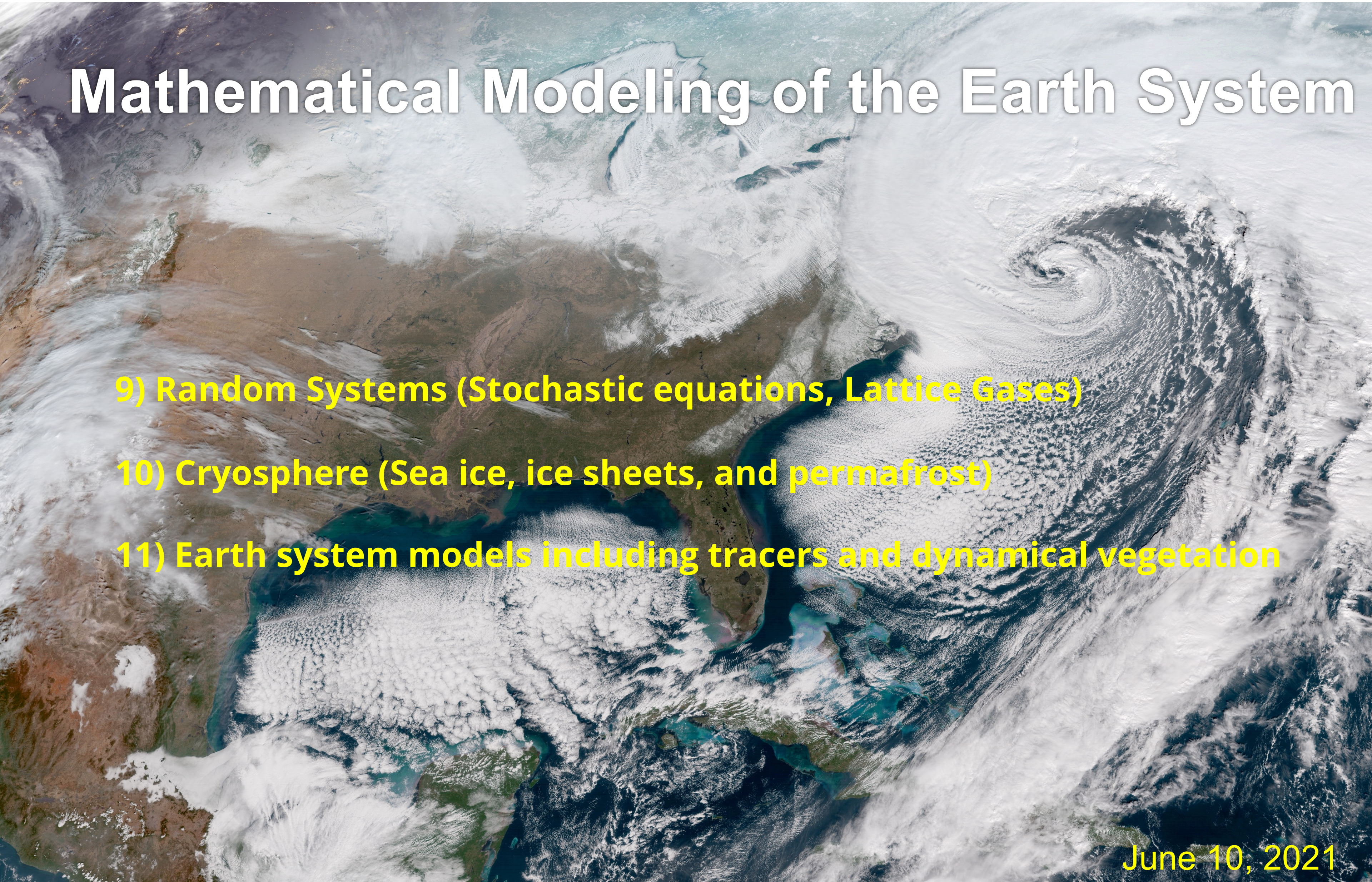
Lecture 9: Gerrit Lohmann

ALFRED-WEGENER-INSTITUT
HELMHOLTZ-ZENTRUM FÜR POLAR-
UND MEERESFORSCHUNG



Mathematical Modeling of the Earth System

9) Random Systems (Stochastic equations, Lattice Gases)

10) Cryosphere (Sea ice, ice sheets, and permafrost)

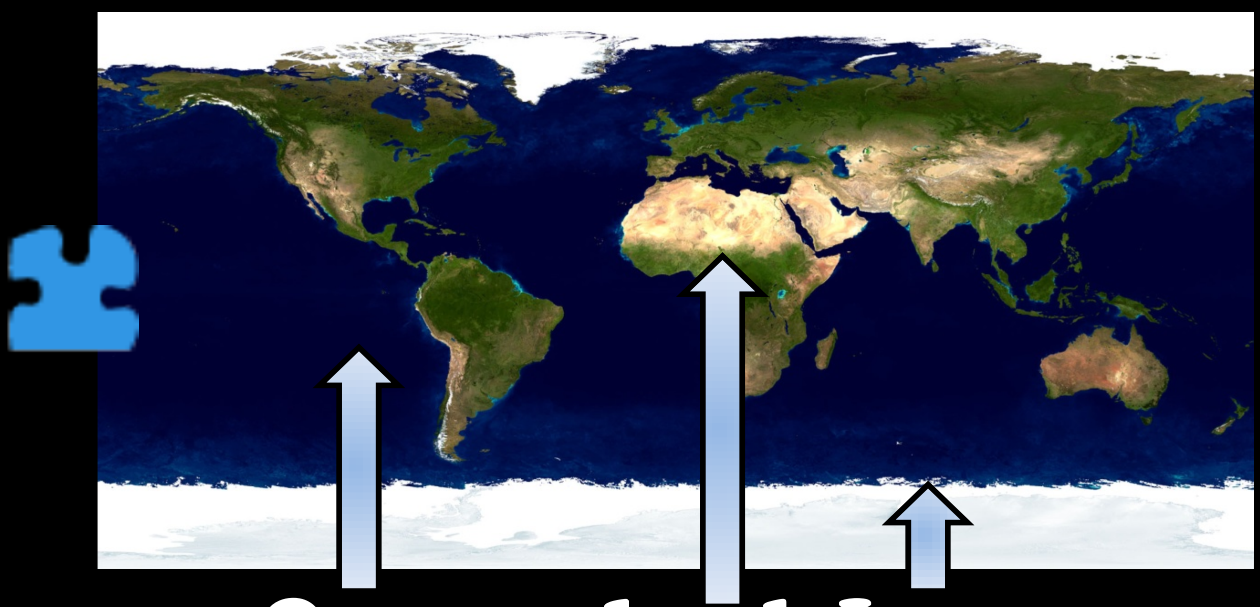
11) Earth system models including tracers and dynamical vegetation

June 10, 2021

How well do we understand Earth?

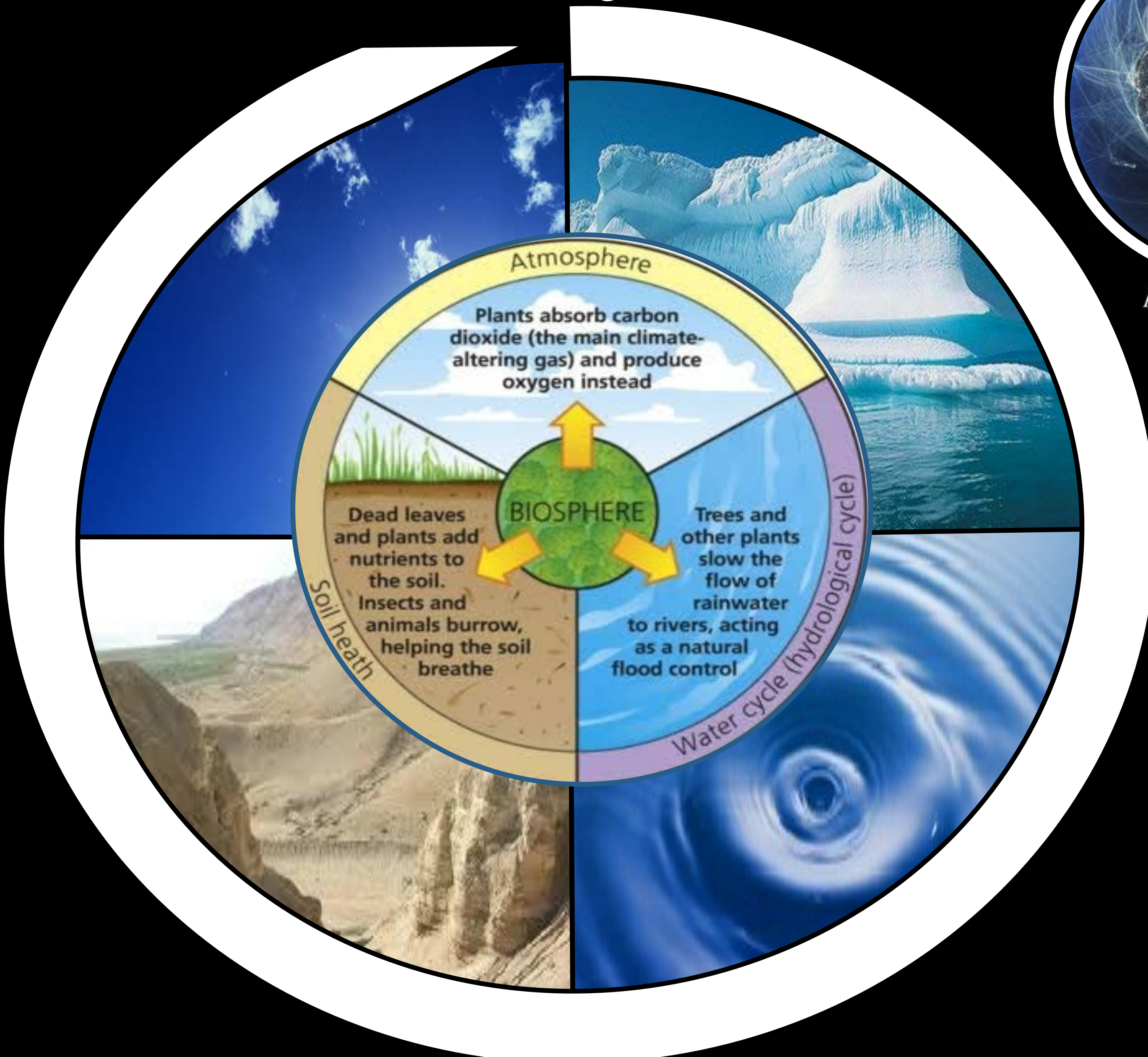
Weather

Climate

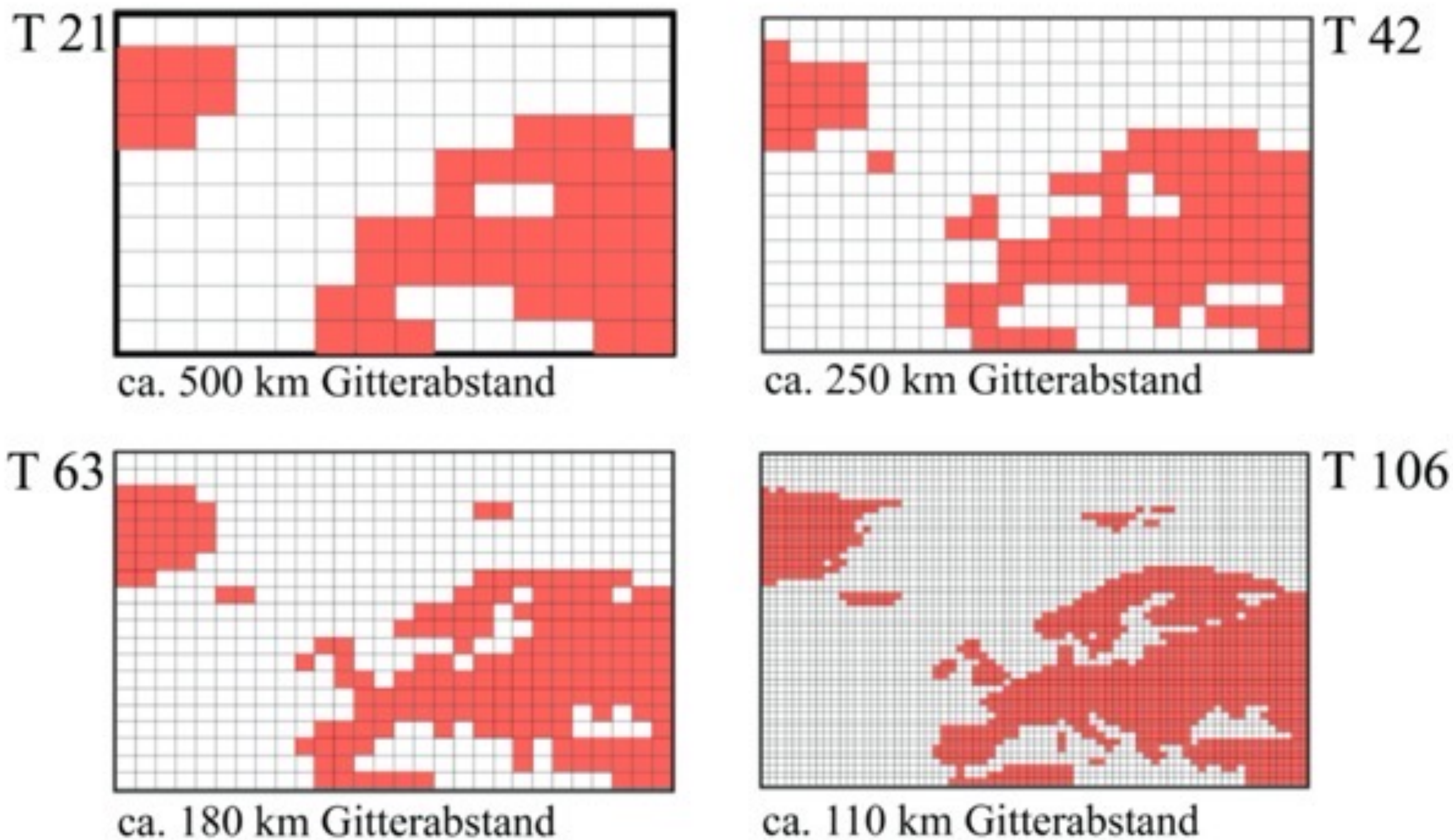


Oceans, land, Ice

Earth System

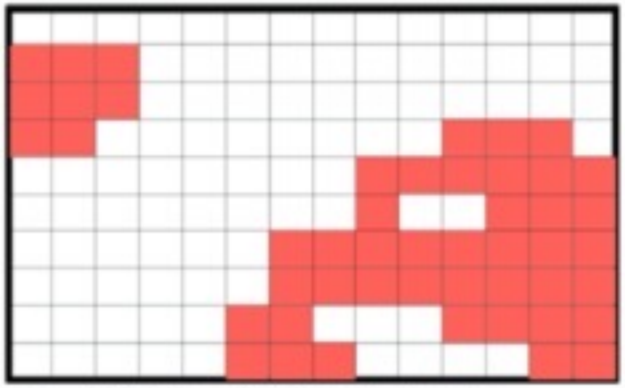


Examples of Resolution
(global spectral model, zoom onto Europe)



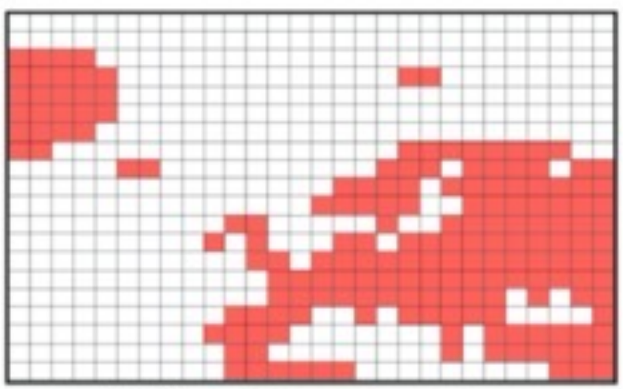
Examples of Resolution
(global spectral model, zoom onto Europe)

T 21

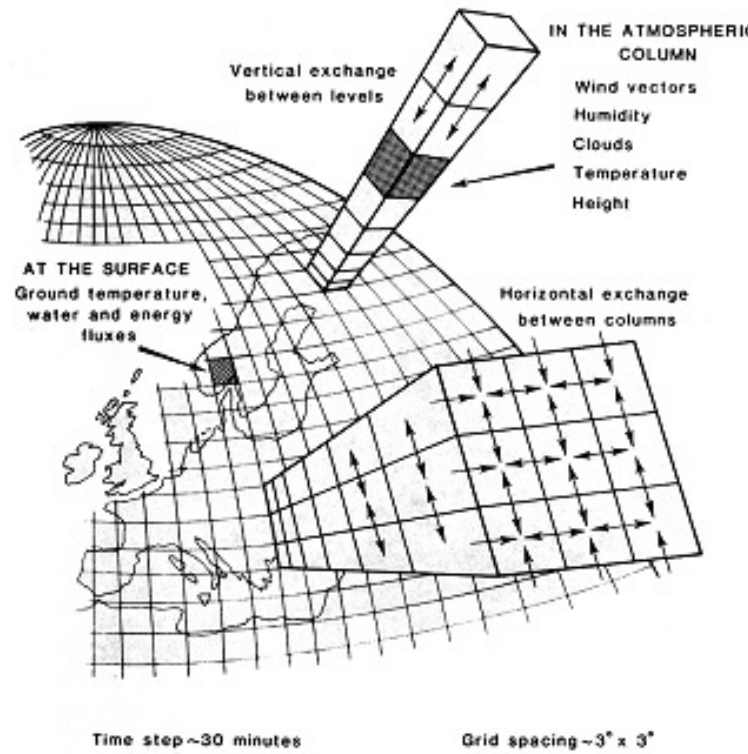


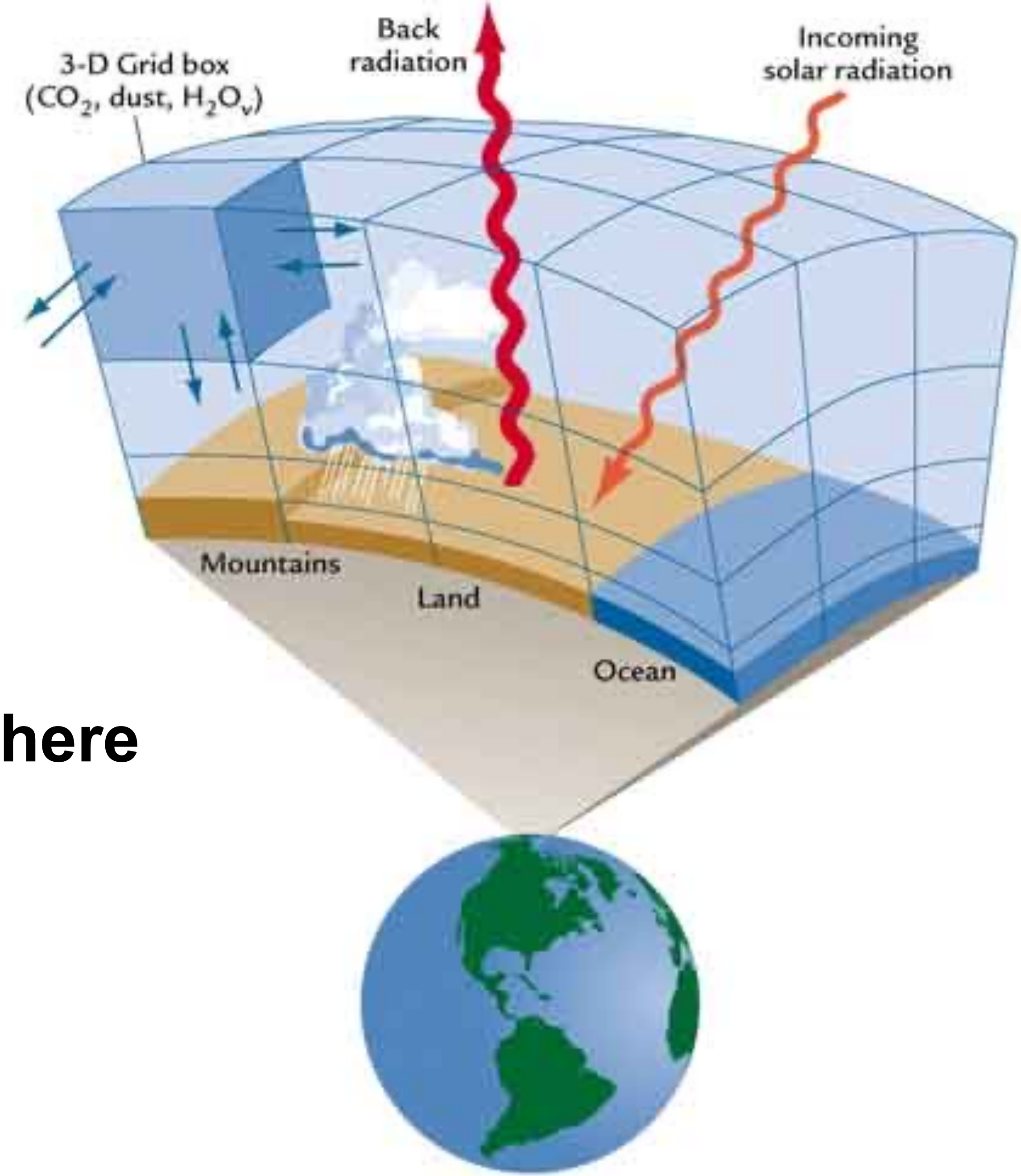
ca. 500 km Gitterabstand

T 63



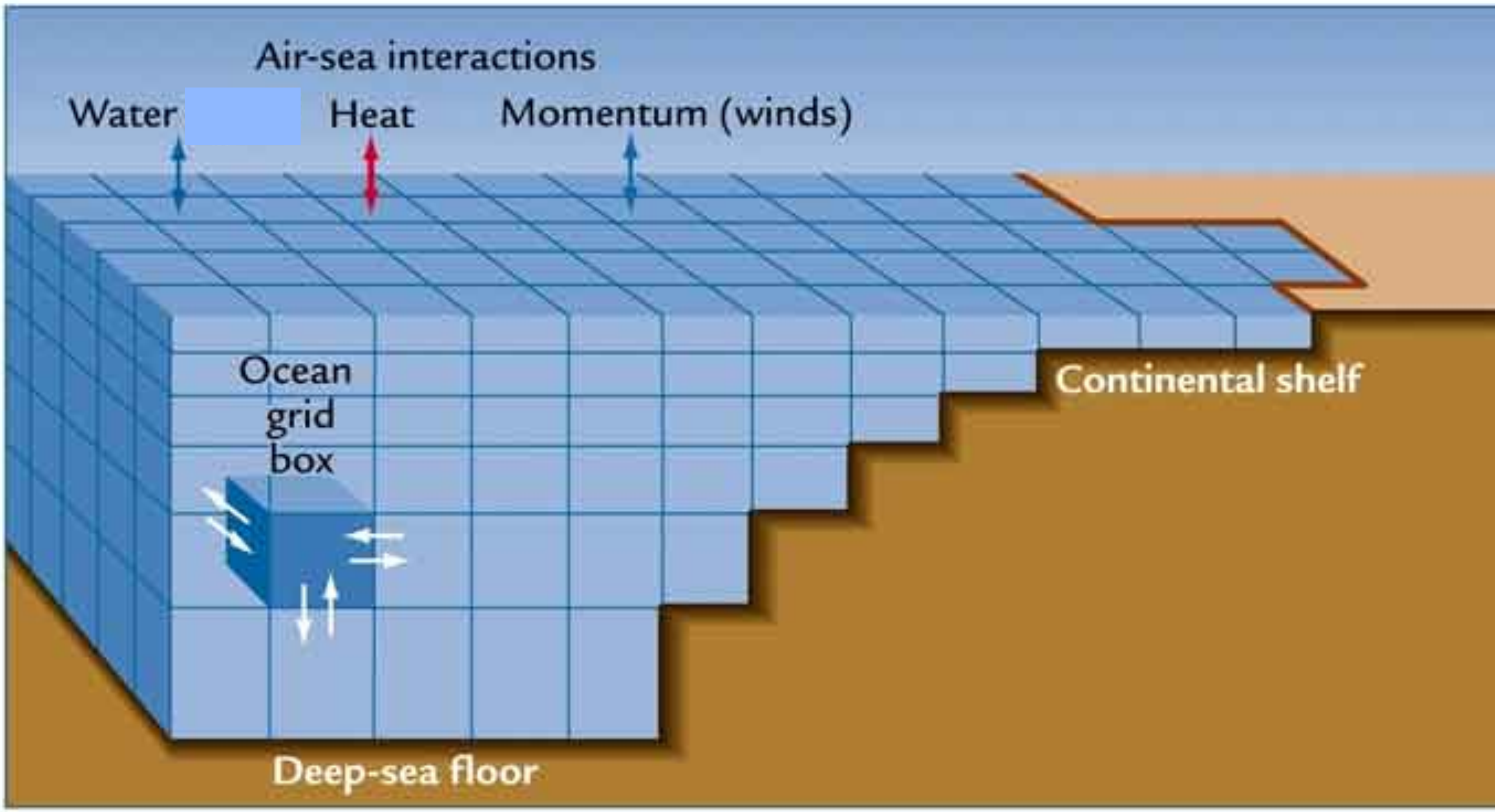
ca. 180 km Gitterabstand

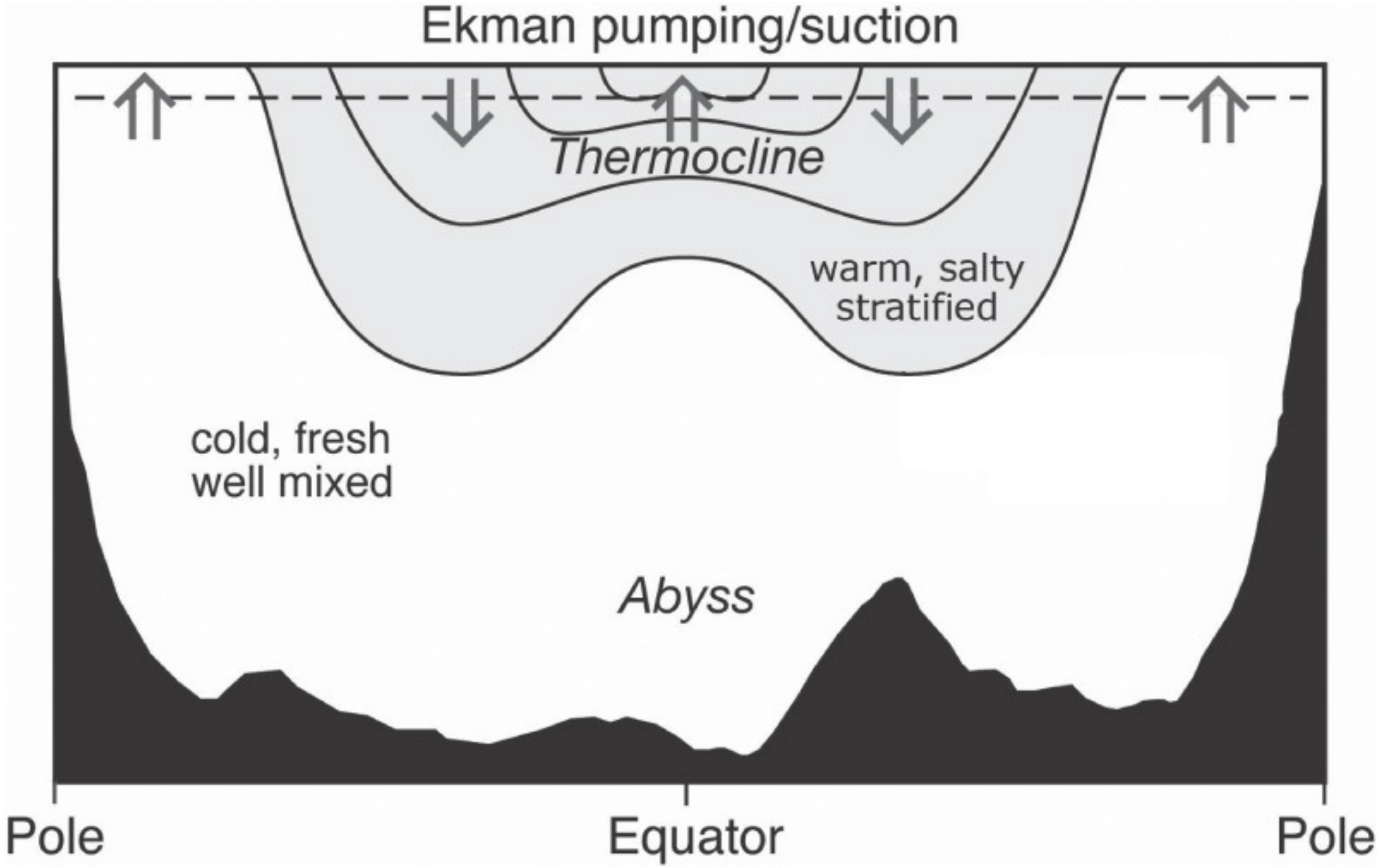




Atmosphere Model

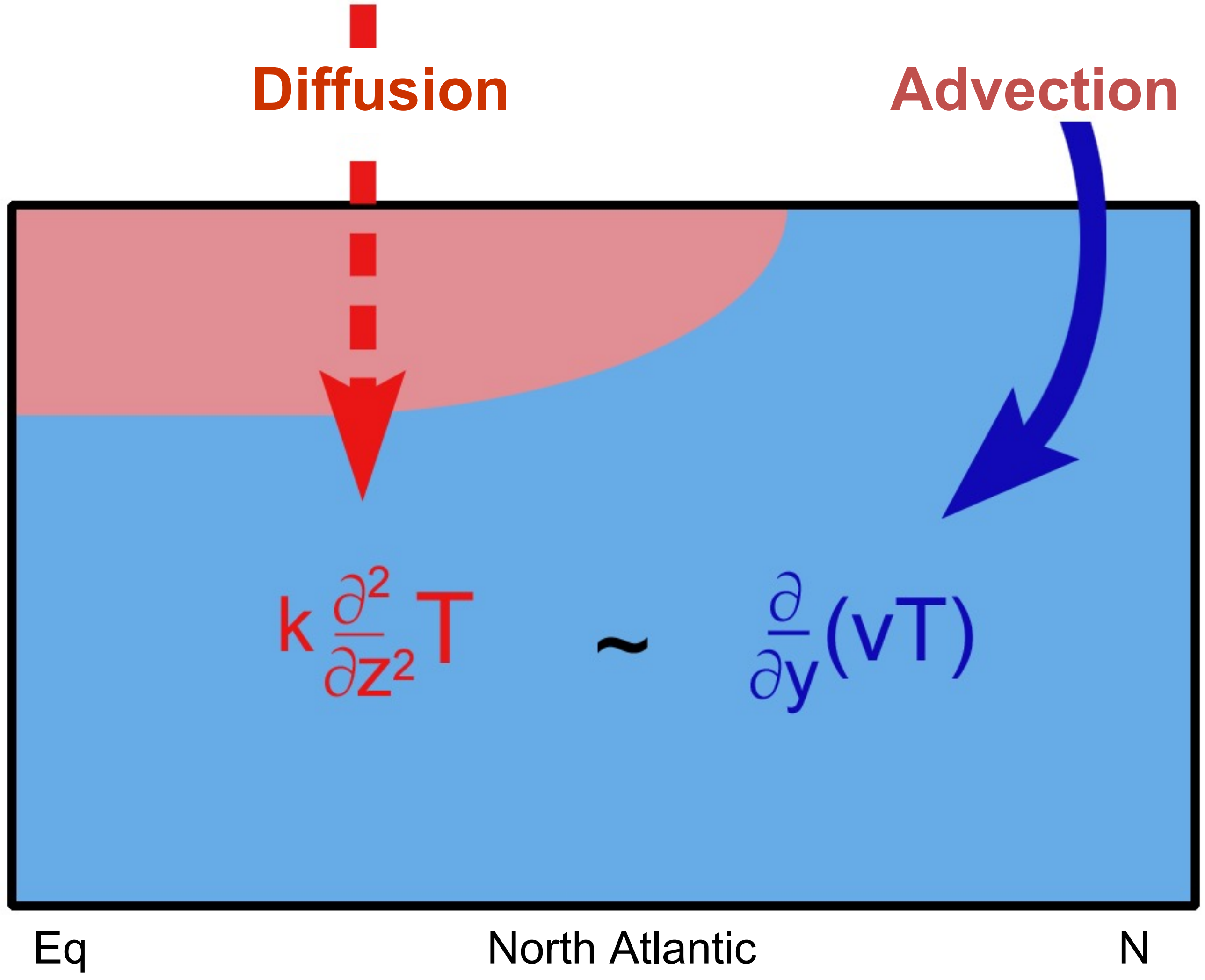
Ocean circulation models and boundary conditions





Ekman: winds blowing on the sea surface produce a thin, horizontal boundary layer, *(~100 m)*

Ocean Dynamics: Temperature change



Mathematical Modelling

Classes of models

Ordinary diffential eq. (Box models)
Partial diffential eq. (Diffusion & Advection)
Stochastic (different time & length scales)
Discrete dynamics (e.g., Population dynamics)

Finite differences: Diffusion-advection

$$\frac{\partial C(x,t)}{\partial t} + u\frac{\partial C(x,t)}{\partial x} = \kappa\frac{\partial^2 C(x,t)}{\partial x^2} \qquad (1)$$

Finite differences: Diffusion-advection

$$\frac{\partial C(x,t)}{\partial t} + u\frac{\partial C(x,t)}{\partial x} = \kappa \frac{\partial^2 C(x,t)}{\partial x^2} \qquad (1)$$

Approximate the derivatives using the centered difference scheme:

$$\frac{\partial C_{i,j}}{\partial t} \approx \frac{C_{i,j+1} - C_{i,j-1}}{2\Delta t} \qquad (2)$$

$$\frac{\partial C_{i,j}}{\partial x} \approx \frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} \qquad (3)$$

$$\frac{\partial^2 C_{i,j}}{\partial x^2} \approx \frac{C_{i+1,j} - 2C_{i,j} + C_{i-1,j}}{\Delta x^2} \qquad (4)$$

Finite differences Diffusion-advection

$$\frac{\partial C(x,t)}{\partial t} + u\frac{\partial C(x,t)}{\partial x} = \kappa\frac{\partial^2 C(x,t)}{\partial x^2} \qquad (1)$$

Approximate the derivatives using the centered difference scheme:

$$\frac{\partial C_{i,j}}{\partial t} \approx \frac{C_{i,j+1} - C_{i,j-1}}{2\Delta t} \qquad (2)$$

$$\frac{\partial C_{i,j}}{\partial x} \approx \frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} \qquad (3)$$

$$\frac{\partial^2 C_{i,j}}{\partial x^2} \approx \frac{C_{i+1,j} - 2C_{i,j} + C_{i-1,j}}{\Delta x^2} \qquad (4)$$

Inserting (2)-(4) into (1) gives

$$\frac{C_{i,j+1} - C_{i,j-1}}{2\Delta t} + u\frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} \approx \kappa\frac{C_{i+1,j-1} - 2C_{i,j-1} + C_{i-1,j-1}}{\Delta x^2} \qquad (5)$$

(Note that we evaluate the diffusion term at time $j-1$ instead of time j to avoid numerical instability.) Rephrasing

Finite differences: Diffusion-advection

$$\frac{\partial C(x,t)}{\partial t} + u\frac{\partial C(x,t)}{\partial x} = \kappa \frac{\partial^2 C(x,t)}{\partial x^2} \tag{1}$$

Approximate the derivatives using the centered difference scheme:

$$\frac{\partial C_{i,j}}{\partial t} \approx \frac{C_{i,j+1} - C_{i,j-1}}{2\Delta t} \tag{2}$$

$$\frac{\partial C_{i,j}}{\partial x} \approx \frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} \tag{3}$$

$$\frac{\partial^2 C_{i,j}}{\partial x^2} \approx \frac{C_{i+1,j} - 2C_{i,j} + C_{i-1,j}}{\Delta x^2} \tag{4}$$

Inserting (2)-(4) into (1) gives

$$\frac{C_{i,j+1} - C_{i,j-1}}{2\Delta t} + u\frac{C_{i+1,j} - C_{i-1,j}}{2\Delta x} \approx \kappa \frac{C_{i+1,j-1} - 2C_{i,j-1} + C_{i-1,j-1}}{\Delta x^2} \tag{5}$$

(Note that we evaluate the diffusion term at time $j-1$ instead of time j to avoid numerical instability.) Rephrasing

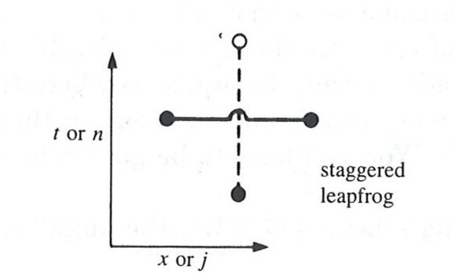
Leapfrog scheme

$$C_{i,j+1} \approx C_{i,j-1} - u\frac{\Delta t}{\Delta x}(C_{i+1,j} - C_{i-1,j}) + \kappa \frac{2\Delta t}{\Delta x^2}(C_{i+1,j-1} - 2C_{i,j-1} + C_{i-1,j-1}) \tag{6}$$

Finite differences: Diffusion-advection

$$\frac{\partial C(x,t)}{\partial t} + u\frac{\partial C(x,t)}{\partial x} = \kappa \frac{\partial^2 C(x,t)}{\partial x^2}$$

Leap-Frog Scheme (CTCS Scheme)



The leap-frog scheme is second order in time and space,

$$T_i^{n+1} = T_i^{n-1} - \frac{V\Delta t}{\Delta x}\left(T_{i+1}^n - T_{i-1}^n\right)$$

but it requires that the two last time levels are kept in memory.

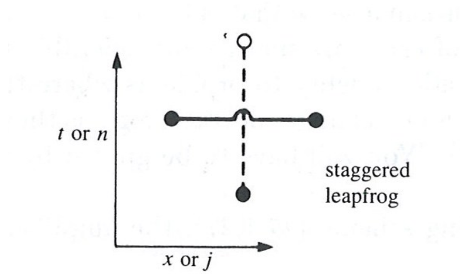
Leapfrog scheme

$$C_{i,j+1} \approx C_{i,j-1} - u\frac{\Delta t}{\Delta x}(C_{i+1,j} - C_{i-1,j}) + \kappa\frac{2\Delta t}{\Delta x^2}(C_{i+1,j-1} - 2C_{i,j-1} + C_{i-1,j-1})$$

Finite differences: Diffusion-advection

$$\frac{\partial C(x,t)}{\partial t} + u\frac{\partial C(x,t)}{\partial x} = \kappa\frac{\partial^2 C(x,t)}{\partial x^2} \qquad (1)$$

Leap-Frog Scheme (CTCS Scheme)

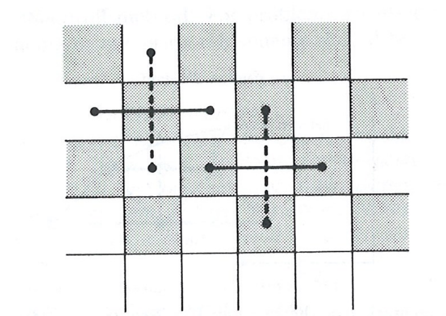


The leap-frog scheme is second order in time and space,

$$T_i^{n+1} = T_i^{n-1} - \frac{V\Delta t}{\Delta x}\left(T_{i+1}^n - T_{i-1}^n\right)$$

but it requires that the two last time levels are kept in memory.

Chess Board for Leap-Frog



Leap-frog is stable, provided CFL-criterium is fulfilled, less diffusive than upwind, but leads to oscillations, especially at sharp gradients. There can be a decoupling of two solutions!

Leapfrog scheme

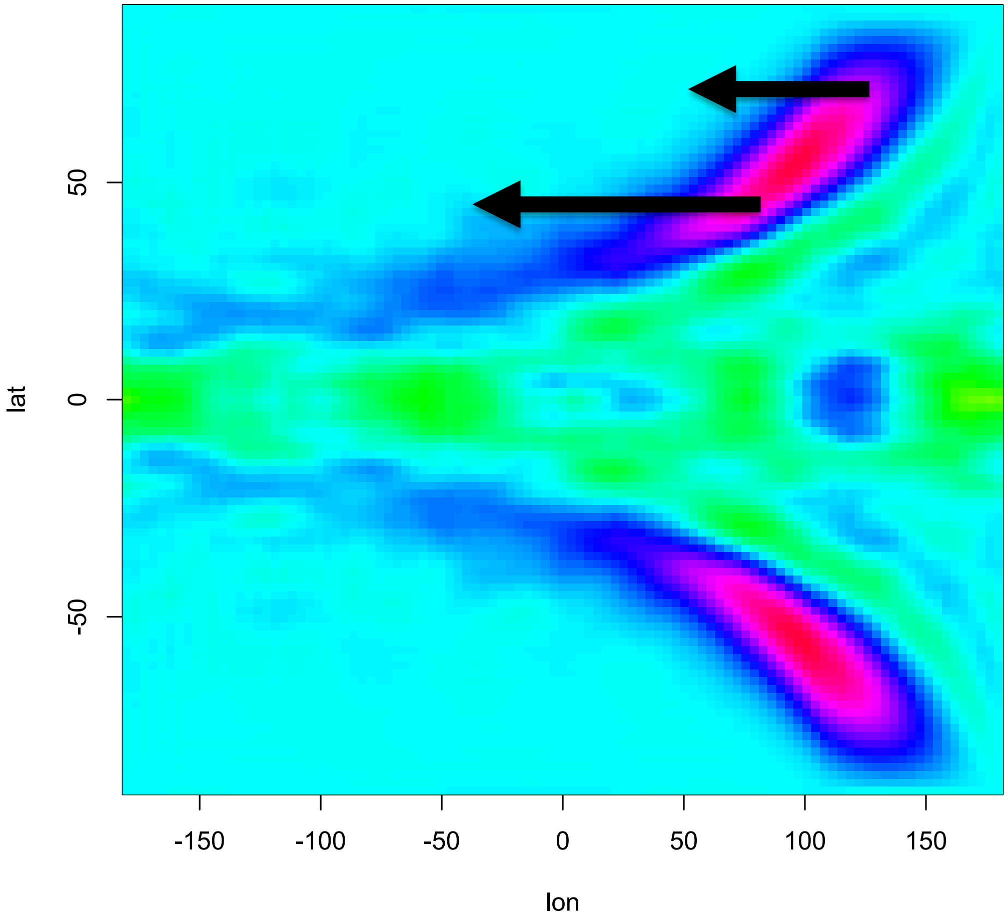
$$C_{i,j+1} \approx C_{i,j-1} - u\frac{\Delta t}{\Delta x}(C_{i+1,j} - C_{i-1,j}) + \kappa\frac{2\Delta t}{\Delta x^2}(C_{i+1,j-1} - 2C_{i,j-1} + C_{i-1,j-1}) \qquad (6)$$

Shallow Water Model

Rossby and gravity waves

$$
\begin{aligned}
\partial_t u &= f\, v - g\, \partial_x \eta \\
\partial_t v &= -f\, u - g\, \partial_y \eta \\
\partial_t \eta &= -\partial_x (H u) - \partial_y (H v)
\end{aligned}
$$

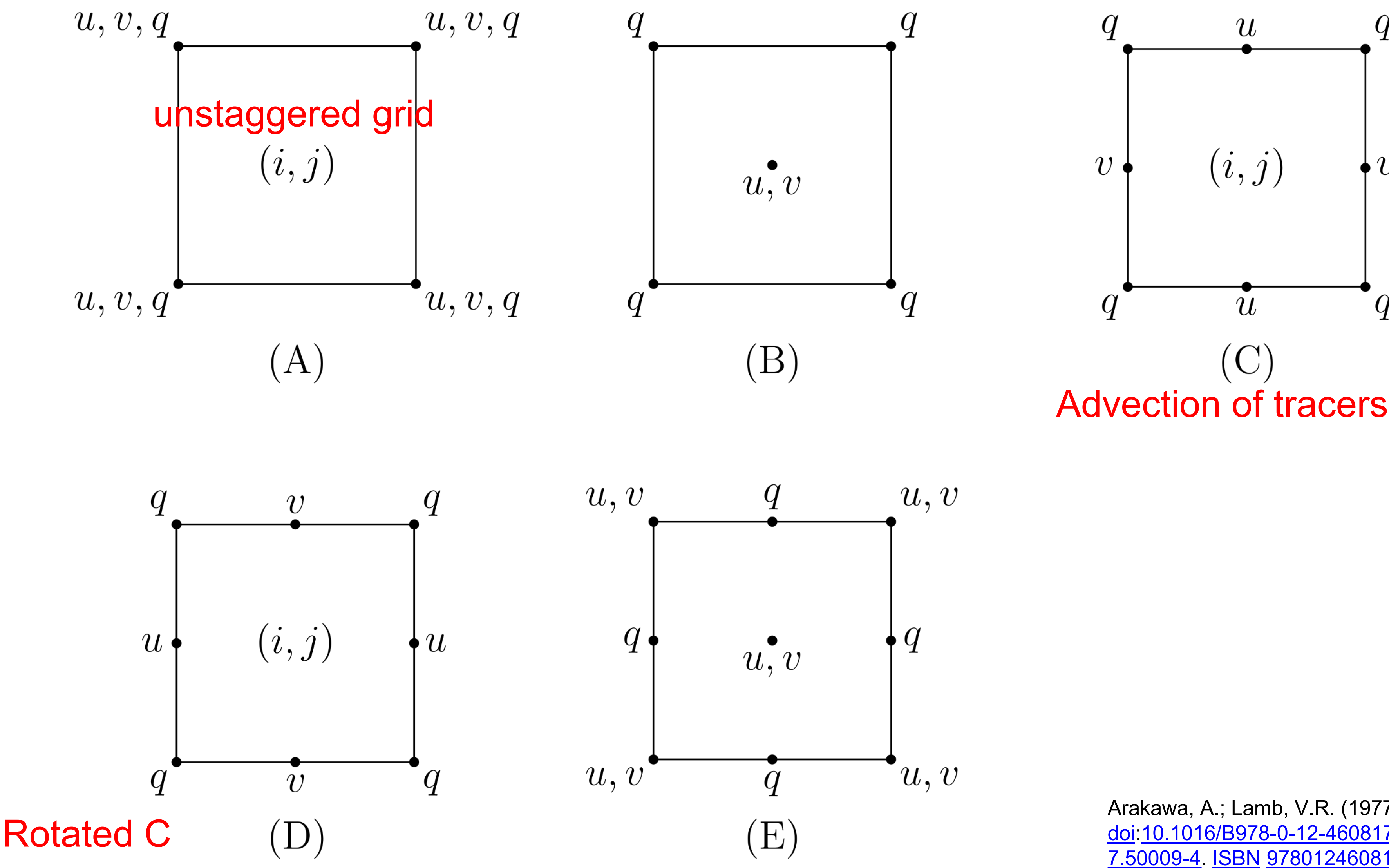
```
u.new[ia.0,ia.0]<-u.old[ia.0,ia.0]-g*dt/dx*(h[ia.p1,ia.0]-h[ia.m1,ia.0])+dt*f*v
v.new[ia.0,ia.0]<-v.old[ia.0,ia.0]-g*dt/dy*(h[ia.0,ia.p1]-h[ia.0,ia.m1])-dt*f*u
h.new[ia.0,ia.0]<-h.old[ia.0,ia.0]-H*dt*((u[ia.p1,ia.0]-u[ia.m1,ia.0])/dx + (v[ia.0,ia.p1]-
v[ia.0,ia.m1])/dy)
```



$$\omega(k,l) = -\frac{\beta k}{k^2 + l^2},$$

Figure 7.3: Global Rossby and Kelvin wave signatures in the exercise 49.

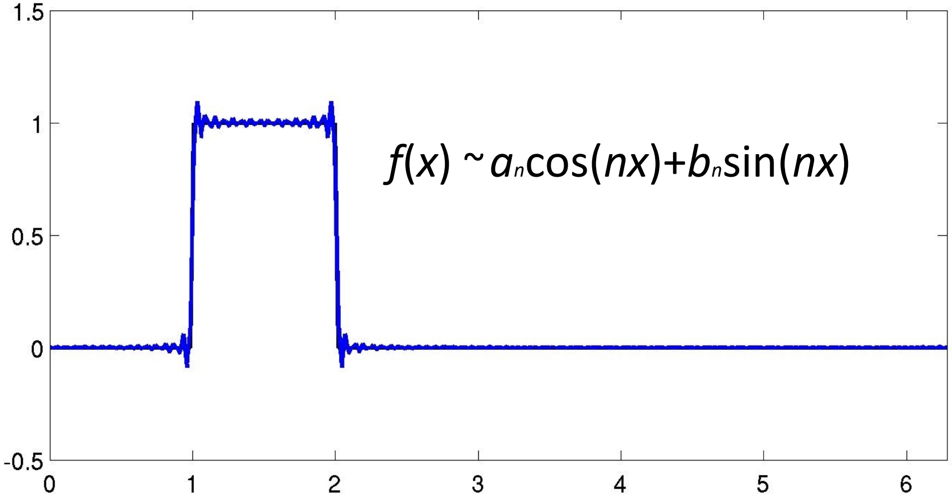
2D Staggered grids: Arakawa



Arakawa, A.; Lamb, V.R. (1977)
doi:10.1016/B978-0-12-460817-7.50009-4. ISBN 9780124608177.

Spectral methods

Rayleigh Bernard System
in lecture Dynamics II

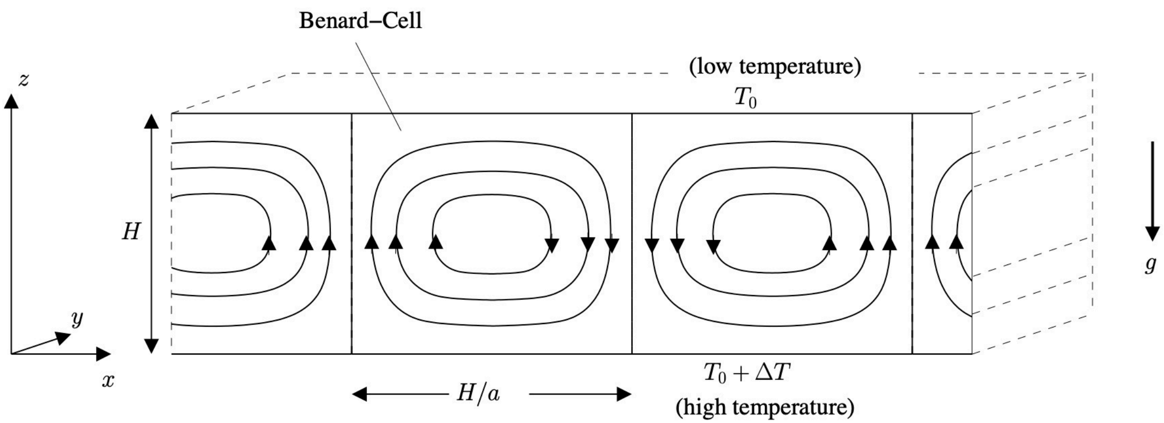


solve the diffusion equation
$\partial_t T = D\, \partial^2_x T$

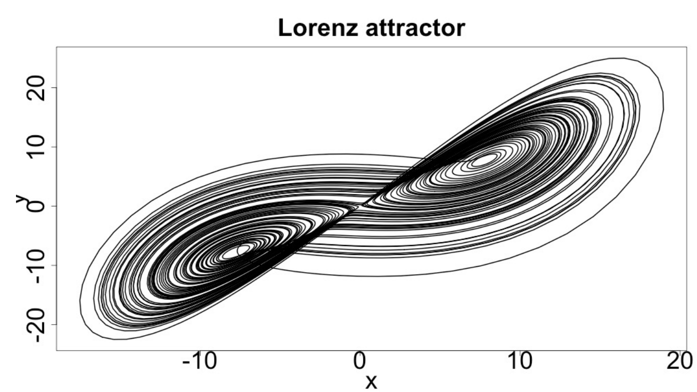
HELMHOLTZ

Fourier–Galerkin method

used for the basis functions (the famous chaotic Lorenz set of differential equations were found as a Fourier- Galerkin approximation to atmospheric convection (Lorenz, 1963)



$$\Psi(x,z,t)=\sum k \sum l \; \Psi k,l(t)\sin\left(k\pi a H x\right)\times\sin\left(l\pi H z\right)$$



Finite differences and finite element methods

finite differences: approximate partial differential equations.
Questions to analyze and improve the stability and accuracy

Finite-difference Approximation of Derivatives

There are many ways to replace differential quotients with difference quotients, e.g.:

$$\frac{\partial T}{\partial x} \rightarrow \frac{T_j - T_{j-1}}{\Delta x} \quad \text{(backward)}$$
$$\rightarrow \frac{T_{j+1} - T_j}{\Delta x} \quad \text{(forward)}$$
$$\rightarrow \frac{T_{j+1} - T_{j-1}}{2\Delta x} \quad \text{(centered)}$$

All these tend to $\frac{\partial T}{\partial x}$ for $\Delta x \rightarrow 0$

A 2nd-order accurate Difference Quotient

in the centered difference approximation, however, we obtain from Taylor's theorem

$$\frac{T(x+\Delta x) - T(x-\Delta x)}{2\Delta x} = \frac{\partial T}{\partial x} + \frac{1}{3!}\frac{\partial^3 T}{\partial x^3}(\Delta x)^2 + \ldots$$

because the terms proportional to the second derivative cancel out. The error in the centered finite-difference approximation is of the order $(\Delta x)^2$.

Finite differences and finite element methods

As powerful as these ideas are, there are two important cases where they do not directly apply: problems that are

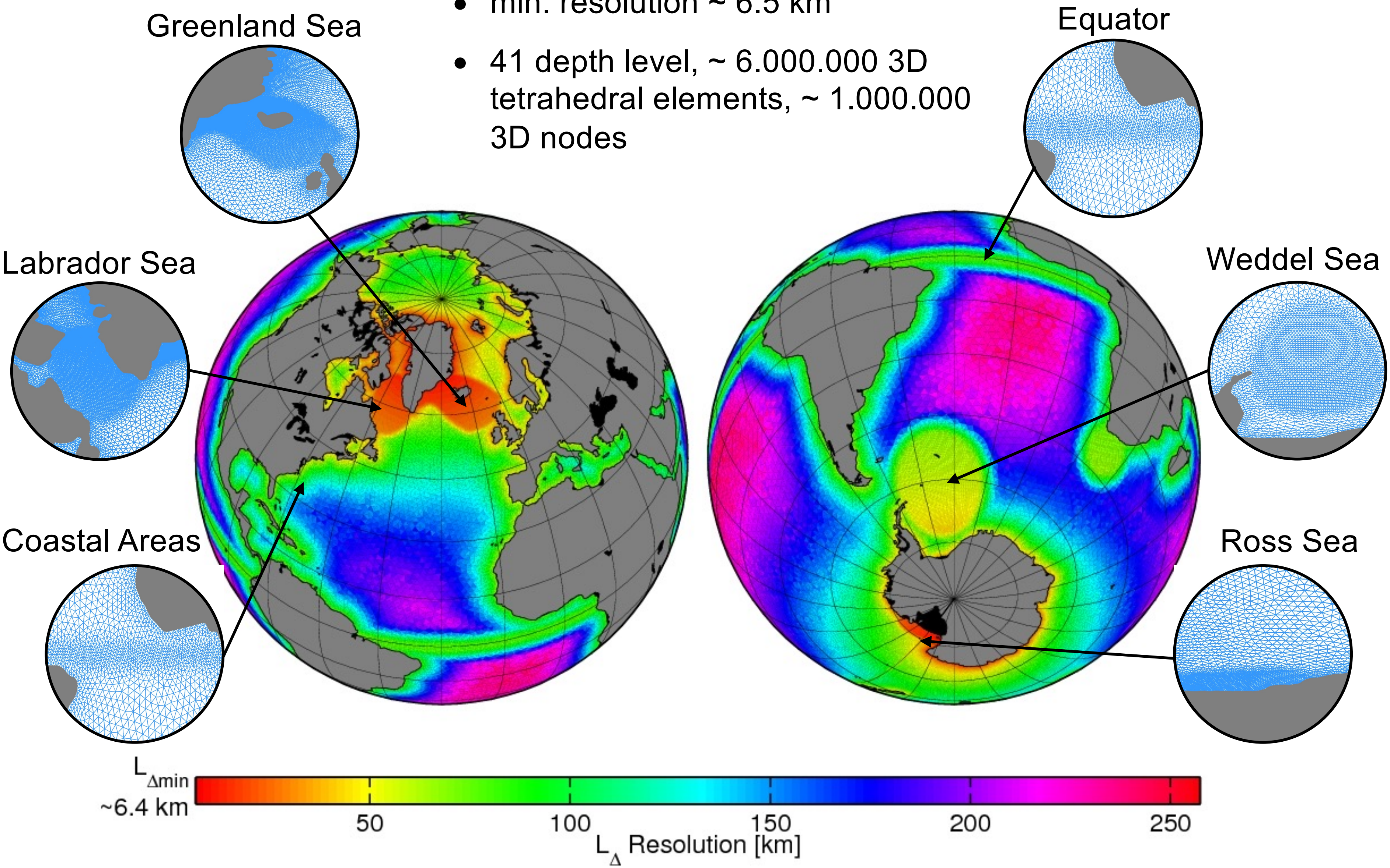
- described in terms of a **spatially inhomogeneous grid,**
- posed in **terms of a variational principle**.

For example, in studying the deformations, it can be most natural to describe it in terms of **finding the minimum energy configuration** instead of a partial differential equation, and for computational efficiency it is certainly important to match the location of the solution nodes to the shape of the body.

These limitations with finite differences can be solved by the use of **finite element methods.**

Ocean Model Setup in finite elements

- mesh with high resolution at deep water mass formation areas
- min. resolution ~ 6.5 km
- 41 depth level, ~ 6.000.000 3D tetrahedral elements, ~ 1.000.000 3D nodes



Mathematical Modelling

Classes of models

Ordinary diffential eq. (Box models)

Partial diffential eq. (Diffusion & Advection)

Stochastic (different time & length scales)

Discrete dynamics (e.g., Population dynamics)

Euler integrator

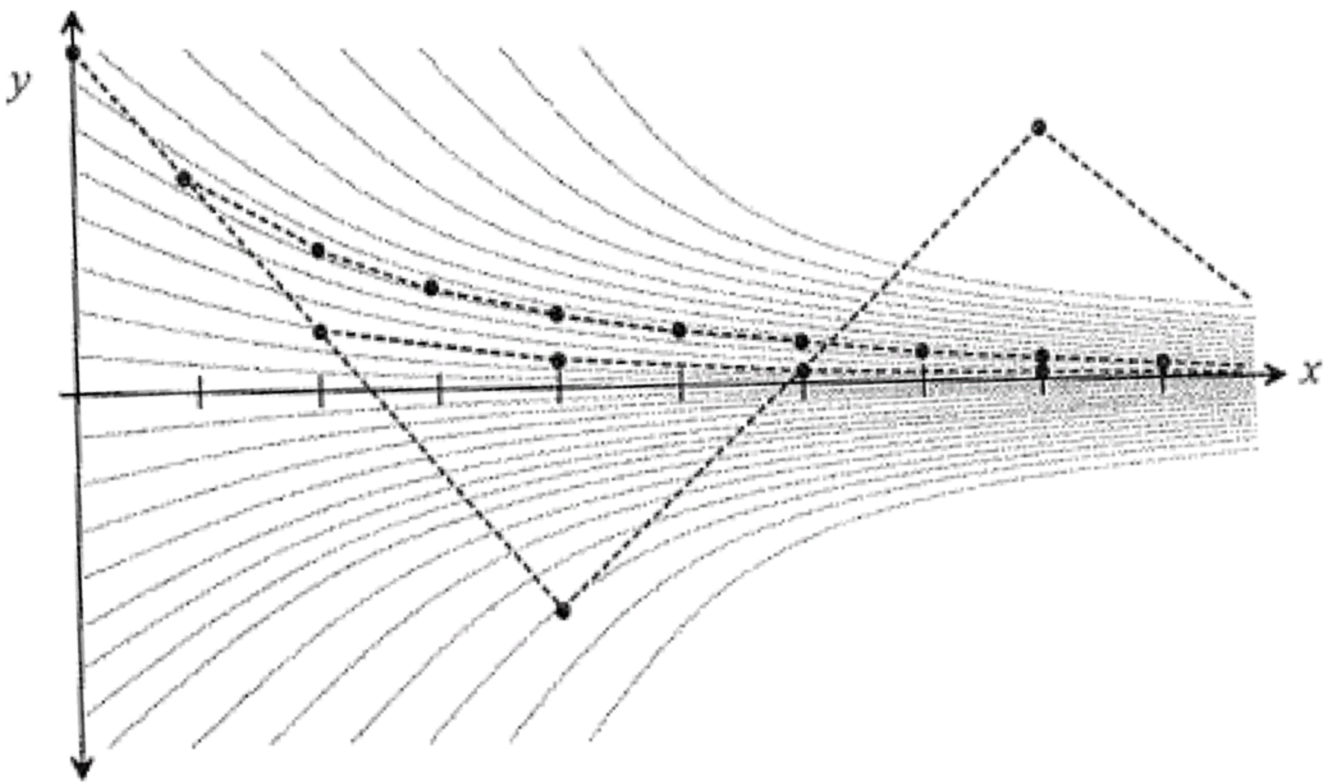


Figure 6.2. Origin of oscillation in the Euler method. The gray lines show the family of solutions of $dy/dx = Ay$, and the dotted lines show the numerical solution for various step sizes.

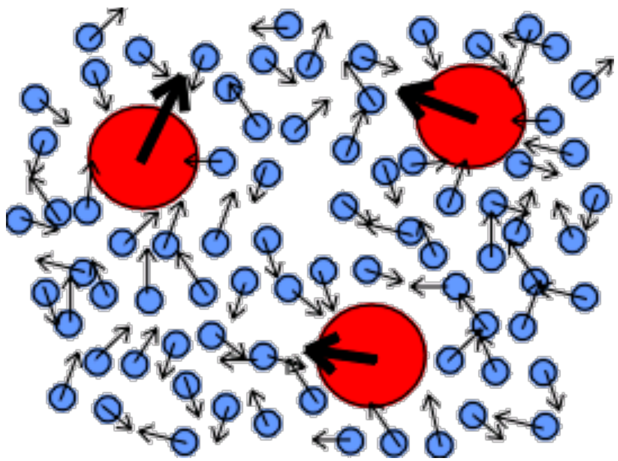
This algorithm approximates the point computation by this formula

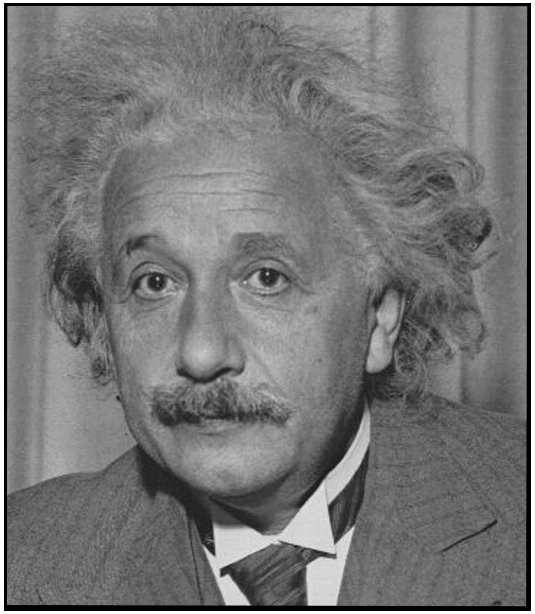
$p_{k+1} = p_k + hv(p_k)$

where *h* specifies the *integration step.*

The streamline is then constructed by successive integration.

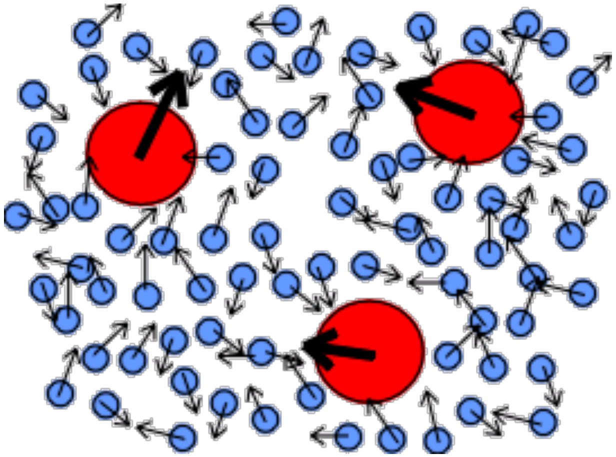
Brownian Motion





Einstein,Orstein, Uhlenbeck, Wiener, Fokker, Planck et al.

dx/dt = f(x) + g(x) dw/dt

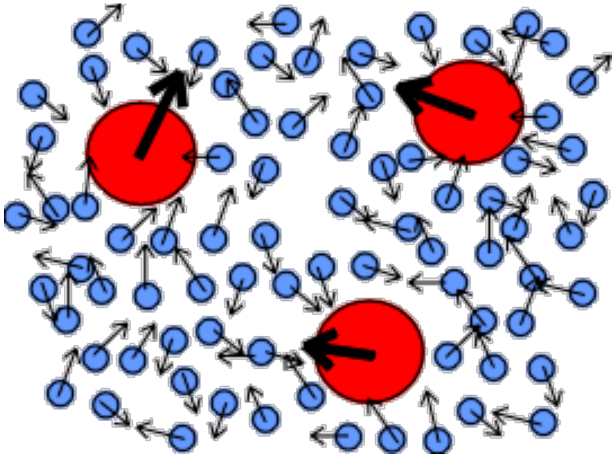


Brownian motion

is the random movement of particles, caused by their bombardment on all sides by bigger molecules.

This motion can be seen in the behavior of pollen grains placed in a glass of water

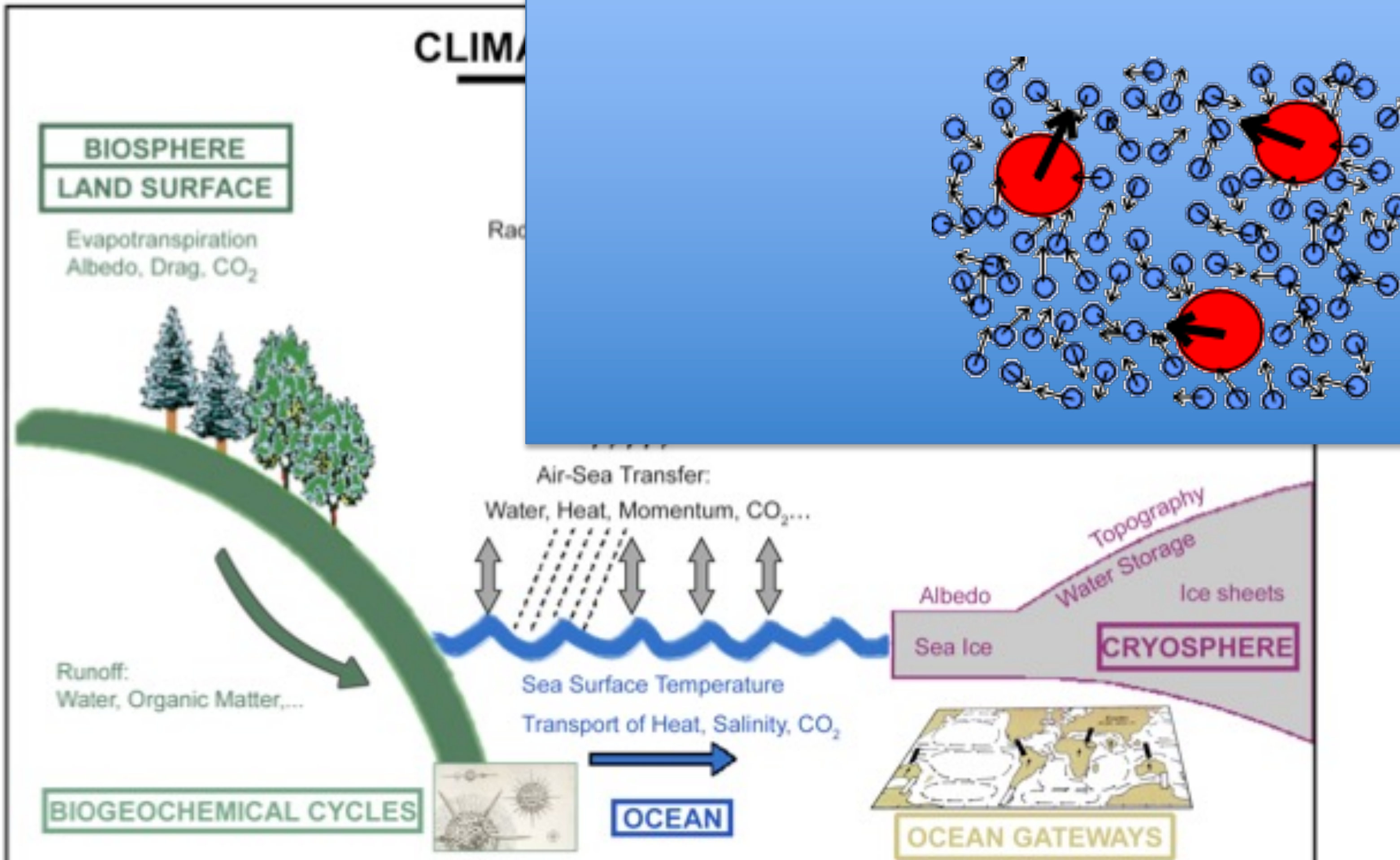
Because this motion often drives the interaction of time and spatial scales, it is important in several fields.



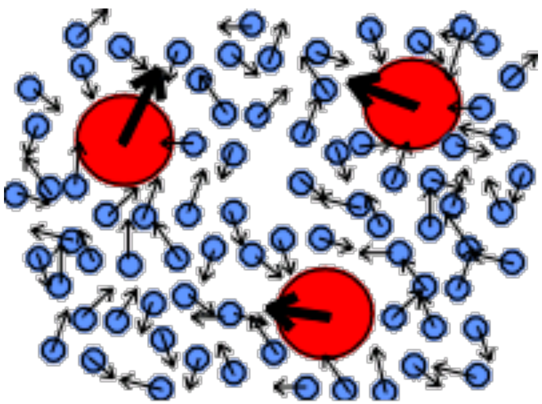
Following an idea of Hasselmann one can divide the climate dynamics into two parts. These two parts are the slowly changing climate part and rapidly changing weather part. **The** weather part can be modeled by a stochastic process such as white noise

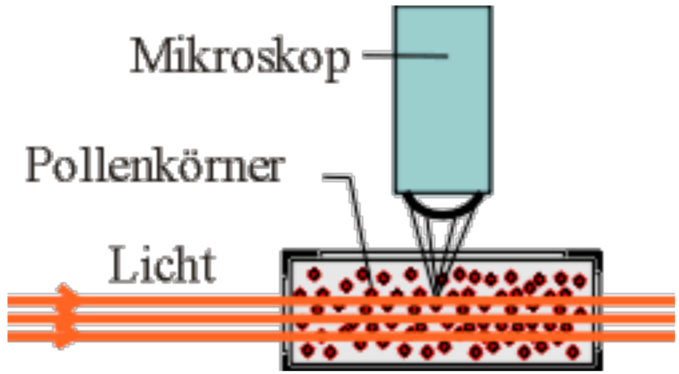
Climate variability

- Brownian Particle: Climate
- Molecules: Weather



Brownian Motion: visible under the Mikroscope: Motion of particles

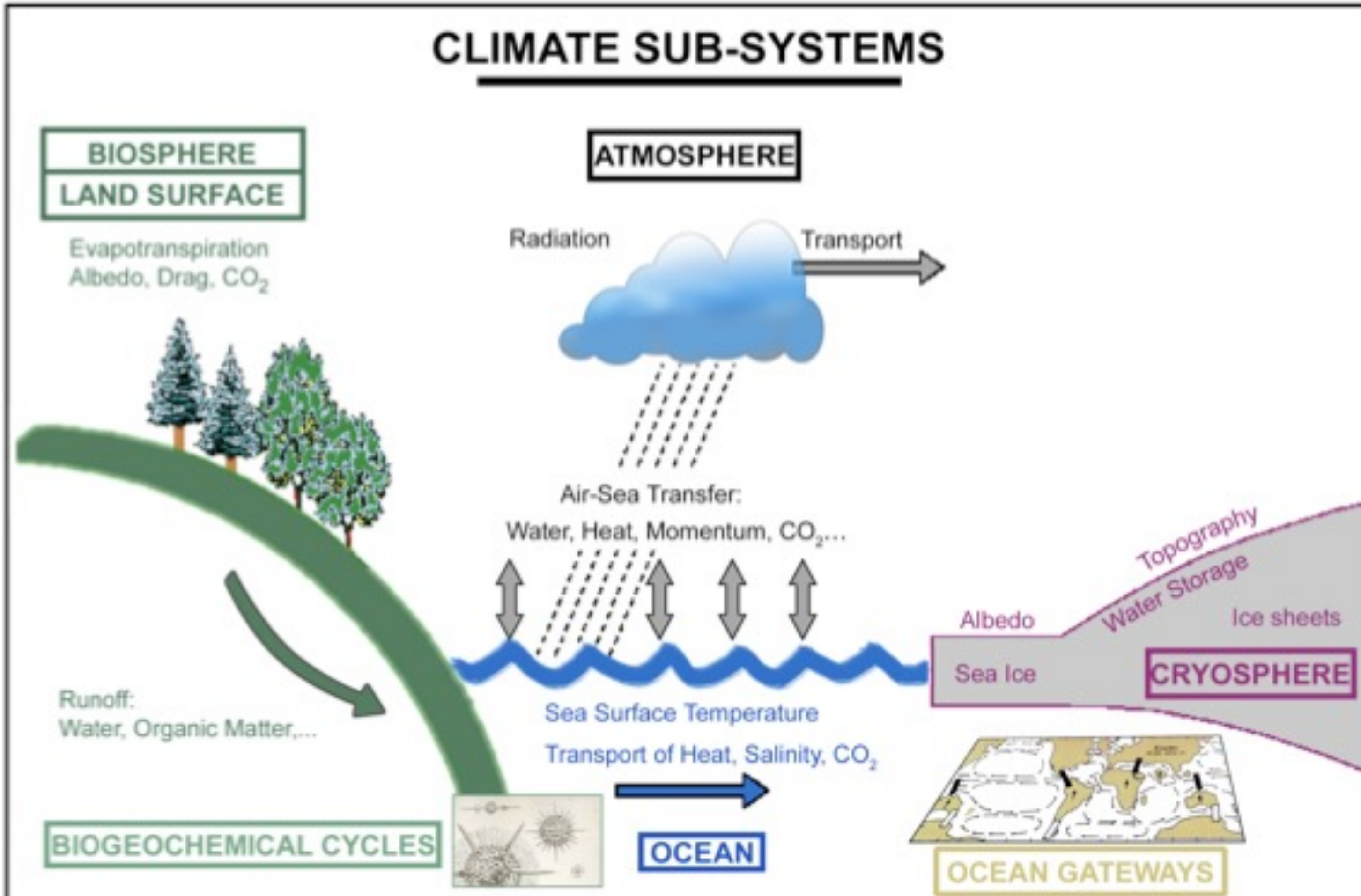




pulses, irregular
Living?
Pulses from all directions, random

Physics of the 20th century

- The matter the world is made of
- views: Elementary particles, quantum mechanics, relativity theory
- Limit of divisibility (Democritus, Aristotle): Matter is not a continuous whole: "The world cannot be composed of infinitely small particles".


CLIMATE SUB-SYSTEMS
BIOSPHERE
LAND SURFACE
Evapotranspiration
Albedo, Drag, CO_2
ATMOSPHERE
Radiation
Transport
Air-Sea Transfer:
Water, Heat, Momentum, CO_2,...
Topography
Water Storage
Albedo
Ice sheets
Sea Ice
CRYOSPHERE
Runoff:
Water, Organic Matter,...
Sea Surface Temperature
Transport of Heat, Salinity, CO_2
BIOGEOCHEMICAL CYCLES
OCEAN
OCEAN GATEWAYS

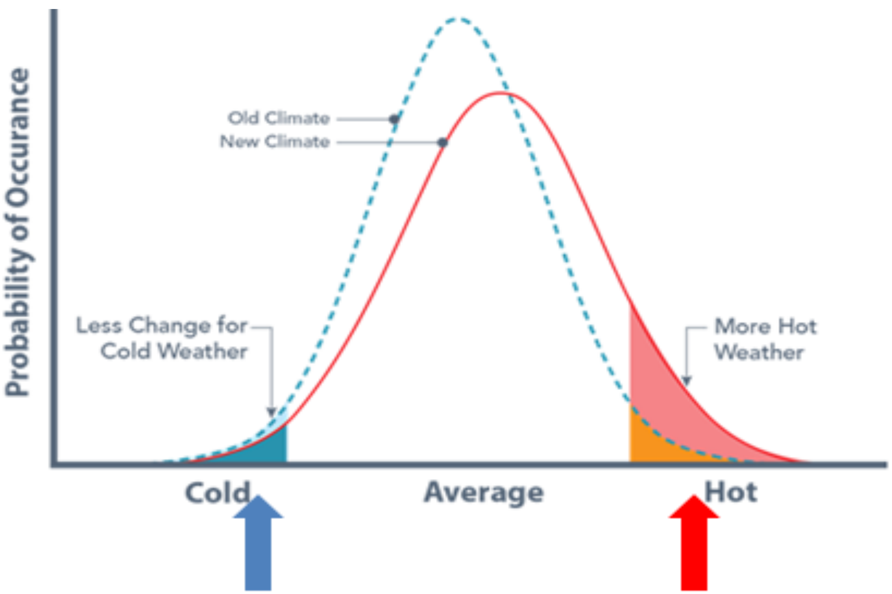
Das Klimaproblem aus physikalischer Sicht

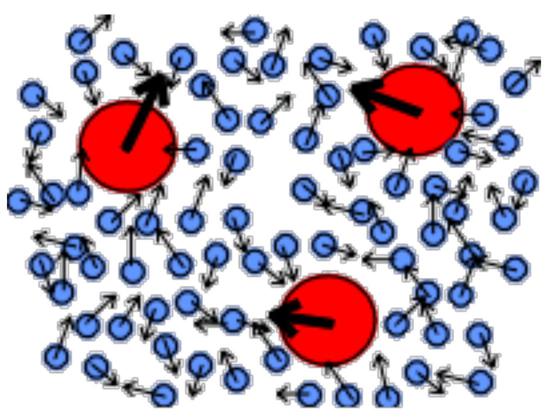
Climate

Brownsche Partikel: Klim

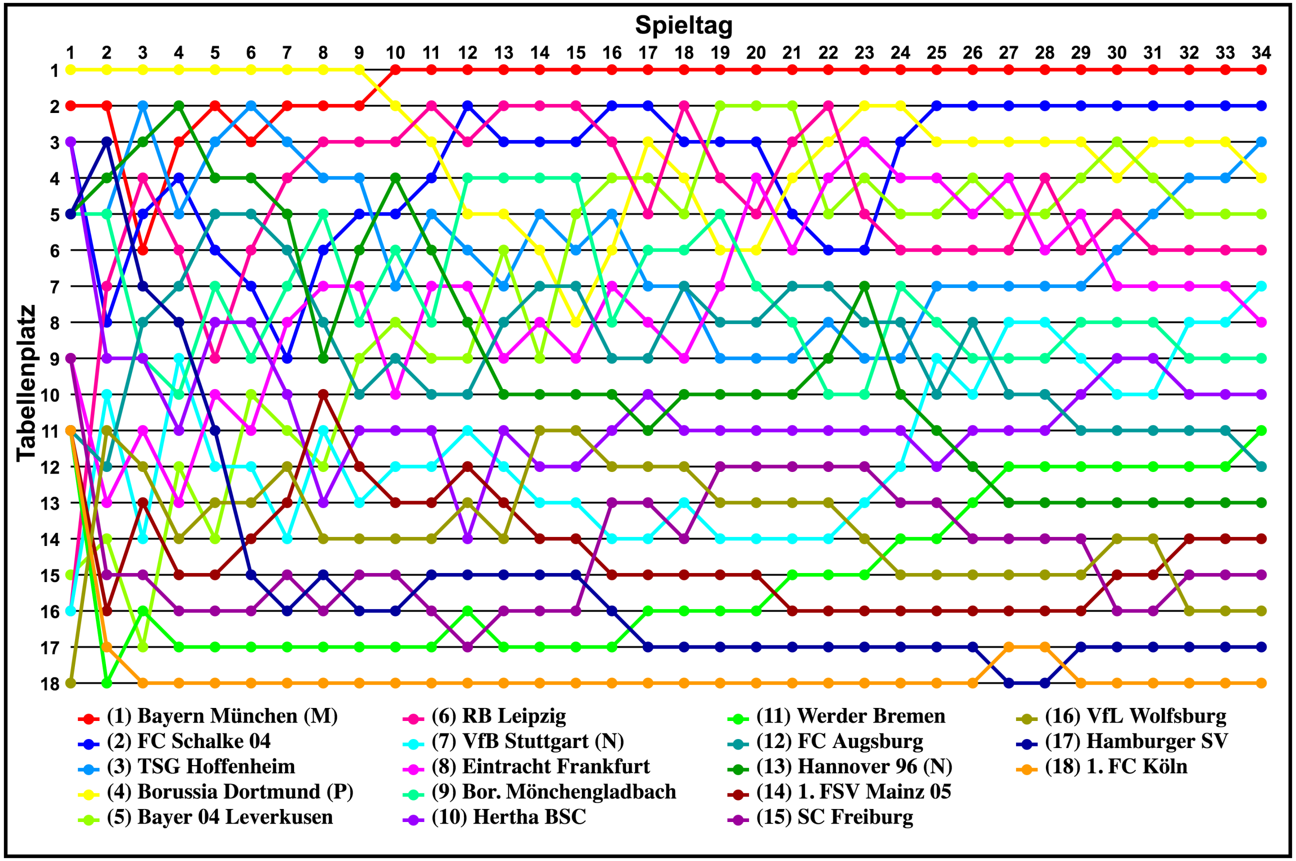
Moleküle: Wetter

Probabilities





Predictability



Coarse graining -> Stochastic

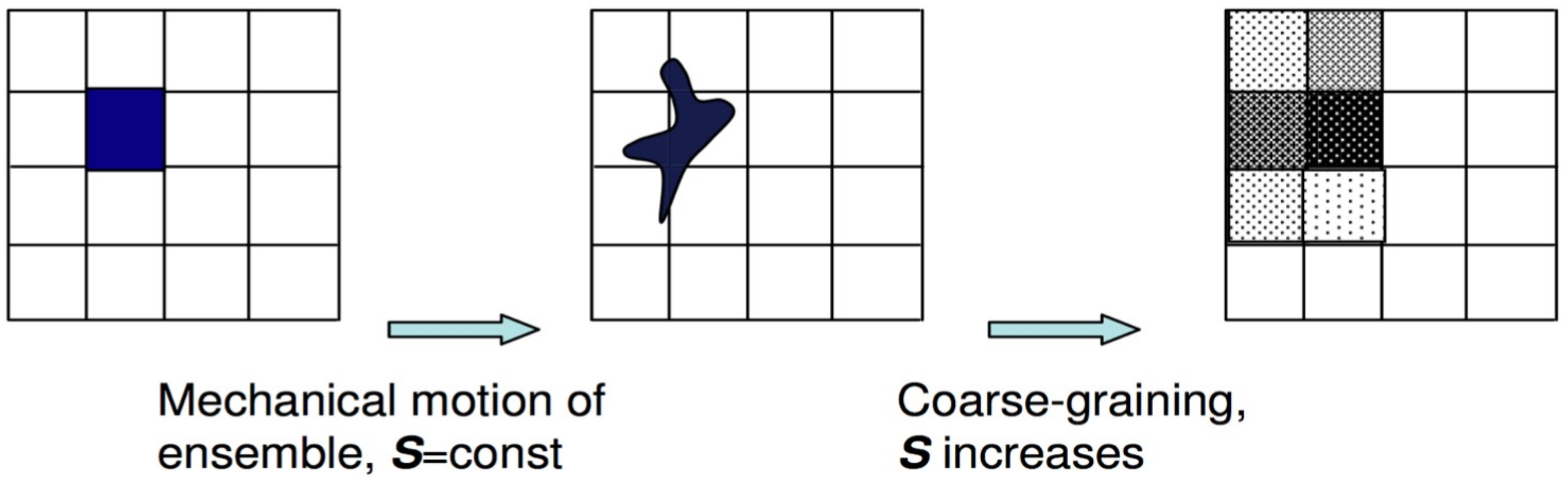


Figure 8.11: The Ehrenfests coarse-graining: two motion - coarse-graining cycles in 2D (values of probability density are presented by hatching density).