Dynamics II, Summer semester 2024 — Exercise 7
Lecturer: Prof. Dr. G. Lohmann, Dr. M. Ionita — 27.5.2024
Tutors: Hanna Knahl, Alexander Thorneloe — Due date: 3.6.2024

1. **Laplace transform** *(2 points)*

is given by $$\mathcal{L}\{x(t)\} = L(s) = \int_0^\infty e^{-st} x(t) dt \tag{1}$$

a) Show that

$$\mathcal{L}\left\{\frac{d}{dt}x(t)\right\} = sL(s) - x(0) \tag{2}$$

through integration by parts.
b) Show furthermore that

$$\mathcal{L}\{\exp(-at)\} = \frac{1}{s+a} \tag{3}$$

2. **Laplace transformation of mixed layer model** *(4 points)*
Imagine that the temperature of the ocean mixed layer is governed by

$$\frac{dT}{dt} = -\lambda T + Q(t)\,, \tag{4}$$

where λ is the typical damping rate of a temperature anomaly and $Q(t)$ a forcing.

(a) Use the Laplace transformation to show

$$L(s) = \frac{Q(s) + T(0)}{s+\lambda} \quad . \tag{5}$$

where $Q(s) = \mathcal{L}\{Q(t)\}$

(b) Consider the special case $Q(t) = \exp(i\omega_0 t)$, then $Q(s) = \frac{1}{s - i\omega_0}$. The forcing and the temperature is of course a real number, but by representing it as a complex number we can simultaneously keep track of both phase components. Show that

$$L(s) = \frac{T(0) + Q(s)}{s+\lambda} = \frac{T(0)}{s+\lambda} + \frac{1}{(s+\lambda)}\frac{1}{(s-i\omega_0)} \tag{6}$$

and via the Laplace back-transformation and (3) of the exercise above as well as (11) that

$$T(t) = \exp(-\lambda t)T(0) + \frac{[\exp(i\omega_0 t) - \exp(-\lambda t)]}{\lambda + i\omega_0} \quad . \tag{7}$$

Calculate the real and imaginary part of (7).

(c) Take the real part. Show: At low frequencies, the output $T(t)$ is similar to the forcing $Q(t)$. At high frequencies it rolls off as $1/\omega$ (it is a low-pass filter) and is out of phase by 90°.

(d) Instead of b), consider now the special case $Q(t) = c \cdot u(t)$ with $u(t)$ as unit step or the so-called Heaviside step function (step forcing). Show via Laplace transform that

$$< T(t) > \; = \; \mathcal{L}^{-1}\{L(s)\}(t) = \mathcal{L}^{-1}\left\{ \frac{< T(0) >}{s+\lambda} + \frac{c}{s} \cdot \frac{1}{s+\lambda} \right\} \tag{8}$$

$$= \; T(0) \cdot \exp(-\lambda t) \; + \; \frac{c}{\lambda}\left(1 - \exp(-\lambda t)\right) \tag{9}$$

the equilibrium response

$$\Delta T = \lim_{t\to\infty} < T(t) > = \frac{c}{\lambda} \, . \tag{10}$$

Hint:

$$\mathcal{L}\left\{ -\exp(-at) + \exp(-bt) \right\} \; = \; \frac{-1}{s+a} + \frac{1}{s+b} = \frac{a-b}{(s+a)} \frac{1}{(s+b)} \tag{11}$$

3. **Stochastic climate model** *(4 points)(1,1,2)*

Imagine that the temperature of the ocean mixed layer of depth h is governed by

$$\frac{dT}{dt} = -\lambda T + Q \, , \tag{12}$$

where λ is the typical damping rate of a temperature anomaly, and Q is the air-sea fluxes due to weather systems are represented by a white-noise process with zero average $< Q > = 0$ and δ-correlated in time

$$Cov_Q(\tau) = < Q(t)Q(t+\tau) > = c \cdot \delta(\tau) \quad . \tag{13}$$

The Fourier transform of the auto-correlation function $Cov_Q(\tau)$ is called spectrum

$$S_Q(\omega) = \int_{\mathsf{R}} Cov_Q(\tau) e^{i\omega\tau} \, d\tau \quad = \int_{\mathsf{R}} c \cdot \delta(\tau) e^{i\omega\tau} \, d\tau \quad = \quad c \tag{14}$$

a) Solve Eq. (12) for the temperature response $T = \hat{T}(\omega) e^{-i\omega t}$ and hence show that:

$$\hat{T}(\omega) = \frac{\hat{Q}(\omega)}{(\lambda - i\omega)} \tag{15}$$

b) Show that it has a spectral density $\hat{T}(\omega)\,\hat{T}^*(\omega)$ is given by:

$$\hat{T}\,\hat{T}^* = \frac{\hat{Q}\,\hat{Q}^*}{(\lambda^2 + \omega^2)} \tag{16}$$

where $\hat{Q}^*$ is the complex conjugate.

c) Show that the spectrum of the T is

$$S_T(\omega) =< \hat{T}\,\hat{T}^* >= \frac{< \hat{Q}\,\hat{Q}^* >}{(\lambda^2 + \omega^2)} = \frac{c}{(\lambda^2 + \omega^2)}. \qquad (17)$$

The brackets $< \ldots >$ denote the ensemble mean. Make a sketch of the spectrum S_T using a log-log plot.

4. ***Covariance and spectrum*** *(additional 3 points)*
Show that the definition of the spectrum via $S(\omega) = \langle\, \hat{x}\hat{x}^* \,\rangle$ and the Fouriertransformation as it is used in (14) are equivalent.

Hint: Use the **ergodic hypothesis** where the ensemble average $S(\omega) = \langle\, \hat{x}\hat{x}^* \,\rangle$ can be expressed as the time average

$$\lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} dt \quad \hat{x}\hat{x}^* \qquad (18)$$

where

$$x(t) = \frac{1}{2\pi} \int_{\mathsf{R}} \hat{x}(\omega) e^{-i\omega t}\, d\omega \qquad (19)$$

Notes on submission form of the exercises: *Working in study groups is encouraged, but each student is responsible for his/her own solution. The answers to the questions can be send until the due date (12:00) to Hanna Knahl (hanna.knahl@awi.de), Alexander Thorneloe (alexander.thorn@awi.de).*