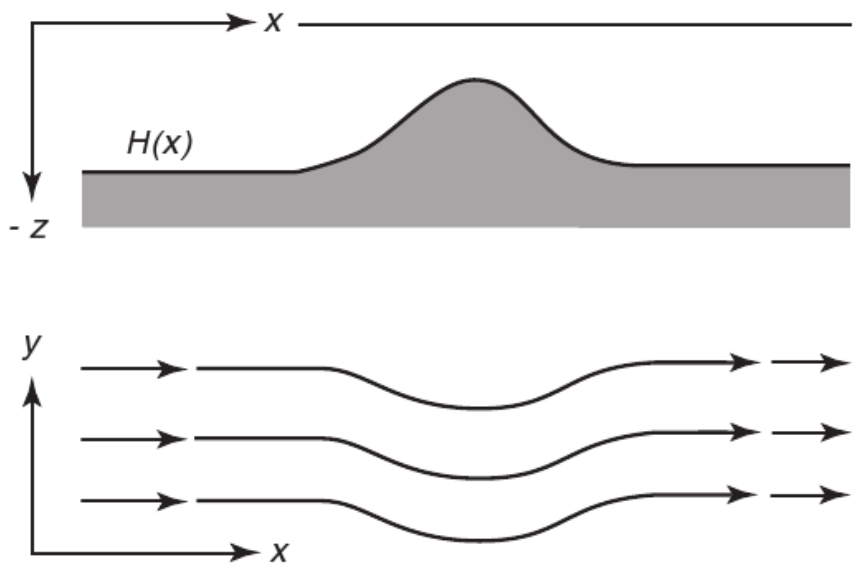

Figure 3.11: Barotropic flow over a sub-sea ridge is turned equatorward to conserve potential vorticity. After Dietrich et al. (1980: 333).

implies $f/h =$ constant along the flow. In this case, we have a streamfunction Ψ and pressure p that are functions of f/h:

$$\Psi = \Psi(f/h) \quad ; \qquad p = p(f/h). \tag{3.66}$$

This requires that the flow in an ocean of constant depth be zonal. Of course, depth is not constant, but in general, currents tend to be east-west rather than north-south. Wind makes small changes in ζ, leading to a small meridional component of the flow (see figure 3.10). The geostrophic contours f/h turn out to be an interesting combination of latitude circles and bottom topographic contours. Over small horizontal distances[8] and at high latitude topography, h tends to dominate (as in the example in Fig.3.13), but over longer distances or in the tropics, the latitude-variation of f dominates.

[8]Then $\frac{D}{Dt}\left(\frac{f}{h}\right) = 0$ can be transformed into $\frac{D}{Dt}h = 0$.

Exercise 27 – Differential operators for the potential vorticity equation

Deriving the vorticity equation

$$\frac{D}{Dt}\left(\frac{\zeta+f}{h}\right)=0 \quad ,$$

we need to evaluate the terms $\partial_y \frac{D}{Dt} u$ and $\partial_x \frac{D}{Dt} v$. Write down the explicit terms!

Exercise 28 – Calculation of potential vorticity in the atmosphere

An air column at $53°$N with $\zeta = 0$ initially streches from the surface to a fixed tropopause at 10 km height. If the air column moves until it is over a mountain barrier 2.5 km hight at $30°$N, what is its absolute vorticity and relative vorticity as it passes the mountain top?
Assume: $\sin 53° = 0.8; \sin 30° = 0.5$. The angular velocity of the Earth $\Omega = 2\pi/(1day)$.

Exercise 29 – f/h contours

Geostrophic contours using available topography data. Barotropic flows are diverted by sea floor features. Consider what happens when a flow that extends from the surface to the bottom encounters a sub-sea ridge.

1. Show the f/h contours for the North Atlantic Ocean! See Fig. 3.12.
2. Show it for low latitudes regions: region around $20°S$ to $20°N$ in the Atlantic and Pacific Ocean. One problem is that the geostrophic contours bump into continents, so that ocean currents running along them have a serious difficulty there. Actually all such f/h contours head toward the Equator as they run up into shallow water (as $h \to 0 \quad f \to 0$ also, hence $\varphi \to 0)$. This shows that we need more terms in the vorticity dynamics to describe the ocean circulation.
3. The examination of tidal rhythmites and theories about the Earth-Moon dynamics suggest that the length of day 900 million years ago was 18 h instead of 24h. How are the results of the vorticity dynamics are affected?

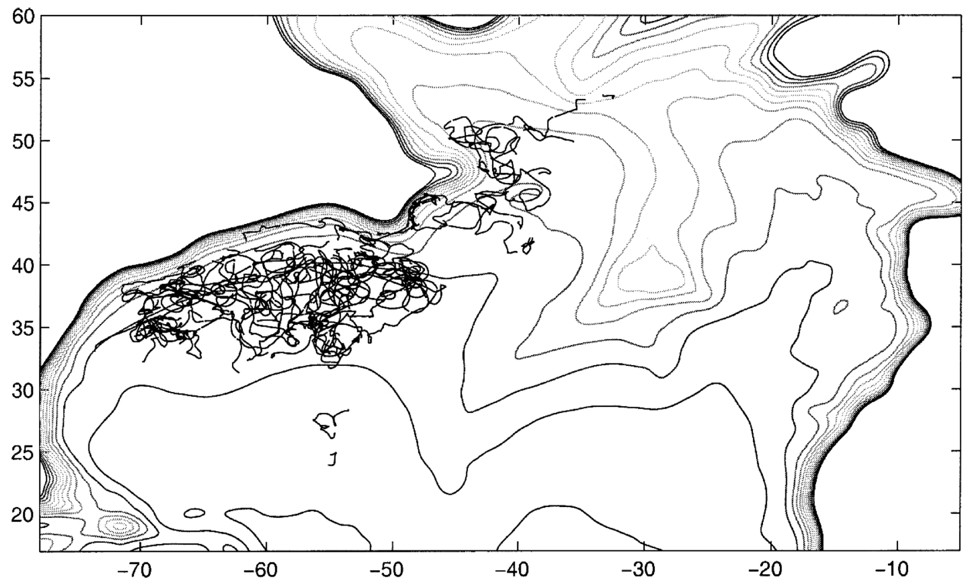

Figure 3.12: Floats in the northwestern NorthAtlantic below 1000m. The trajectories, superimposed on the smoothed f/h contours (LaCasce, 2000).

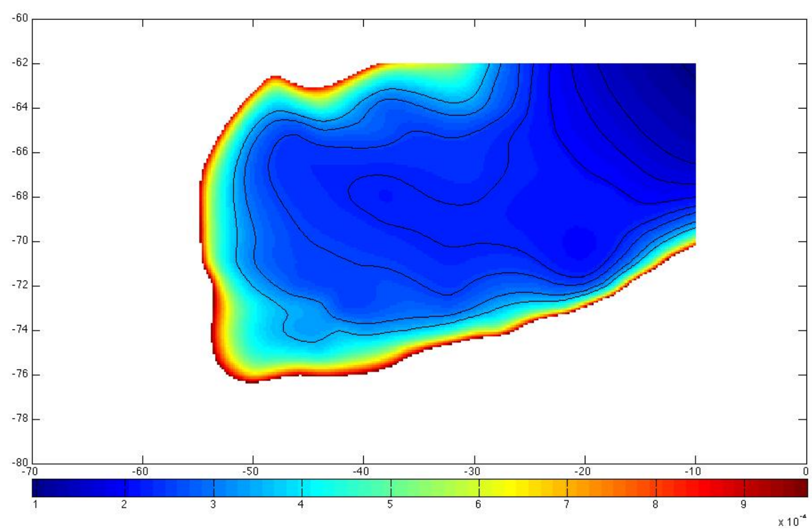

Figure 3.13: f/h countour in the Weddell Sea for 34 Ma $(34 \cdot 10^6$ years before present).

4. For the Miocene (about 34 Million years ago), the topography data were provided in the course. Calculate the f/h-contours! The length of the day was nearly as today. See Fig. **??**.

Baroclinic flow in a continuously stratified fluid

For baroclinic flow in a continuously stratified fluid, the potential vorticity can be written (Pedlosky, 1987)

$$\Pi = \frac{\zeta + f}{\rho} \cdot \nabla \lambda \tag{3.67}$$

where λ is any conserved quantity for each fluid element. In particular, if $\lambda = \rho$ then:

$$\Pi = \frac{\zeta + f}{\rho} \frac{\partial \rho}{\partial z} \tag{3.68}$$

assuming the horizontal gradients of density are small compared with the vertical gradients, a good assumption in the thermocline. In most of the interior of the ocean, $f \gg \zeta$ and (3.68) is written (Pedlosky, 1996)

$$\Pi = \frac{f}{\rho} \frac{\partial \rho}{\partial z} \tag{3.69}$$

This allows the potential vorticity of various layers of the ocean to be determined directly from hydrographic data without knowledge of the velocity field.

3.7.4 Taylor-Proudman Theorem

The influence of vorticity due to Earth's rotation is most striking for geostrophic flow of a fluid with constant density ρ_0 on a plane with constant rotation $f = f_0$. The components of the geostrophic and hydrostatic pressure equations are:

$$
\begin{aligned}
-f\,v &= -\frac{1}{\rho_0}\frac{\partial p}{\partial x} && (3.70)\\
f\,u &= -\frac{1}{\rho_0}\frac{\partial p}{\partial y} && (3.71)\\
g &= -\frac{1}{\rho_0}\frac{\partial p}{\partial z} && (3.72)
\end{aligned}
$$

and the continuity equation is:

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \tag{3.73}$$

Taking the z derivative of (3.70) and using (3.72) gives:

$$-f_0\frac{\partial v}{\partial z} = -\frac{1}{\rho_0}\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{1}{\rho_0}\frac{\partial p}{\partial z}\right) = \frac{\partial g}{\partial x} = 0 \tag{3.74}$$

Therefore for $f_0 \neq 0$

$$\frac{\partial v}{\partial z} = 0$$

Similarly, for the u-component of velocity (3.71). Thus, the vertical derivative of the horizontal velocity field must be zero.

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0 \tag{3.75}$$

The flow is two-dimensional and does not vary in the vertical direction. This is the *Taylor-Proudman Theorem*, which applies to slowly varying flows in a homogeneous, rotating, inviscid fluid. The theorem places strong constraints on the flow[9]. The physical origin of this strangely

[9]Taylor (1921): If therefore any small motion be communicated to a rotating fluid the resulting motion of the fluid must be one in which any two particles originally in a line parallel to the axis of rotation must remain so, except for

constrained flow is in the stiffness endowed to the fluid by rapid rotation of the Earth, which has a peculiarly strong sense along the axis of rotation. Taylor's laboratory experiments showed how homogeneous fluid tends to move in vertical columns. Dye in the water forms curtains, and viewing the dye from above shows fine twists and whirls that are vertically coherent.

Hence, rotation greatly stiffens the flow! Geostrophic flow cannot go over a seamount, it must go around it. Taylor [1917] explicitly derived (3.75) and (3.77) below. Proudman [1916] independently derived the same theorem but not as explicitly.

Laboratory experiments showing the formation of a Taylor column, go to 2:50, other material: vorticity and circulation, boundary layers, good introduction, Taylor column

Vertical velocity in the the Taylor-Proudman theorem

Further consequences of the theorem can be obtained by eliminating the pressure terms from (3.70, 3.71) to obtain:

$$\begin{aligned}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= -\frac{\partial}{\partial x}\left(\frac{1}{f_0\rho_0}\frac{\partial p}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{1}{f_0\rho_0}\frac{\partial p}{\partial x}\right) \\ &= \frac{1}{f_0\rho_0}\left(-\frac{\partial^2 p}{\partial x\partial y} + \frac{\partial^2 p}{\partial x\partial y}\right) = 0 \end{aligned} \tag{3.76}$$

Because the fluid is incompressible, the continuity equation (3.73) requires

$$\frac{\partial w}{\partial z} = 0 \tag{3.77}$$

Furthermore, because $w = 0$ at the sea surface and at the sea floor, if the bottom is level, there can be no vertical velocity on an f-plane.

possible small oscillations about that position.

Geostrophic flow: Vertical velocity leads to north-south currents

If the Taylor-Proudman theorem in (3.77) is true, the flow cannot expand or contract in the vertical direction, and it is indeed as rigid as a steel bar. Since we observe gradients of vertical movements, one of the constraints used in deriving (3.77) must be violated, i.e. our assumption that $f = f_0$ can not be a good approximation.

Going back to (3.56):

$$\frac{D}{Dt}(\zeta + f) + (\zeta + f)\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad . \tag{3.78}$$

we obtain

$$\beta\, v \quad + \quad f\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0 \quad . \tag{3.79}$$

Using the continuity equation, we obtain

$$f\frac{\partial w_g}{\partial z} = \beta\, v \tag{3.80}$$

where we have used the subscript $_g$ to emphasize that (3.80) applies to the ocean's interior, geostrophic flow. Thus the variation of Coriolis force with latitude allows vertical velocity gradients in the geostrophic interior of the ocean, and the vertical velocity leads to north-south currents.

Part III

Third part: Stochastic climate model and Mesoscopic Dynamics

Part IV

Fourth part: Programming and tools

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